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Cohomology operations in the space of loops on the exceptional Lie group E_6

By

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Let E_6 be the compact 1-connected exceptional Lie group of rank 6. In [9] we determined the Hopf algebra structure of $H_*(\Omega E_6;\mathbb{Z})$ by the generating variety approach of R. Bott [1]. In this case, as a generating variety we can take $EIII$, the irreducible Hermitian symmetric space of exceptional type. Then as Bott pointed out in [1], §6, we can determine the action of the mod p Steenrod algebra $\mathcal{A}_p$ on $H^*(\Omega E_6;\mathbb{Z}_p)$ from that on $H^*(EIII;\mathbb{Z}_p)$ for all primes p.

In this paper, for ease of algebraic description, we compute the action of $\mathcal{A}_{p_*}$, the dual of $\mathcal{A}_p$, on $H_*(\Omega E_6;\mathbb{Z}_p)$ for $p = 2, 3$ (For larger primes the description is easy). In the course of computation we also determine the action of $\mathcal{A}_3$ on $H^*(E_6/T;\mathbb{Z}_3)$, where T is a maximal torus of E_6.

The paper is constructed as follows: In Section 2 we recall some results concerning the cohomology of some homogeneous spaces of E_6. In Section 3 by considering the action of the Weyl group on E_6/T, we determine the cohomology operations in $EIII$. Using the results obtained, in Section 4 we shall determine the cohomology operations in ΩE_6.

Throughout this paper $\sigma_i(x_1,\dots,x_n)$ denotes the i-th elementary symmetric function in the variables $x_1,\dots,x_n$.

1. Preliminaries

Let T be a maximal torus of E_6 and we use the root system $\{\alpha_i\}_{1\le i\le 6}$ given in [2]. We denote the corresponding fundamental weights by $\{w_i\}_{1\le i\le 6}$. As usual we may regard roots and weights as elements of $H^1(T;\mathbb{Z}) \xrightarrow{\sim} H^2(BT;\mathbb{Z})$. Then $\{w_i\}_{1\le i\le 6}$ forms a basis of $H^2(BT;\mathbb{Z})$ and $H^*(BT;\mathbb{Z}) = \mathbb{Z}[w_1, w_2, \dots, w_6]$.

Let C_1 (resp. C_2) be the centralizer of the 1-dimensional torus determined by $\alpha_j = 0$ $(j \ne 1)$ (resp. $\alpha_j = 0$ $(j \ne 2)$). Then as is well known

$$\begin{aligned} C_1 &= T^1 \cdot Spin(10), & T^1 \cap Spin(10) &\cong \mathbb{Z}_4,\\ C_2 &= T^1 \cdot SU(6), & T^1 \cap SU(6) &\cong \mathbb{Z}_2. \end{aligned}$$

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Let R_i denote the reflection to the hyperplane $\alpha_i = 0$, then the Weyl groups $W(\cdot)$ of E_6, C_i $(i = 1, 2)$ are finite groups generated by these reflections:

$$\begin{aligned} W(E_6) &= \langle R_i \ (1 \le i \le 6) \rangle, \\ W(C_1) &= \langle R_i \ (i \ne 1) \rangle, \\ W(C_2) &= \langle R_i \ (i \ne 2) \rangle. \end{aligned}$$

Following [10], we introduce elements of $H^2(BT; \mathbb{Z})$ by

$$\begin{aligned} &t_6 = w_6, \quad t_i = R_{i+1}(t_{i+1}) \ (2 \le i \le 5), \quad t_1 = R_1(t_2), \\ &c_i = \sigma_i(t_1, \ldots, t_6), \quad t = \frac{1}{3} c_1 = w_2 \end{aligned} \tag{1.1}$$

and denote by the same symbols for the images of t_i's and t under the cohomology homomorphism induced by the natural map $E_6/T \longrightarrow BT$. Then we have the following isomorphism and the table of the action of $W(E_6)$ on these elemeмts:

$$H^*(BT; \mathbb{Z}) = \mathbb{Z}[t_1, t_2, \ldots, t_6, t]/(c_1 - 3t).$$

	R_1	R_2	R_3	R_4	R_5	R_6
t_1	t_2	$t - b_1 + t_1$				
t_2	t_1	$t - b_1 + t_2$	t_3			
t_3		$t - b_1 + t_3$	t_2	t_4		
t_4				t_3	t_5	
t_5					t_4	t_6
t_6						t_5
t		$-t + a_1$				

Table 1.

where $b_1 = t_1 + t_2 + t_3$, $a_1 = t_4 + t_5 + t_6$ and blanks indicate the trivial action.

