J. Math. Kyoto Univ. (JMKYAZ)
44-1 (2004), 1–32

Analyticity of solutions of the Korteweg-de Vries equation

By

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Abstract

We consider the analytic smoothing effect for the KdV equation. That is to say, if the initial data given at $t = 0$ decays very rapidly, the solution to the Cauchy problem becomes analytic with respect to the space variable for $t > 0$. In this paper we show this effect by using the inverse scattering method which transforms the KdV equation to a linear dispersive equation whose analytic smoothing effect is shown through the properties of the Airy function.

1. Introduction

The smoothing effect is one of the important properties of the dispersive equations such as the Schrödinger equation and the KdV equation. This effect says that solutions gain smoothness if the initial data decays rapidity. In this paper we treat only the analytic smoothing effect, that is to say, the phenomenon that solutions become analytic with respect to the space variables.

There are already many works on this analytic smoothing effect for linear dispersive equations and also for nonlinear dispersive equations (see for example de Bouard, Hayashi and Kato [1], Hayashi and Kato [5], [6], Kato-Ogawa [7] for nonlinear case). But concerning the KdV equation, comparing to the results on its linearlized equation, we have still problems to be studied. (See also Craig, Kappeler and Strauss [3] for smoothing effect for the general KdV-type equation.)

First consider the linear dispersive equation

$$\frac{\partial}{\partial t}v(t,x) + A\frac{\partial^3}{\partial x^3}v(t,x) = 0 \tag{1.1}$$

with a positive constant A.

The fundamental solution $E_A(t,x)$ to the Cauchy problem for (1.1) is given

2000 *Mathematics Subject Classification(s).* 35Q53
Received July 10, 2000

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$$E_A(t,x) = \begin{cases} \dfrac{1}{(3At)^{\frac{1}{3}}} Ai\left(\dfrac{x}{(3At)^{\frac{1}{3}}}\right), & t > 0, \\ \delta(x), & t = 0, \end{cases}$$

where the function $Ai(w)$ is the Airy function defined by

$$Ai(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(wz + \frac{1}{3}z^3)} d\, z.$$

Then we see that any solution $v(t,x)$ of the equation (1.1) belonging to $C([0,+\infty), L^2(R))$ has the following expression: for $t \geq 0$

$$v(t,x) = \int_{-\infty}^{\infty} E_A(t, x-w) v(0,w) d\, w. \tag{1.2}$$

The Airy function defined above has a holomorphic extension on the whole complex plane and the following estimates (see Ch. 4 §4 of Olver [14]): for any $C_0 > 0$ and $\gamma > 0$,

$$|Ai(z)| \leq Ce^{-\frac{2}{3}\Re(z^{\frac{3}{2}})} \quad \text{and} \quad |Ai'(z)| \leq C|z|^{\frac{1}{2}} e^{-\frac{2}{3}\Re(z^{\frac{3}{2}})} \tag{1.3}$$

when $|\arg z| < \pi - \gamma$ and $|z| \geq C_0$, while

$$|Ai(-z)| \leq Ce^{\frac{2}{3}|\Im(z^{\frac{3}{2}})|} \quad \text{and} \quad |Ai'(-z)| \leq C|z|^{\frac{1}{2}} e^{\frac{2}{3}|\Im(z^{\frac{3}{2}})|} \tag{1.4}$$

when $|\arg z| < \frac{1}{3}\pi - \gamma$ and $|z| \geq C_0$.

Since

$$\Re(x+iy)^{\frac{3}{2}} \geq \frac{1}{2} x^{\frac{3}{2}} \quad \text{for } x > |y|$$

and

$$|\Im(x+iy)^{\frac{3}{2}}| \leq 2|x|^{\frac{1}{2}} |y| \quad \text{for } x > |y|,$$

then it follows from (1.3) and (1.4) that when t belongs a compact interval in $(0,\infty)$ and $|y| \leq 1$ we get

$$\left| Ai\left(\frac{x+iy}{(3At)^{\frac{1}{3}}}\right) \right| \leq \begin{cases} Ce^{-C_1 t^{-\frac{1}{2}} x^{\frac{3}{2}}}, & x \geq 1, \\ Ce^{C_2 t^{-\frac{1}{2}} |y||x|^{\frac{1}{2}}}, & x \leq -1 \end{cases} \tag{1.5}$$

and

$$\left| Ai'\left(\frac{x+iy}{(3At)^{\frac{1}{3}}}\right) \right| \leq \begin{cases} C|x|^{\frac{1}{2}} e^{-C_1 t^{-\frac{1}{2}} x^{\frac{3}{2}}}, & x \geq 1, \\ C|x|^{\frac{1}{2}} e^{C_2 t^{-\frac{1}{2}} |y||x|^{\frac{1}{2}}}, & x \leq -1. \end{cases} \tag{1.6}$$

Therefore we see from (1.2) that if $v(0,x)$ has the estimate, with $\delta > 0$,

$$|v(0,x)| \leq \begin{cases} Ce^{-\delta x^{\frac{1}{2}}} & \text{for } x > 0, \\ C & \text{for } x < 0, \end{cases}$$

then the solution $v(t,x)$ is real analytic with respect to x. For by choosing $\delta_0 \le \frac{t^{\frac{1}{2}}}{2C_2}\delta$ we see that for any $x_0 \in \mathbb{R}$, on $\{x+iy \in \mathbb{C}\ ;\ |(x+iy)-x_0| \le \delta_0\}$ $E_A(t, x-w+iy)v(0,w)$ is uniformly integrable with respect to the variable w.

More generally

Proposition 1.1. *If $v(x)$ is given by*

$$v(x) = \left(\frac{d}{dx} - 1\right) R(x),$$

where a continuous function $R(x)$ has the estimate, with $\delta > 0$,

$$|R(x)| \le \begin{cases} Ce^{-\delta x^{\frac{1}{2}}} & \text{for } x > 0, \\ Ce^{C_1|x|} & \text{for } x \le 0, \end{cases}$$

then the function $v(t,x)$ defined by

$$v(t,x) = \int_{-\infty}^{\infty} E_A(t, x-w)v(w)d\,w$$

is real analytic with respect to x when $t > 0$.