Consider the two fibrations

$$\begin{aligned} SO(10)/T' \cong C_1/T &\xrightarrow{i} E_6/T \xrightarrow{p} E_6/C_1 = EIII, \\ SU(6)/T'' \cong C_2/T &\xrightarrow{j} E_6/T \xrightarrow{q} E_6/C_2, \end{aligned}$$

where T', T'' are standard maximal tori of $SO(10), SU(6)$ respectively. By the classical results of R. Bott, both the fibre and the base have no odd dimensional cohomology in either case. Hence the Serre spectral sequences of these fibrations collapse for any coefficient ring Λ and we have

Lemma 1.1.

$$\begin{aligned} p^* &: H^*(EIII; \Lambda) \longrightarrow H^*(E_6/T; \Lambda), \\ q^* &: H^*(E_6/C_2; \Lambda) \longrightarrow H^*(E_6/T; \Lambda) \end{aligned}$$

are split monomorphisms for any coefficient ring Λ.

The integral cohomology ring of E_6/T (resp. $EIII$) is determined in [10], Theorem B (resp. Corollary C). The results are as follows:

Theorem 1.1.
(i)

$$H^*(E_6/T;\mathbb{Z}) = \mathbb{Z}[t_1,\dots,t_6,t,\gamma_3,\gamma_4]/(\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6,\rho_8,\rho_9,\rho_{12}),$$

where $t_1,\dots,t_6,t$ are as in (1.1)*, $\gamma_3 \in H^6$, $\gamma_4 \in H^8$ and*

$$\begin{aligned}
&\rho_1 = c_1 - 3t, \quad \rho_2 = c_2 - 4t^2, \quad \rho_3 = c_3 - 2\gamma_3, \quad \rho_4 = c_4 + 2t^4 - 3\gamma_4,\\
&\rho_5 = c_5 - 3t\gamma_4 + 2t^2\gamma_3, \quad \rho_6 = {\gamma_3}^2 + 2c_6 - 3t^2\gamma_4 + t^6,\\
&\rho_8 = 3{\gamma_4}^2 - 6t\gamma_3\gamma_4 - 9t^2c_6 + 15t^4\gamma_4 - 6t^5\gamma_3 - t^8,\\
&\rho_9 = t_0^9 - 3t_0w^2, \quad \rho_{12} = w^3 + 15t_0^4w^2 - 9t_0^8w
\end{aligned}$$

for

$$\begin{aligned}
&c_i = \sigma_i(t_1,\dots,t_6),\ t_0 = t - t_1,\\
&w = \gamma_4 + (-2t_1 - t_0)\gamma_3 + 2t_1^4 + 6t_1^3t_0 + 7t_1^2t_0^2 + 3t_1t_0^3 + t_0^4.
\end{aligned}$$

(ii)

$$H^*(EIII;\mathbb{Z}) = \mathbb{Z}[t_0,w]/(t_0^9 - 3t_0w^2, w^3 + 15t_0^4w^2 - 9t_0^8w),$$

where $t_0 \in H^2, w \in H^8$ and the generator w can be chosen so that it coincides with the above w of (i) *under the natural injection $p^* : H^8(EIII;\mathbb{Z}) \longrightarrow H^8(E_6/T;\mathbb{Z})$.*

2. The cohomology operations in $H^*(EIII;\mathbb{Z}_p)$ for $p = 2,3$

The Case $p = 2$. The mod 2 cohomology of $EIII$ is easily obtained from Theorem 1.1. Furthermore the squaring operations in $H^*(EIII;\mathbb{Z}_2)$ are also determined in [6], Theorem 2.4. The results are as follows:

Theorem 2.1.

$$H^*(EIII;\mathbb{Z}_2) = \mathbb{Z}_2[t_0,w]/(t_0^9 + t_0w^2, w^3 + t_0^4w^2 + t_0^8w),$$

where $\deg(t_0) = 2$*,* $\deg(w) = 8$ *and*

$$\begin{aligned}
&Sq^2(t_0) = t_0^2,\\
&Sq^2(w) = t_0^5 + t_0w, \qquad Sq^4(w) = t_0^6, \qquad Sq^8(w) = w^2.
\end{aligned}$$

The Case $p = 3$. The rest of this section and the next section are devoted to the determination of the reduced power operations in $H^*(EIII;\mathbb{Z}_3)$.

From Lemma 1.1

$$p^* : H^*(EIII;\mathbb{Z}_3) \longrightarrow H^*(E_6/T;\mathbb{Z}_3)$$

is injective. Therefore the action of the reduced power operations $\mathcal{P}^i$ on $EIII$ is deduced from that on E_6/T.

From Theorem 1.1 the mod 3 cohomology of E_6/T is given by

$$\begin{aligned} H^*(E_6/T;\mathbb{Z}_3) = \mathbb{Z}_3[t_1,\dots,t_6,t,\gamma_4] \\ /(c_1, c_2-t^2, c_4-t^4, c_5+t^2c_3, c_3^2-c_6+t^6, t^8, t_0^9, w^3), \end{aligned} \tag{2.1}$$

where

$$t_0 = t - t_1,\ w \equiv \gamma_4 + (-t_1+t_0)c_3 - t_1^4 + t_1^2t_0^2 + t_0^4 \mod 3.$$

Note that in $H^*(E_6/T;\mathbb{Z}_3)$

$$\begin{aligned} t_0^9 &\equiv c_3c_6, \\ w^3 &\equiv {\gamma_4}^3 - t^6c_6, \end{aligned}$$

so that the relations t_0^9, w^3 are replaced with $c_3c_6, {\gamma_4}^3 - t^6c_6$ respectively.

Therefore the problem is to determine the action of $\mathcal{P}^i$ on γ_4. For this purpose we consider the action of the Weyl group $W(E_6)$ on $H^*(E_6/T;\Lambda), \Lambda = \mathbb{Z}$ or $\mathbb{Z}_3$ (for this account see also [7, §3]). From Table 1 R_i $(i \neq 2)$ act trivially on t and $\{c_n\}_{1\le n\le 6}$. Therefore they act trivially on γ_4 by the definition of $\gamma_4, 3\gamma_4 = c_4 + 2t^4$.

Next consider the action of R_2 on $\{c_n\}_{1\le n\le 6},\ \gamma_4$. From now on we use the notation

$$R = R_2 \quad \text{and} \quad \bar{R} = R - id.$$

We put

$$b_i = \sigma_i(t_1,t_2,t_3) \quad \text{and} \quad a_j = \sigma_j(t_4,t_5,t_6) \in H^*(E_6/T;\mathbb{Z})$$

so that

$$c_n = \sum_{i+j=n} b_ia_j. \tag{2.2}$$

Substituting $c_1 = 3t,\ c_2 = 4t^2$ in $H^*(E_6/T;\mathbb{Z})$ into (2.2) we obtain

$$\begin{aligned} b_1 &= 3t - a_1, \\ b_2 &= 4t^2 - 3a_1t + a_1^2 - a_2, \\ b_3 &= c_3 - 4a_1t^2 + (3a_1^2 - 3a_2)t - a_3 + 2a_1a_2 - a_1^3. \end{aligned} \tag{2.3}$$

From (2.2), (2.3) we can write $c_n, n = 4,5,6$ in terms of t, c_3, a_j's. Applying the mod 3 reduction we obtain

$$\begin{aligned} c_4 &\equiv c_3a_1 - a_2^2 + a_1a_3 - a_1^4 \mod (3,t), \\ c_5 &\equiv c_3a_2 + a_2a_3 - a_1a_2^2 + a_1^2a_3 - a_1^3a_2 \mod (3,t), \\ c_6 &\equiv c_3a_3 - a_3^2 - a_1a_2a_3 - a_1^3a_3 \mod (3,t). \end{aligned} \tag{2.4}$$