More precisely, if $t > 0$, $v(t,x)$ has the analytic extension on $\{(x+iy) :\ x \in \mathbb{R}$ and $|y| \le \delta_0\ \}$ with some $\delta_0 > 0$ and satisfies, if $x+iy$ belongs to the set above, $x \ge 1$ and t belongs to a compact interval in $(0,\infty)$,

$$|v(t, x+iy)| \le C_0 e^{-C_1|x|^{\frac{1}{2}}}. \tag{1.7}$$

Proof. First note that from (1.5) and (1.6) we see

$$v(t,x) = -\int_{-\infty}^{\infty} E_A(t,x-w)R(w)d\,w + \int_{-\infty}^{\infty} \partial_x E_A(t,x-w)R(w)d\,w.$$

Then we see the analyticity of $v(t, x+iy)$ in $\{(x+iy) :\ x \in \mathbb{R}$ and $|y| \le \delta_0\ \}$ when δ_0 satisfies

$$0 < \delta_0 \le \frac{t^{\frac{1}{2}}}{4C_2}\delta.$$

In order to show (1.7) we consider the estimate of

$$\int_{-\infty}^{\infty} \partial_x E_A(t,x-w)R(w)d\,w.$$

The estimate of the other term follows similarly. Let be $t_1 > t_2 > 0$ and assume $t_1 \ge t \ge t_2$, $|y| < 1$ and $x \ge 1$. We see that for $w > x+1$

$$|\partial_x E_A(t, x+iy-w)v(w)| \le C|x-w|^{\frac{1}{2}} e^{C_2|x-w|^{\frac{1}{2}} t^{-\frac{1}{2}}|y|} e^{-\delta|w|^{\frac{1}{2}}}$$

and for $0 \le w \le x-1$

$$|\partial_x E_A(t, x+iy-w)v(w)| \le C|x-w|^{\frac{1}{2}} e^{-C_1|x-w|^{\frac{3}{2}} t^{-\frac{1}{2}}} e^{-\delta|w|^{\frac{1}{2}}}$$

and for $w \le 0$

$$|\partial_x E_A(t, x+iy-w)v(w)| \le C|x-w|^{\frac{1}{2}} e^{-C_1|x-w|^{\frac{3}{2}} t^{-\frac{1}{2}}} e^{C|w|}$$

When $x \ge 1$, we obtain

$$\begin{aligned} 2|w|^{\frac{1}{2}} &\ge |w-x|^{\frac{1}{2}} + |x|^{\frac{1}{2}} && \text{for} \quad w > x+1, \\ |x|^{\frac{1}{2}} &\le |w-x|^{\frac{1}{2}} + |w|^{\frac{1}{2}} && \text{for} \quad x+1 \ge w \ge 0, \\ |x| + |w| &= |w-x| && \text{for} \quad w < 0. \end{aligned}$$

Then we see that if $x > 1$, $|x-w| > 1$ and $|y| \le (\frac{t^{\frac{1}{2}}}{4C_2})\delta$,

$$|\partial_x E_A(t, x+iy-w)v(w)| \le C|x-w|^{\frac{1}{2}} e^{-C_3(|x-w|^{\frac{1}{2}} + |x|^{\frac{1}{2}})},$$

while for $|x-w| \le 1$, we obtain form $x > 1$

$$|\partial_x E_A(t, x+iy-w)v(w)| \le Ce^{-\delta|x|^{\frac{1}{2}}}.$$

Then when $|y| \le (\frac{t^{\frac{1}{2}}}{4C_2})\delta$ we obtain

$$\left| \int_{-\infty}^{\infty} \partial_x E_A(t, x+iy-w)v(w)d\,w \right| \le C_0 e^{-C_1|x|^{\frac{1}{2}}}.$$

$\square$

Now we consider the nonlinear case. According to Bourgain [2] (see also Kenig, Ponce and Vega [10], [11]) the Cauchy problem for the KdV equation

$$(1.8) \qquad \begin{cases} \dfrac{\partial}{\partial t} u(x,t) - 6u(x,t) \dfrac{\partial}{\partial x} u(x,t) + \dfrac{\partial^3}{\partial x^3} u(x,t) = 0, \\ u(x,0) = u_0(x) \end{cases}$$

is L^2-well posed, that is to say, for any $u_0(x) \in L^2(\mathbb{R})$ there exists one and only one solution belonging to $\{\ u(t,x) \in L^2(\mathbb{R}^2)\ ;\ (1+|\tau-\xi^3|)^b \widehat{\phi u}(\tau,\xi) \in L^2(\mathbb{R}^2)$ for any $\phi(t) \in C_0^\infty(\mathbb{R})\}$ with $b > \frac{1}{2}$, which implies $u(t,x) \in C([0,\infty), L^2(\mathbb{R}))$. Furthermore when the initial data $u_{0,n}(x) \in L^2(\mathbb{R})$ converges to $u_{0,\infty}(x)$ in $L^2(\mathbb{R})$, its solution converges to the solution with the initial data $u_{0,\infty}(x)$ in $C([0,\infty), L^2(\mathbb{R}))$.

In this paper we show that the analytic smoothing effect, that is similar to the one discussed above for the linear equation, holds for the KdV equation.

Theorem 1.1. *If the real-valued initial data $u_0(x) \in L^2(\mathbb{R})$ satisfies*

$$\int_{-\infty}^{\infty} (1+|x|)|u_0(x)|d\,x < \infty$$

and with some positive constant δ

$$\text{(D)} \qquad \int_0^\infty e^{\delta|x|^{\frac{1}{2}}}|u_0(x)|^2 d\,x < \infty,$$

then the solution to the Cauchy problem (1.8) *becomes analytic with respect to the variable* x *for* $t > 0$.

Since through the inverse scattering method we can transform the KdV equation to the linear dispersive equation for which the analytic smoothing effect can be seen by the argument above, we use this method to prove Theorem. In the next two sections we review the inverse scattering method following Marchenko [12]. Then we study the properties of scattering data for the Strum-Liouville operator with a potential satisfying the decay condition (D). After one section is devoted to the proof of two lemmas, we complete the proof of Theorem in Section 6.

In the following we use the following notations; we denote by $\mathcal{S}$ the set of all rapidly decreasing smooth functions on $\mathbb{R}$, by $C_0^\infty(\mathbb{R})$ the set of all compactly supported smooth functions on $\mathbb{R}$ and by $L^2(D)$ [resp. $L^\infty(D)$] the space of square integrable functions on D [resp. the space of essentially bounded measurable functions on D] whose norm is denoted by $\|\cdot\|_{L^2(D)}$ [resp. $\|\cdot\|_{L^\infty(D)}$]. But in the case where $D = \mathbb{R}$ it is denoted by $\|\cdot\|_2$ [resp. $\|\cdot\|_\infty$]. We denote by $\mathbb{C}^+$ the upper half plane $\{\lambda \in \mathbb{C} : \Im\lambda > 0\}$. In order to denote several constants we use the same notation. Then they may be different line by line.

2. Review of Scattering theory 1. Transformation operator.

In this and next section we review the scattering theory for the Strum-Liouville operator on the full line and its relation with the KdV equation following Marchenko [12] (see also Melin [13]).