Since

$$\begin{aligned}\sum_{i=0}^{3} R(b_i) &= R\left(\sum_{i=0}^{3} b_i\right) = R\left(\prod_{i=1}^{3}(1+t_i)\right) = \prod_{i=1}^{3}(1+R(t_i)) \\ &= \prod_{i=1}^{3}(1+t-b_1+t_i) = \sum_{i=0}^{3}(1+t-b_1)^{3-i}b_i,\end{aligned} \tag{2.5}$$

we have

$$\begin{aligned}\bar{R}(b_1) &= -6t+3a_1, \\ \bar{R}(b_2) &= -2a_1t+a_1^2, \\ \bar{R}(b_3) &= -4t^3+6a_1t^2+(-4a_1^2+2a_2)t-a_1a_2+a_1^3.\end{aligned} \tag{2.6}$$

Since $R(a_j)=a_j$ by Table 1

$$\bar{R}(c_n) = \sum_{i+j=n} \bar{R}(b_i)a_j. \tag{2.7}$$

From (2.6), (2.7) we can write $\bar{R}(c_n), n=3,4,5,6$ in terms of t, a_j's. In particular

$$\bar{R}(c_4+2t^4) = 3\{-4a_1t^3+6a_1^2t^2+(-4a_1^3-2a_3)t+a_1^4+a_1a_3\},$$

which implies

$$\begin{aligned}\bar{R}(\gamma_4) &= -4a_1t^3+6a_1^2t^2+(-4a_1^3-2a_3)t+a_1^4+a_1a_3 \\ &\equiv a_1^4+a_1a_3 \mod (t)\end{aligned}$$

by the definition of γ_4. Applying the mod 3 reduction we obtain the following results:

	$\bar{R}(x) \mod (t)$
t	a_1
c_3	$-a_1a_2-a_1^3$
c_4	a_1^4
c_5	$-a_1a_2^2+a_1^2a_3+a_1^3a_2$
c_6	$-a_1a_2a_3+a_1^3a_3$
γ_4	$a_1a_3+a_1^4$

Table 2.

3. The action of $\mathcal{P}^i$ on γ_4

The purpose of this section is to determine $\mathcal{P}^i(\gamma_4)$ for $i=1,3$ (the other cases follow from the axioms of the reduced power operations).

From Lemma 1.1

$$q^* : H^*(E_6/C_2;\mathbb{Z}_3) \longrightarrow H^*(E_6/T;\mathbb{Z}_3)$$

is injective and we can identify $H^*(E_6/C_2;\mathbb{Z}_3)$ with $\operatorname{Im} q^*$ and regard it as a subalgebra of $H^*(E_6/T;\mathbb{Z}_3)$.

Notation. $A = H^*(E_6/C_2;\mathbb{Z}_3) \hookrightarrow B = H^*(E_6/T;\mathbb{Z}_3)$

On the other hand the integral cohomology ring of E_6/C_2 is determined in [5], Theorem 3.2. From this the following is easily obtained:

$$A = \mathbb{Z}_3[t, c_3, \gamma_4, c_6]/(c_3^2 - c_6 + t^6, t^8, c_3c_6, {\gamma_4}^3 - t^6c_6). \tag{3.1}$$

An additive basis of A as a $\mathbb{Z}_3$-vector space for degree ≤ 20 is given by

deg	0	2	4	6	8	10	12	14	16	18	20
	1	t	t^2	t^3	t^4	t^5	t^6	t^7			
				c_3	tc_3	t^2c_3	t^3c_3	t^4c_3	t^5c_3	t^6c_3	t^7c_3
					γ_4	$t\gamma_4$	$t^2\gamma_4$	$t^3\gamma_4$	$t^4\gamma_4$	$t^5\gamma_4$	$t^6\gamma_4$
							c_6	tc_6	t^2c_6	t^3c_6	t^4c_6
								$c_3\gamma_4$	$tc_3\gamma_4$	$t^2c_3\gamma_4$	$t^3c_3\gamma_4$
									${\gamma_4}^2$	$t{\gamma_4}^2$	$t^2{\gamma_4}^2$

Table 3.

Now we regard $\bar{R}$ as a homomorphism

$$\bar{R} : B \longrightarrow B \longrightarrow B/(t)$$

and restrict it to the subalgebra A (also denoted by $\bar{R}$). Then since the ideal $(t) \subset B$ generated by t is closed under the action of $\mathcal{P}^i$, we have the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{q^*} & B & \xrightarrow{\bar{R}} & B/(t) \\ {\scriptstyle\mathcal{P}^i}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{P}^i} & & \big\downarrow{\scriptstyle\mathcal{P}^i} \\ A & \xrightarrow[q^*]{} & B & \xrightarrow[\bar{R}]{} & B/(t)\ . \end{array} \tag{3.2}$$

Now let us determine the action of $\mathcal{P}^i$ on γ_4 for $i = 1, 3$. Using Tables 2 and 3 $\bar{R}$ becomes a monomorphism on degree 12. On the other hand in the expression $\bar{R}(\mathcal{P}^1(\gamma_4)) \equiv \mathcal{P}^1(\bar{R}(\gamma_4))$, the right hand side is computed by Table 2 and the next lemma, which is easily obtained.

Lemma 3.1. *For $a_j = \sigma_j(t_4, t_5, t_6) \in H^*(E_6/T;\mathbb{Z}_3)$ we have*

$$\mathcal{P}^1(a_1) \equiv a_1^3,$$
$$\mathcal{P}^1(a_2) \equiv a_2^2 + a_1^2a_2 - a_1a_3,$$
$$\mathcal{P}^1(a_3) \equiv a_2a_3 + a_1^2a_3.$$

Then by the injectivity of $\bar{R}$ we obtain

$$\mathcal{P}^1(\gamma_4) \equiv -c_6 + t^6.$$

Since $\bar{R}(t^7c_3) \equiv 0$, $\bar{R}$ does not become a monomorphism on degree 20. But similar computation yields

$$\mathcal{P}^3(\gamma_4) \equiv -t^4c_6 + m \cdot t^7c_3$$

for some $m \in \mathbb{Z}_3$. Hence under the monomorphism p^*

$$\begin{aligned} \mathcal{P}^3(w) &\equiv \mathcal{P}^3(\gamma_4 + (-t_1 + t_0)c_3 - t_1^4 + t_1^2t_0^2 + t_0^4) \\ &\equiv m \cdot (t_1^7 + t_1^6t_0 - t_1^4t_0^3 - t_1^3t_0^4 + t_1t_0^6 + t_0^7)c_3. \end{aligned} \tag{3.3}$$

On the other hand from Theorem 1.1 we can put

$$\begin{aligned} \mathcal{P}^3(w) &\equiv k \cdot t_0^6w + l \cdot t_0^2w^2 \\ &\equiv l \cdot t_0^2{\gamma_4}^2 + (k-l) \cdot t_0^6\gamma_4 + \cdots \end{aligned} \tag{3.4}$$

for some $k, l \in \mathbb{Z}_3$. From (3.3), (3.4) we deduce $k = l = m = 0$ by the linearly independence of monomials in $H^{20}(E_6/T; \mathbb{Z}_3)$.

Remark 1. In the above computations note that the following relations hold in $B/(t)$ which are derived from (2.1), (2.4):

$$\begin{aligned} &a_2^3 \equiv a_1^2a_2^2 - a_1^3a_3, \quad a_1^7 \equiv 0, \\ &a_3^3 \equiv a_1^2a_2^2a_3 + a_1^3a_3^2 - a_1^4a_2a_3 - a_1^6a_3. \end{aligned}$$

Summarizing these we obtain the following results:

Proposition 3.1.
(i) *The action of $\mathcal{P}^i$ on γ_4 is given by*

$$\begin{aligned} &\mathcal{P}^1(\gamma_4) = -c_6 + t^6, \quad \mathcal{P}^2(\gamma_4)\ (= -\mathcal{P}^1\mathcal{P}^1(\gamma_4)) = t^2c_6, \\ &\mathcal{P}^3(\gamma_4) = -t^4c_6, \quad \mathcal{P}^4(\gamma_4) = {\gamma_4}^3 = t^6c_6. \end{aligned}$$