We denote by $\mathcal{P}(1)$ the space of real valued measurable functions $q(x)$ on $\mathbb{R}$ satisfying

$$\int_{-\infty}^\infty (1+|x|)|q(x)|\,dx < +\infty$$

whose value is the norm of $q(x)$ in $\mathcal{P}(1)$.

Let $L_{q(x)}$ be the Strum-Liouville operator with a potential $q(x) \in \mathcal{P}(1)$ defined by

$$L_{q(x)} = -\frac{d^2}{dx^2} + q(x). \tag{2.1}$$

We denote by $e^+(x,\lambda)$ and $e^-(x,\lambda)$ with $\lambda \in \mathbb{R}\setminus\{0\}$ the soluitons of

$$L_{q(x)} f = \lambda^2 f \tag{2.2}$$

satisfying the following asymptotic behavior respectively;

$$e^+(x,\lambda) \to e^{i\lambda x} \quad \text{and} \quad e^{+\prime}(x,\lambda) \to i\lambda e^{i\lambda x} \tag{2.3}$$

as $x \to \infty$ and

$$e^-(x,\lambda) \to e^{i\lambda x} \quad \text{and} \quad {e^-}'(x,\lambda) \to i\lambda e^{i\lambda x}$$

as $x \to -\infty$.

These Jost solutions $e^+(x,\lambda)$ and $e^-(x,\lambda)$ can be represented by the following way;

$$e^+(x,\lambda) = e^{i\lambda x} + \int_x^\infty K^+(x,y)e^{i\lambda y}\,dy, \tag{2.4}$$

$$e^-(x,\lambda) = e^{i\lambda x} + \int_{-\infty}^x K^-(x,y)e^{i\lambda y}\,dy, \tag{2.5}$$

where $K^+(x,y)$ [resp. $K^-(x,y)$] is a solution of the integral equation;

$$K^+(x,y) = \frac{1}{2}\int_{\frac{x+y}{2}}^\infty q(s)\,ds + \int_{\frac{x+y}{2}}^\infty d\alpha \int_0^{\frac{y-x}{2}} q(\alpha-\beta)K^+(\alpha-\beta,\alpha+\beta)\,d\beta \tag{2.6}$$

[resp.
(2.7)

$$K^-(x,y) = \frac{1}{2}\int_{-\infty}^{\frac{x+y}{2}} q(s)\,ds + \int_{-\infty}^{\frac{x+y}{2}} d\alpha \int_0^{\frac{x-y}{2}} q(\alpha+\beta)K^-(\alpha+\beta,\alpha-\beta)\,d\beta].$$

Remark 1. We remark that $K^-(-x,-y)$ satisfies the equation (2.6) with $q(-x)$ in the place of $q(x)$.

Remark 2. We remark also that $K^+(x,y)$ and $K^-(x,y)$ satisfy the wave equation

$$-\partial_x^2 w(x,y) + \partial_y^2 w(x,y) + q(x)w(x,y) = 0,$$

$2\frac{d}{dx}K^+(x,x) = -q(x)$ and $2\frac{d}{dx}K^-(x,x) = q(x)$.

In the following we show the existence of solutions of (2.6) and (2.7) and their properties. First we note that for a $q(x) \in \mathcal{P}(1)$, the non-increasing function $\sigma(x)$ defined by

$$\sigma(x) = \int_x^\infty |q(w)|\,dw \tag{2.8}$$

satisfies that for any $x_o \in \mathbb{R}$

$$\int_{x_o}^\infty \sigma(x)d\,x < +\infty.$$

For $q(x) \in \mathcal{P}(1)$ we consider the operator $\mathrm{Op}^+(q)$ defined by

$$\mathrm{Op}^+(q)[k(\cdot,\cdot)](s,v) = \int_s^\infty d\alpha \int_0^v q(\alpha-\beta)k(\alpha,\beta)\,d\beta.$$

Lemma 2.1. *For $v \geq 0$*

$$e^{-2(\sigma_1(s-v)-\sigma_1(s))}|\mathrm{Op}^+(q)[k(\cdot,\cdot)](s,v)| \leq \frac{1}{2} \sup_{\alpha \geq s,\ v \geq \beta \geq 0} e^{-2(\sigma_1(\alpha-\beta)-\sigma_1(\alpha))}|k(\alpha,\beta)|, \tag{2.9}$$

where

$$\sigma_1(x) = \int_x^\infty \sigma(w)d\,w. \tag{2.10}$$

Proof. By definition, the function $\sigma_1(x)$ is non-increasing. Then for $v \geq \beta \geq 0$,

$$|k(\alpha,\beta)| \leq e^{2(\sigma_1(\alpha-v)-\sigma_1(\alpha))} \sup_{v \geq \gamma \geq 0} e^{-2(\sigma_1(\alpha-\gamma)-\sigma_1(\alpha))}|k(\alpha,\gamma)|.$$

Since

$$\begin{aligned}\int_0^v |q(\alpha-\beta)|d\,\beta &= \sigma(\alpha-v)-\sigma(\alpha) \\ &= -\frac{d}{d\alpha}\left(\sigma_1(\alpha-v)-\sigma_1(\alpha)\right),\end{aligned}$$

$$\int_s^\infty d\,\alpha \int_0^v |q(\alpha-\beta)|e^{2(\sigma_1(\alpha-v)-\sigma_1(\alpha))}d\,\beta = \frac{1}{2}e^{2(\sigma_1(s-v)-\sigma_1(s))}.$$

Then we obtain the estimate (2.9). □

Remark 3. The proof of Lemma 2.1 implies that the estimate (2.9) is still valid even if $\sigma_1(s)$ is replced by

$$\int_s^\infty d\,x \int_x^\infty r(w)d\,w,$$

where $r(x) \in \mathcal{P}(1)$ satifies $r(x) \geq |q(x)|$.

For the later use, we note that for $s_1 > s_2$

$$\left|\int_{s_2}^{s_1} q(s)d\,s\right| \leq \sigma(s_2)-\sigma(s_1)$$

and that, since

$$\sigma_1(s-v)-\sigma_1(s) = \int_0^v \sigma(s-\alpha)d\,\alpha$$

and $\sigma(s)$ is non-increasing, we have for $v \geq 0$ and $s_1 > s_2$

$$\sigma_1(s_2-v)-\sigma_1(s_2) \geq \sigma_1(s_1-v)-\sigma_1(s_1).$$

We denote by D_{s_o,v_o}, where $s_o \in \mathbb{R}$ and $v_o \geq 0$, the closed domain

$$D_{s_o,v_o} = \{(\alpha,\beta) \in \mathbb{R}\times\mathbb{R} \ :\ \alpha \geq s_o,\ v_o \geq \beta \geq 0\ \}.$$

For a function $l \in L^\infty(D_{s_o,v_o})$, since $(\sigma_1(\alpha-\beta)-\sigma_1(\alpha)) \geq 0$ on D_{s_o,v_o}, we see from (2.9) that for any integer $n > 0$ and any $(s,v) \in D_{s_o,v_o}$

$$(\mathrm{Op}^+(q))^n[l(\cdot,\cdot)](s,v) \leq \frac{1}{2^n} e^{2(\sigma_1(s-v)-\sigma_1(s))} \|l(\cdot,\cdot)\|_{L^\infty(D_{s,v})}.$$

Then we obtain

Proposition 2.1. *For any function $l(s,v) \in L^\infty(D_{s_o,v_o})$, there exists one and only one solution $k^+(s,v)$ of the equation*

$$k^+(s,v) = l(s,v) + \mathrm{Op}(q)[k^+(\cdot,\cdot)](s,v) \qquad \text{on } D_{s_o,v_o} \tag{2.11}$$

which satisties, for any $(s,v) \in D_{s_o,v_o}$,

$$|k^+(s,v)| \leq 2e^{2(\sigma_1(s-v)-\sigma_1(s))} \|l(\cdot,\cdot)\|_{L^\infty(D_{s,v})}. \tag{2.12}$$

Remark 4. Since $\mathrm{Op}^+(q)[k(\cdot,\cdot)](s,v)$ is continuous for any $k(s,v) \in L^\infty(D_{s_o,v_o})$, $k^+(s,v)$ is continuous in D_{s_o,v_o}if $l(s,v)$ is continuous. In this case $\mathrm{Op}^+(q)[k(\cdot,\cdot)](s,v)$ becomes continuously differentiable. Then $k^+(s,v) - l(s,v)$ is continuously differentiable if $l(s,v)$ is continuous. Furthermore if $l(s,v)$ admits bounded continuous derivatives and $q(s)$ is continuous, $k^+(s,v) - l(s,v)$ admits second order continuous derivatives.

Next we show that the solution above $k^+(s,v)$ depends continuously on $q(x)$. First we note that for $q_1(x)$ and $q_2(x)$ in $\mathcal{P}(1)$,

$$\begin{aligned} &e^{-2(\tilde\sigma_1(s-v)-\tilde\sigma_1(s))} |\{\mathrm{Op}^+(q_1) - \mathrm{Op}^+(q_2)\}[k(\cdot,\cdot)](s,v)| \\ &\qquad \leq \int_{s-v}^\infty dx \int_x^\infty |q_1(w) - q_2(w)| dw \\ &\qquad\qquad \times \sup_{\alpha \geq s,\ v \geq \beta \geq 0} e^{-2(\tilde\sigma_1(\alpha-\beta)-\tilde\sigma_1(\alpha))} |k(\alpha,\beta)|, \end{aligned}$$

where

$$\tilde\sigma_1(x) = \int_x^\infty ds \int_s^\infty (|q_1(w)| + |q_2(w)|) dw. \tag{2.13}$$

Hence it follows from Remark 2.3 that

$$\begin{aligned} &e^{-2(\tilde\sigma_1(s-v)-\tilde\sigma_1(s))} |\{\mathrm{Op}^+(q_1)^n - \mathrm{Op}^+(q_2)^n\}[l(\cdot,\cdot)](s,v)| \\ &\qquad \leq \frac{n}{2^{n-1}} \int_{s-v}^\infty dx \int_x^\infty |q_1(w) - q_2(w)| dw \\ &\qquad\qquad \times \sup_{\alpha \geq s,\ v \geq \beta \geq 0} e^{-2(\tilde\sigma_1(\alpha-\beta)-\tilde\sigma_1(\alpha))} |l(\alpha,\beta)|. \end{aligned}$$

Hence we obtain

Lemma 2.2. *For a function $l(s,v) \in L^\infty(D_{s_o,v_o})$, let $k_1^+(s,v)$ and k_2^+ (s,v) be solutions of the equations*

$$k_1^+(s,v) = l(s,v) + \mathrm{Op}(q_1)[k_1^+(\cdot,\cdot)](s,v) \qquad \text{on } D_{s_o,v_o}$$

and

$$k_2^+(s,v) = l(s,v) + \mathrm{Op}(q_2)[k_2^+(\cdot,\cdot)](s,v) \qquad \text{on } D_{s_o,v_o}.$$

Then we see that for any $(s,v) \in D_{s_o,v_o}$

$$\begin{aligned}|k_1^+(s,v) - k_2^+(s,v)| \\ &\le 4\int_{s-v}^\infty dx \int_x^\infty |q_1(w) - q_2(w)| dw \\ &\qquad \times e^{2(\tilde\sigma_1(s-v)-\tilde\sigma_1(s))}\|l(\cdot,\cdot)\|_{L^\infty(D_{s,v})},\end{aligned}$$

where $\tilde\sigma_1(x)$ is defined by (2.13).

Therefore we see that the following proposition is valid.

Proposition 2.2. *When the kernel q in the equation* (2.11) *depends continuously on a parameter $t \in [0,T]$ with some $T > 0$, that is to say $q(x,t) \in C([0,T], \mathcal{P}(1))$, its solution $k^+(s,v,t)$ is a $L^\infty(D_{s_o,v_o})$-valued continuous function on $[0,T]$. Furthermore in the case that $q(x,t) \in C^1([0,T], \mathcal{P}(1))$, $k^+(s,v,t)$ is a $L^\infty(D_{s_o,v_o})$-valued C^1 function on $[0,T]$.*

For a given $q(x) \in \mathcal{P}(1)$ we denote by $k_q^+(s,v)$ the solution of the equation

$$k_q^+(s,v) = \frac{1}{2}\int_s^\infty q(x)dx + \mathrm{Op}(q)[k_q^+(\cdot,\cdot)](s,v). \tag{2.14}$$

Then we see from Remark 4 that $k_q^+(s,v)$ is continuous and $k_q^+(s,v) - \frac{1}{2}\int_s^\infty q(x)$ dx is continuously differentiable and it follows from (2.12) that for $(s,v) \in D_{s_o,v_o}$

$$|k_q^+(s,v)| \le C_{s_o-v_o}\sigma(s), \tag{2.15}$$

where $\sigma(s)$ is defined by (2.8) and a positive constant $C_{s_o-v_o}$ depends only on $\sigma_1(s_o - v_o)$ which is given by (2.10). Since

$$\left|\int_s^\infty q(\alpha - v)k(\alpha,v)d\alpha\right| \le \sigma(s-v)\sup_{\alpha\ge s}|k(\alpha,v)|$$

and

$$\left|\int_0^v q(s-\beta)k(s,\beta)d\beta\right| \le \sigma(s-v)\sup_{0\le\beta\le v}|k(s,\beta)|,$$

we see that for $(s,v) \in D_{s_o,v_o}$

$$\left|\frac{\partial}{\partial s}\mathrm{Op}(q)[k_q^+(\cdot,\cdot)](s,v)\right| + \left|\frac{\partial}{\partial v}\mathrm{Op}(q)[k_q^+(\cdot,\cdot)](s,v)\right| \le C_{s_o-v_o}\sigma(s), \tag{2.16}$$

where a positive constant $C_{s_o-v_o}$ depends only on $\sigma_1(s_o-v_o)$ and $\sigma(s_o-v_o)$.