(ii) *The reduced power operations in $H^*(EIII; \mathbb{Z}_3)$ are given as follows*:

$$\begin{aligned} &\mathcal{P}^1(t_0) = t_0^3, \\ &\mathcal{P}^1(w) = -t_0^6, \quad \mathcal{P}^2(w)(= -\mathcal{P}^1\mathcal{P}^1(w)) = 0, \quad \mathcal{P}^3(w) = 0, \\ &\mathcal{P}^4(w) = w^3 = 0. \end{aligned}$$

4. Cohomology operations in $H_*(\Omega E_6; \mathbb{Z}_p)$ for $p = 2, 3$

In this section, using the results obtained so far we determine the cohomology operations in ΩE_6. Hereafter we use the notations and the results of [9] without specific references.

First consider the case $p = 2$: From [9], Theorem 1.1, the mod 2 homology of ΩE_6 is given by

Theorem 4.1.

$$H_*(\Omega E_6;\mathbb{Z}_2)=\mathbb{Z}_2[\sigma_1,\sigma_2,\sigma_4,\sigma_5,\sigma_7,\sigma_8,\sigma_{11}]/({\sigma_1}^2),$$

where $\deg(\sigma_i)=2i$. *Moreover* $\sigma_1,\tilde{\sigma}_5=\sigma_1{\sigma_2}^2+\sigma_5,\tilde{\sigma}_7=\sigma_2\sigma_5+\sigma_7,\tilde{\sigma}_{11}=\sigma_1{\sigma_5}^2+\sigma_2\sigma_7+\sigma_{11}$ *are primitive and* $\tilde{\psi}(\sigma_2)=\sigma_1\otimes\sigma_1$.

From Theorem 4.1 the primitive elements of $H_*(\Omega E_6;\mathbb{Z}_2)$ which appear in degree ≤ 22 are given by

deg	2	8	10	14	16	20	22
	σ_1	${\sigma_2}^2$	$\tilde{\sigma}_5$	$\tilde{\sigma}_7$	${\sigma_2}^4$	$\tilde{\sigma}_5^2$	$\tilde{\sigma}_{11}$

Table 4.

Let $Sq^i_*\in\mathcal{A}_{2*}$ be the dual of the squaring operation $Sq^i\in\mathcal{A}_2$, that is

$$\langle a,Sq^i_*(\alpha)\rangle=\langle Sq^i(a),\alpha\rangle,$$

where $a\in H^*,\alpha\in H_*$ and $\langle\ ,\ \rangle$ is the Kronecker pairing (For the properties of Sq^i_*, see [11, §3]).

Let us determine the squaring operations in $H_*(\Omega E_6;\mathbb{Z}_2)$. By Theorem 4.1 we have only to determine the $Sq^i_*(\ )$ on the elements $\sigma_1,\sigma_2,\sigma_4,\tilde{\sigma}_5,\tilde{\sigma}_7,\sigma_8,\tilde{\sigma}_{11}$.

(1) Since $Sq^2(a_1)=a_1^2=a_2$, $Sq^2_*(\sigma_2)=\sigma_1$.

(2) By Theorem 4.1 we can put

$$Sq^2_*(\sigma_4)=k\cdot\sigma_1\sigma_2$$

for some $k\in\mathbb{Z}_2$. On the other hand since $Sq^2(a_3)=Sq^2(a_1^3)=a_1^4\equiv b_4$ we have

$$k=\langle a_3,Sq^2_*(\sigma_4)\rangle=\langle Sq^2(a_3),\sigma_4\rangle=\langle b_4,\sigma_4\rangle=1.$$

Thus

$$Sq^2_*(\sigma_4)=\sigma_1\sigma_2.$$

Since $Sq^4(a_2)=a_2^2\equiv b_4$ we obtain

$$Sq^4_*(\sigma_4)=\sigma_2.$$

(3) Since $Sq^i_*(\ )$ sends primitive elements to primitive elements, we make use of a pattern of computation stated in [11], p. 476 for $\tilde{\sigma}_5,\tilde{\sigma}_7,\tilde{\sigma}_{11}$. So details are omitted.