We see from (2.14) that $k_q^+(\frac{x+y}{2},\frac{y-x}{2})$ satisfies (2.6), while it follows from Remark 1.1 that $K^-(x,y)$ is given by $k_{\check{q}}^+(-\frac{x+y}{2},\frac{x-y}{2})$ where $\check{q}(x)=q(-x)$. That is to say, we see

$$K^+(x,y)=k_q^+\left(\frac{x+y}{2},\frac{y-x}{2}\right), \tag{2.17}$$

$$K^-(x,y)=k_{\check{q}}^+\left(-\frac{x+y}{2},\frac{x-y}{2}\right). \tag{2.18}$$

We denote by Δ^+ [resp. Δ^-] the subset of $\mathbb{R}^2$ given by $\{(x,y)\in\mathbb{R}^2\ ;\ y\ge x\}$ [resp. $\{(x,y)\in\mathbb{R}^2\ ;\ y\le x\ \}$] and by $\Delta^+_{x_o}$ [resp. $\Delta^-_{x_o}$] $\{(x,y)\in\mathbb{R}^2\ ;\ y\ge x\ge x_o\ \}$ [resp. $\{(x,y)\in\mathbb{R}^2\ ;\ y\le x\le x_o\ \}$].

Then Propositions 2.1 and 2.2, the estimates (2.15) and (2.16) and the relations (2.17) and (2.18) imply the following;

Proposition 2.3. *For a $q(x)\in\mathcal{P}(1)$, the equation* (2.6) [*resp.* (2.7)] *has one and only one continuous solution $K^+(x,y)$ on Δ^+* [*resp. $K^-(x,y)$ on Δ^-*] *that is bounded on $\Delta^+_{x_o}$* [*resp. $\Delta^-_{x_o}$*] *for any $x_o\in\mathbb{R}$. Furthermore $K^+(x,y)-Q^+(\frac{x+y}{2})$ and $K^-(x,y)-Q^-(\frac{x+y}{2})$ have first order continuous derivatives where*

$$Q^+(s)=\frac{1}{2}\int_s^\infty q(x)d\,x,$$
$$Q^-(s)=\frac{1}{2}\int_{-\infty}^s q(x)d\,x$$

and we have for $(x,y)\in\Delta^+_{x_o}$

$$\begin{aligned}\Bigg(|K^+(x,y)|&+\left|\partial_x\left(K^+(x,y)-Q^+\left(\frac{x+y}{2}\right)\right)\right|\\&+\left|\partial_y\left(K^+(x,y)-Q^+\left(\frac{x+y}{2}\right)\right)\right|\Bigg)\le C^+_{x_o}\int_{\frac{x+y}{2}}^\infty|q(s)|d\,s\end{aligned} \tag{2.19}$$

and for $(x,y)\in\Delta^-_{x_o}$

$$\begin{aligned}\Bigg(|K^-(x,y)|&+\left|\partial_x\left(K^-(x,y)-Q^-\left(\frac{x+y}{2}\right)\right)\right|\\&+\left|\partial_y\left(K^-(x,y)-Q^-\left(\frac{x+y}{2}\right)\right)\right|\Bigg)\le C^-_{x_o}\int_{-\infty}^{\frac{x+y}{2}}|q(s)|d\,s,\end{aligned} \tag{2.20}$$

where the constant $C^+_{x_o}$ [*resp. $C^-_{x_o}$*] *depends only on $\int_{x_o}^\infty|q(s)|d\,s$ and $\int_{x_o}^\infty d\,s\int_s^\infty|q(w)|d\,w$* [*resp. $\int_{-\infty}^{x_o}|q(s)|d\,s$ and $\int_{-\infty}^{x_o}d\,s\int_{-\infty}^s|q(w)|d\,w$*].

When $q_n(x) \to q(x)$ *in* $\mathcal{P}(1)$*, we see that, letting be* $K_n^+(x,y)$ *[resp.* $K_n^-(x,y)$*] the solution of* (2.6) *[resp.* (2.7)*] with* $q_n(x)$ *for* $q(x)$*,*

$$\begin{aligned}(2.21)\qquad \int_x^\infty (|K_n^+(x,y) - K^+(x,y)| + |\partial_x K_n^+(x,y) - \partial_x K^+(x,y)| \\ + |\partial_y K_n^+(x,y) - \partial_y K^+(x,y)|)dy \to 0,\end{aligned}$$

$$\begin{aligned}(2.22)\qquad \int_{-\infty}^x (|K_n^-(x,y) - K^-(x,y)| + |\partial_x K_n^-(x,y) - \partial_x K^-(x,y)| \\ + |\partial_y K_n^-(x,y) - \partial_y K^-(x,y)|)dy \to 0.\end{aligned}$$

Finally when $q(x,t) \in C^1([0,T], \mathcal{P}(1))$*,* $K^+(x,y,t)$ *and* $K^-(x,y,t)$*, the solution of* (2.6) *and that of* (2.7) *respectively with* $q(x,t)$ *for* $q(x)$*, are continuously differentiable with respect to the variable* t *and* $K^+(0,y,t)$*,* $\partial_x K^+(0,y,t)$ *and* $\partial_y K^+(0,y,t)$ *belong to* $C^1([0,T], L^1((0,+\infty))$ *and* $K^-(0,y,t)$*,* $\partial_x K^-(0,y,t)$ *and* $\partial_y K^-(0,y,t)$ *belong to* $C^1([0,T], L^1((-\infty,0))$*.*

Remark 5. We see from (2.19) and (2.20) that

$$\begin{aligned}&\int_0^\infty (|K^+(0,y)| + |\partial_x K^+(0,y)| + |\partial_y K^+(0,y)|)dy \le C,\\ (2.23)\qquad &\int_{-\infty}^0 (|K^-(0,y)| + |\partial_x K^-(0,y)| + |\partial_y K^-(0,y)|)dy \le C\end{aligned}$$

with the constant C depending only on $\int_{-\infty}^\infty (1+|x|)|q(x)|dx$.