(4) By Table 4 we can put

$$Sq^2_*(\sigma_8)=k\cdot\sigma_1{\sigma_2}^3+l\cdot\sigma_1\sigma_2\sigma_4+m\cdot\sigma_2\sigma_5+n\cdot\sigma_7$$

for some $k,l,m,n\in\mathbb{Z}_2$. Dualizing this gives

$$Sq^2(a_7)=b_8+c_8+d_8+k\cdot e_8,\tag{4.1}$$

$$Sq^2(b_7)=l\cdot e_8,\tag{4.2}$$

$$Sq^2(c_7)=m\cdot e_8,\tag{4.3}$$

$$Sq^2(d_7)=n\cdot e_8.\tag{4.4}$$

Applying g_s^* on both sides of (4.4), then using Theorem 2.1

$$\begin{aligned} l.h.s. &= g_s^* Sq^2(d_7) = Sq^2 g_s^*(d_7) = Sq^2(d) \equiv Sq^2(t_0^3 w) = t_0^8, \\ r.h.s. &= n \cdot e \equiv n \cdot t_0^8. \end{aligned}$$

Therefore $n = 1$. Similarly from (4.3), (4.2)

$$\begin{aligned} g_s^* Sq^2(c_7) &= Sq^2 g_s^*(c_7) = Sq^2(d') \equiv Sq^2(t_0^7) = t_0^8, \\ g_s^* Sq^2(b_7) &= Sq^2 g_s^*(c_7) = Sq^2(-d') \equiv Sq^2(t_0^7) = t_0^8. \end{aligned}$$

Therefore $m = 1, l = 1$. Finally applying g_s^* on both sides of (4.1), then

$$\begin{aligned} l.h.s. &= g_s^* Sq^2(a_7) = Sq^2 g_s^*(a_7) = 0, \\ r.h.s. &= g_s^*(b_8) + g_s^*(c_8) + g_s^*(d_8) + k \cdot g_s^*(e_8) \\ &= (e' + e'') + (2e' + 8e'') + (-e' - 3e'') + k \cdot (-e' - 3e'') \\ &\equiv k \cdot t_0^4 w. \end{aligned}$$

Therefore $k = 0$. Thus

$$Sq_*^2(\sigma_8) = \sigma_1\sigma_2\sigma_4 + \sigma_2\sigma_5 + \sigma_7 = \sigma_1\sigma_2\sigma_4 + \tilde{\sigma}_7.$$

Similar computations give the results for $Sq_*^4(\sigma_8), Sq_*^6(\sigma_8), Sq_*^8(\sigma_8)$.

Thus we obtain the following results:

Theorem 4.2. *The squaring operations in*

$$H_*(\Omega E_6; \mathbb{Z}_2) = \mathbb{Z}_2[\sigma_1, \sigma_2, \sigma_4, \tilde{\sigma}_5, \tilde{\sigma}_7, \sigma_8, \tilde{\sigma}_{11}]/({\sigma_1}^2)$$

are given as follows:

$$\begin{aligned} &Sq_*^2(\sigma_2) = \sigma_1, \\ &Sq_*^2(\sigma_4) = \sigma_1\sigma_2, \quad Sq_*^4(\sigma_4) = \sigma_2, \\ &Sq_*^2(\tilde{\sigma}_5) = {\sigma_2}^2, \quad Sq_*^4(\tilde{\sigma}_5) = 0, \\ &Sq_*^2(\tilde{\sigma}_7) = 0, \quad Sq_*^4(\tilde{\sigma}_7) = \tilde{\sigma}_5, \quad Sq_*^6(\tilde{\sigma}_7) = 0, \\ &Sq_*^2(\sigma_8) = \sigma_1\sigma_2\sigma_4 + \tilde{\sigma}_7, \quad Sq_*^4(\sigma_8) = \sigma_2\sigma_4, \\ &Sq_*^6(\sigma_8) = \sigma_5, \quad Sq_*^8(\sigma_8) = \sigma_4, \\ &Sq_*^2(\tilde{\sigma}_{11}) = \tilde{\sigma}_5^2, \quad Sq_*^4(\tilde{\sigma}_{11}) = 0, \quad Sq_*^6(\tilde{\sigma}_{11}) = 0, \\ &Sq_*^8(\tilde{\sigma}_{11}) = \tilde{\sigma}_7, \quad Sq_*^{10}(\tilde{\sigma}_{11}) = 0. \end{aligned}$$

The computations for $p = 3$ are similar and therefore we exhibit the data and the results. From [9], Theorem 1.1, the mod 3 homology of ΩE_6 is given by

Theorem 4.3.

$$H_*(\Omega E_6; \mathbb{Z}_3) = \mathbb{Z}_3[\sigma_1, \sigma_3, \sigma_4, \sigma_5, \sigma_7, \sigma_8, \sigma_{11}]/({\sigma_1}^3),$$

where $\deg(\sigma_i) = 2i$. *Moreover* $\sigma_1, \tilde{\sigma}_4 = -\sigma_1\sigma_3 + \sigma_4, \tilde{\sigma}_5 = {\sigma_1}^2\sigma_3 - \sigma_5, \tilde{\sigma}_7 = -\sigma_1{\sigma_3}^2 + {\sigma_1}^2\sigma_5 + \sigma_7, \tilde{\sigma}_8 = {\sigma_1}^2{\sigma_3}^2 - \sigma_3\sigma_5 - {\sigma_4}^2 + \sigma_8, \tilde{\sigma}_{11} = -{\sigma_3}^2\sigma_5 - \sigma_1{\sigma_5}^2 - \sigma_4\sigma_7 - \sigma_{11}$ *are primitive and* $\tilde{\psi}(\sigma_3) = -{\sigma_1}^2 \otimes \sigma_1$.

From Theorem 4.3 the primitive elements of $H_*(\Omega E_6; \mathbb{Z}_3)$ which appear in degree ≤ 22 are given by

deg	2	8	10	14	16	18	22
	σ_1	$\tilde{\sigma}_4$	$\tilde{\sigma}_5$	$\tilde{\sigma}_7$	$\tilde{\sigma}_8$	${\sigma_3}^3$	$\tilde{\sigma}_{11}$

Table 5.

Using Proposition 3.1 we obtain

Theorem 4.4. *The reduced power operations in*

$$H_*(\Omega E_6; \mathbb{Z}_3) = \mathbb{Z}_3[\sigma_1, \sigma_3, \tilde{\sigma}_4, \tilde{\sigma}_5, \tilde{\sigma}_7, \tilde{\sigma}_8, \tilde{\sigma}_{11}]/({\sigma_1}^3)$$

are givern as follows:

$$\begin{aligned}
&\mathcal{P}^1_*(\sigma_3) = \sigma_1, \\
&\mathcal{P}^1_*(\tilde{\sigma}_4) = 0, \\
&\mathcal{P}^1_*(\tilde{\sigma}_5) = 0, \\
&\mathcal{P}^1_*(\tilde{\sigma}_7) = \tilde{\sigma}_5, \qquad \mathcal{P}^2_*(\tilde{\sigma}_7) = 0, \\
&\mathcal{P}^1_*(\tilde{\sigma}_8) = 0, \qquad \mathcal{P}^2_*(\tilde{\sigma}_8) = 0, \\
&\mathcal{P}^1_*(\tilde{\sigma}_{11}) = {\sigma_3}^3, \qquad \mathcal{P}^2_*(\tilde{\sigma}_{11}) = 0, \qquad \mathcal{P}^3_*(\tilde{\sigma}_{11}) = 0.
\end{aligned}$$

Remark 2. The Hopf algebra structure of $H_*(\Omega E_6; \mathbb{Z}_p)$ over $\mathcal{A}_{p*}$ for $p = 2, 3$ is already determined in [8], [3] and [4] without using the generating variety. Therefore our contribution is to make the description of $H_*(\Omega E_6; \mathbb{Z}_p)$ for $p = 2, 3$ explicit in terms of $H_*(\Omega E_6; \mathbb{Z})$.

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