3. Review of Scattering theory 2. Gelfand-Levitan-Marchenko equation.

We continue the review of the scattering theory for the Strum-Liouville operator. Since we see from (2.3) that for $\lambda \in \mathbb{R} \setminus \{0\}$ the solutions $e^+(x,\lambda)$ and $e^+(x,-\lambda)$ of the equation (2.2) are linearly independent, there exist $a(\lambda)$ and $c(\lambda)$ such that

$$(3.1)\qquad e^-(x,-\lambda) = c(\lambda)e^+(x,\lambda) + a(\lambda)e^+(x,-\lambda).$$

Since $q(x)$ is real-valued, $\overline{e^\pm(x,\lambda)} = e^\pm(x,-\lambda)$. Then we have

$$(3.2)\qquad \overline{a(\lambda)} = a(-\lambda) \quad \text{and} \quad \overline{c(\lambda)} = c(-\lambda).$$

Let $W(f(\cdot), g(\cdot))$ be the wronskian of $f(x)$ and $g(x)$ given by

$$W(f,g) = f'(x)g(x) - g'(x)f(x).$$

Since we see from (2.3) that $W(e^+(\cdot,\lambda), e^+(\cdot,-\lambda)) = 2i\lambda$, it follows from (3.1) that

$$(3.3)\qquad a(\lambda) = \frac{1}{2i\lambda} W(e^+(\cdot,\lambda), e^-(\cdot,-\lambda)),$$

$$(3.4)\qquad c(\lambda) = \frac{1}{-2i\lambda} W(e^+(\cdot,-\lambda), e^-(\cdot,-\lambda)).$$

By the integration by parts we see from (2.4) and (2.5) that

$$
\begin{aligned}
a(\lambda) = 1 + \frac{1}{2i\lambda}\Big(&-\int_{-\infty}^{\infty} q(w)dw \\
&+\int_0^{\infty} G^+(y)e^{i\lambda y}dy - \int_{-\infty}^0 G^-(y)e^{-i\lambda y}dy \\
&+\int_0^{\infty} K_x^+(0,y)e^{i\lambda y}dy \int_{-\infty}^0 K^-(0,y)e^{-i\lambda y}dy \\
&-\int_0^{\infty} K^+(0,y)e^{i\lambda y}dy \int_{-\infty}^0 K_x^-(0,y)e^{-i\lambda y}dy\Big),
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
G^+(y) &= K_x^+(0,y) - K_y^+(0,y) - K^-(0,0)K^+(0,y), \\
G^-(y) &= K_x^-(0,y) - K_y^-(0,y) + K^+(0,0)K^-(0,y).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
c(\lambda) = -\frac{1}{2i\lambda}\Big(&\int_0^{\infty} H^+(y)e^{-i\lambda y}dy - \int_{-\infty}^0 H^-(y)e^{-i\lambda y}dy \\
&+\int_0^{\infty} K_x^+(0,y)e^{-i\lambda y}dy \int_{-\infty}^0 K^-(0,y)e^{-i\lambda y}dy \\
&-\int_0^{\infty} K^+(0,y)e^{-i\lambda y}dy \int_{-\infty}^0 K_x^-(0,y)e^{-i\lambda y}dy\Big),
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
H^+(y) &= K_x^+(0,y) + K_y^+(0,y) - K^-(0,0)K^+(0,y), \\
H^-(y) &= K_x^-(0,y) + K_y^-(0,y) + K^+(0,0)K^-(0,y).
\end{aligned}
$$

Taking into account (2.19), (2.20) and (3.5) we see that $e^+(x,\lambda)$ and $e^-(x,-\lambda)$ are well-defined on the upper half plane $\mathbb{C}^+ = \{\lambda \in \mathbb{C} \ : \ \Im\lambda > 0\}$ and that $a(\lambda)$ has the analytic extension on $\mathbb{C}^+$ and satisfies on $\mathbb{C}^+$

$$
|a(\lambda) - 1| \le \frac{C}{|\lambda|}. \tag{3.7}
$$

If $q(x)$ belongs to $C_0^\infty(\mathbb{R})$, then we see from (2.19) and (2.20) that $K^+(0,y)$, $K_x^+(0,y)$ and $K_y^+(0,y)$ [resp. $K^-(0,y)$, $K_x^-(0,y)$ and $K_y^-(0,y)$] vanish for large y [resp. large negative y]. Furthermore $K^\pm(0,y)$ and $K_x^\pm(0,y)$ are smooth. Then $\lambda a(\lambda)$ and $\lambda c(\lambda)$ are entire analytic and we obtain

$$
|\lambda^2 c(\lambda)| \le C_1 e^{C_2|\Im\lambda|}. \tag{3.8}
$$

Remark 6. In (3.6), the terms $\int_{-\infty}^0 H^-(y)e^{-i\lambda y}dy$, $\int_{-\infty}^0 K^-(0,y)e^{-i\lambda y}dy$ and $\int_{-\infty}^0 K_x^-(0,y)e^{-i\lambda y}dy$ have the analytic extensions on the upper half plane $\mathbb{C}^+$ for any $q(x) \in \mathcal{P}(1)$.

The relations (3.3) and (3.4) imply that

$$e^+(x,\lambda) = a(\lambda)e^-(x,\lambda) - c(-\lambda)e^-(x,-\lambda), \tag{3.9}$$

$$e^+(x,-\lambda) = a(-\lambda)e^-(x,-\lambda) - c(\lambda)e^-(x,\lambda). \tag{3.10}$$

Then we obtain from (3.1), (3.2), (3.9) and (3.10) that

$$1 + |c(\lambda)|^2 = |a(\lambda)|^2. \tag{3.11}$$

Since the square of a zero of $a(\lambda)$ in $\mathbb{C}^+$ is an eigenvalue of the operator $L_{q(x)}$ of (2.1), the zeros of $a(\lambda)$ in $\mathbb{C}^+$ lie on on the upper half line $\{z = ix\ ;\ x > 0\}$. Furthermore their number is finite (see LEMMA 3.5.2 of [12]). Let $i\mu$ with $\mu > 0$ be a zero of $a(\lambda)$. Then $e^+(x,i\mu)$ and $e^-(x,-i\mu)$ are linearly dependent. Then there exists a constant c_μ such that

$$e^-(x,-i\mu) = c_\mu e^+(x,i\mu). \tag{3.12}$$

We see from (3.3) that

$$-2\mu a'(\lambda) = W(\dot e^+(x,i\mu), e^-(x,-i\mu)) - W(e^+(x,i\mu), \dot e^-(x,-i\mu)),$$

where

$$\dot e^\pm(x,\lambda) = \partial_\lambda e^\pm(x,\lambda).$$

Since

$$-\frac{d^2}{dx^2}\dot e^+(x,i\mu) + q(x)\dot e^+(x,i\mu) = -\mu^2 \dot e^+(x,i\mu) + 2i\mu e^+(x,i\mu),$$

we obtain

$$\begin{aligned}\frac{d}{dx}(\dot e^{+'}(x,i\mu)e^-(x,-i\mu) - \dot e^+(x,i\mu)e^{-'}(x,-i\mu))&\\ = -2i\mu e^+(x,i\mu)e^-(x,-i\mu).&\end{aligned}$$

Since $\dot e^+(x,i\mu)$ and $\dot e^{+'}(x,i\mu)$ tend to zero as $x \to \infty$, we get

$$W(\dot e^+(x,i\mu), e^-(x,-i\mu)) = 2i\mu \int_x^\infty e^+(x,i\mu)e^-(x,-i\mu)d\,x.$$

Similarly we get

$$-W(e^+(x,i\mu), \dot e^-(x,-i\mu)) = 2i\mu \int_{-\infty}^x e^+(x,i\mu)e^-(x,-i\mu)d\,x.$$

Hence

$$\begin{aligned}-2\mu a'(i\mu) &= 2i\mu \int_{-\infty}^\infty e^+(x,i\mu)e^-(x,-i\mu)d\,x\\ &= 2i\mu c_\mu \int_{-\infty}^\infty e^+(x,i\mu)^2 d\,x.\end{aligned} \tag{3.13}$$

If the potential $q(x)$ belongs to $C_0^\infty(\mathbb{R})$, we see that

$$\begin{aligned} 2\mu c(i\mu) &= W(e^+(\cdot,-i\mu), e^-(\cdot,-i\mu)) \\ &= 2\mu c_\mu, \end{aligned}$$

from which we see that the residue of $\frac{c(\lambda)}{a(\lambda)}$ at $\lambda = i\mu$ is equal to

$$\frac{c(i\mu)}{a'(i\mu)} = i\frac{1}{\int_{-\infty}^{\infty} e^+(x,i\mu)^2 d\,x}. \tag{3.14}$$

In general, when we denote by $\{i\mu_1, \ldots, i\mu_M\}$ $(\mu_1 > \cdots > \mu_M > 0)$ all the zeros of $a(\lambda)$ in $\mathbb{C}^+$, we define $R^+(x)$ and $F^+(x)$ by

$$R^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r^+(\lambda)e^{i\lambda x} d\,\lambda, \tag{3.15}$$

where

$$r^+(\lambda) = \frac{c(\lambda)}{a(\lambda)}, \tag{3.16}$$

and

$$F^+(x) = \sum_{j=1}^{M}(m_j^+)^2 e^{-\mu_j x} + R^+(x), \tag{3.17}$$

where

$$(m_j^+)^2 = \frac{1}{\int_{-\infty}^{\infty} e^+(x,i\mu_j)^2 d\,x}. \tag{3.18}$$

Remark 7. We see from (3.2), (3.5), (3.6) and (3.11), that $\overline{r^+(\lambda)} = r^+(-\lambda)$ and

$$|r^+(\lambda)| \le C\frac{1}{1+|\lambda|}.$$

Hence $R^+(x)$ is real valued and $R^+(x) \in L^2(\mathbb{R})$.

It follows from (3.11) that for any $\lambda \in \mathbb{R}\setminus\{0\}$

$$|r^+(\lambda)| < 1. \tag{3.19}$$

On the other hand, (3.13) implies

$$(m_j^+)^2 = \frac{-ic_\mu}{a'(i\mu)}. \tag{3.20}$$

Now that $K^+(x,y)$ and $F^+(x)$ are defined, we derive the Gelfand-Levitan-Marchenko equation. We see from (3.1) that

$$\left(\frac{1}{a(\lambda)} - 1\right) e^-(x,-\lambda) = e^+(x,-\lambda) - e^-(x,-\lambda) + r^+(\lambda)e^+(x,\lambda).$$

Since $(\frac{1}{a(\lambda)}-1)e^-(x,-\lambda)e^{iy\lambda}$ is a meromorphic function on $\mathbb{C}^+$ whose poles are $i\mu_j$ $(j=1,\dots,M)$ and $|(\frac{1}{a(\lambda)}-1)e^-(x,-\lambda)e^{iy\lambda}| \le C\frac{1}{|\lambda|}$ for $\lambda \in \mathbb{C}^+$ with $|\lambda| \ge 2\mu_1$ and $y \ge x$, then we obtain for $y > x$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}(\frac{1}{a(\lambda)}-1)e^-(x,-\lambda)e^{iy\lambda}d\lambda &= \sum_{j=1}^{M}\frac{i}{a'(i\mu_j)}e^-(x,-i\mu_j)e^{-\mu_j y}\\ &= \sum_{j=1}^{M}\frac{ic_{\mu_j}}{a'(i\mu_j)}e^+(x,i\mu_j)e^{-\mu_j y}\\ &= \sum_{j=1}^{M}-(m_j^+)^2e^+(x,i\mu_j)e^{-\mu_j y},\end{aligned}$$

where we used the boundedness of $\frac{1}{|a(\lambda)|}$ on $\{\lambda \in \mathbb{C}^+ \ ; \ |\lambda| \le \frac{|\mu_M|}{2}\}$ (see Marchenko [12, LEMMA 3.5.2 in Ch. 3, Sec. 5]). On the other hand since

$$e^+(x,-\lambda)-e^-(x,-\lambda) = \int_{-\infty}^{\infty}K(x,w)e^{-iw\lambda}d\,w,$$

where

$$K(x,w) = \begin{cases}K^+(x,w), & w \ge x,\\ K^-(x,w), & w < x\end{cases}$$

and

$$r^+(\lambda)e^+(x,\lambda) = r^+(\lambda)e^{ix\lambda} + r^+(\lambda)\int_x^{\infty}K^+(x,w)e^{iw\lambda}d\,w,$$

we see that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(e^+(x,-\lambda)-e^-(x,-\lambda))e^{iy\lambda}d\,\lambda = K(x,y)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}r^+(\lambda)e^+(x,-\lambda)e^{iy\lambda}d\,\lambda = R^+(x+y) + \int_x^{\infty}R^+(y+w)K^+(x,w)d\,w.$$

Therefore we obtain the Gelfand-Levitan-Marchenko equation; for $y > x$

$$K^+(x,y) + F^+(x+y) + \int_x^{\infty}K^+(x,w)F^+(w+y)d\,w = 0.$$

Next let be $u(t,x) \in C^1([0,\infty),\mathcal{S})$ a solution to Cauchy problem (1.8) with a real initial data $u_0(x) \in \mathcal{S}$. We can apply the argument above for the potential $u(t,x)$. Hence we denote by $K^{\pm}(x,y,t)$, $a(\lambda,t)$, $c(\lambda,t)$, $r^+(\lambda,t)$ and $F^+(x,t)$ the corresponding quantities.

Let $\mathcal{M}$ be operator given by

$$\mathcal{M} = \frac{\partial}{\partial t} - 2(u(t,x)+2\lambda^2)\frac{\partial}{\partial x} + u_x(t,x).$$

Then since $u(t,x)$ satisfies KdV equation,

$$[\mathcal{M}, L_{u(t,x)} - \lambda^2] = 4u_x(t,x)(L_{u(t,x)} - \lambda^2), \tag{3.21}$$

where $[\mathcal{M}, L_{u(t,x)} - \lambda^2]$ is the commuator of $\mathcal{M}$ and $L_{u(t,x)} - \lambda^2 = -\frac{\partial^2}{\partial x^2} + u(t,x) - \lambda^2$.

The solution $e^+(x,\lambda,t)$ [resp. $e^-(x,\lambda,t)$] of

$$(L_{u(t,x)} - \lambda^2)y = 0 \tag{3.22}$$

satisfying $y - e^{i\lambda x}$ and $y' - i\lambda e^{i\lambda x}$ tend to zero when $x \to \infty$ [resp. $y - e^{i\lambda x}$ and $y' - i\lambda e^{i\lambda x}$ tend to zero when $x \to -\infty$], is given by

$$e^+(x,\lambda,t) = e^{ix\lambda} + \int_x^\infty K^+(x,y,t)e^{iy\lambda}dy$$

[resp.

$$e^-(x,\lambda,t) = e^{ix\lambda} + \int_\infty^x K^-(x,y,t), e^{iy\lambda}dy].$$

We see from (3.21) that $\mathcal{M}e^+(x,\lambda,t)$ and $\mathcal{M}e^-(x,\lambda,t)$ are solutions of the equation (3.22). Proposition 2.3 implies

$$\begin{aligned}\mathcal{M}e^+(x,\lambda,t) = \big((-2u(t,x) - 4\lambda^2)(i\lambda - K^+(x,x,t)) \\ + u_x(t,x)\big)e^{ix\lambda} + \int_x^\infty \mathcal{M}K^+(x,y,t)e^{iy\lambda}dy, \\ \mathcal{M}e^-(x,\lambda,t) = \big((-2u(t,x) - 4\lambda^2)(i\lambda + K^-(x,x,t)) \\ + u_x(t,x)\big)e^{ix\lambda} + \int_{-\infty}^x \mathcal{M}K^-(x,y,t)e^{iy\lambda}dy,\end{aligned}$$

from which we obtain, by considering the asymptotic behavior, for any non-zero $\lambda \in \mathbb{R}$

$$\mathcal{M}e^+(x,\lambda,t) = -4i\lambda^3 e^+(x,\lambda,t), \tag{3.23}$$

$$\mathcal{M}e^-(x,\lambda,t) = -4i\lambda^3 e^-(x,\lambda,t). \tag{3.24}$$

Hence for any non-zero $\lambda \in \mathbb{R}$

$$\frac{\partial}{\partial t}a(\lambda,t) = 0, \tag{3.25}$$

$$\frac{\partial}{\partial t}c(\lambda,t) = 8i\lambda^3 c(\lambda,t) \tag{3.26}$$

follow from (3.1) (see Marchenko [12, Ch. 4, Sec. 2]). Then we see that $a(\lambda,t)$ is independent of t, hence its zeros in $\mathbb{C}^+$ are also. Let $i\mu \in \mathbb{C}^+$ be a zero of $a(\lambda,t)$, then there exists a $c_\mu(t)$ such that

$$e^-(x,-i\mu,t) = c_\mu(t)e^+(x,i\mu,t).$$

By applying $\mathcal{M}$ to both sides above, it follows from (3.23) and (3.24) that

$$\frac{d}{dt}c_\mu(t) = 8\mu^3 c_\mu(t). \tag{3.27}$$

Therefore we obtain from (3.25),(3.26), (3.27) and (3.20)

$$r^+(\lambda, t) = e^{8i\lambda^3 t} r^+(\lambda, 0),$$
$$(m_j^+)^2(t)e^{-\mu_j(t)x} = (m_j^+)^2(0)e^{8\mu_j^3 t - \mu_j x}.$$

Hence we see that

$$\partial_t R^+(x,t) + 8\partial_x^3 R^+(x,t) = 0 \tag{3.28}$$

and

$$F^+(x,t) = \sum_{j=1}^{M} (m_j^+)^2(0)e^{8\mu_j^3 t - \mu_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x\lambda + 8\lambda^3 t)} r^+(\lambda, 0)\, d\lambda. \tag{3.29}$$

4. Strum-Liouville operator with a potential satisfying (D).

In this section we study that the behavior of functions $F^+(x)$ and $R^+(x)$, defined by (3.17) and (3.15), associated to a potential $q(x) \in \mathcal{P}(1) \cap L^2(\mathbb{R})$ satisfying (D), i.e $\int_0^\infty |q(x)|^2 e^{\delta\sqrt{x}} dx < \infty$.

First we consider the case where a real potential $q(x)$ belongs to $C_0^\infty(\mathbb{R})$. In this case, as mentioned in Section 3, we see that $r^+(\lambda) = \frac{c(\lambda)}{a(\lambda)}$ is meromorphic and has no poles on $\mathbb{R}$. Let $\{i\mu_1, i\mu_2, \ldots, i\mu_\nu\}$ ($\mu_1 > \mu_2 > \cdots > \mu_\nu > 0$) be all the zeros of $a(\lambda)$ in $\mathbb{C}^+$. We see from (3.14) and (3.18) that the residue of $r^+(\lambda)e^{i\lambda x}$ at $\lambda = i\mu_j$ is equal to

$$i(m_j^+)^2 e^{-\mu_j x},$$

from which, taking into account (3.7) and (3.8), we obtain for any $1 \le k \le \nu$ and any $\sigma \in (\mu_k, \mu_{k-1})$

$$\sum_{j=k}^{\nu} (m_j^+)^2 e^{-\mu_j x} + R^+(x) = \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} r^+(\lambda)e^{i\lambda x} d\lambda. \tag{4.1}$$

Now consider the case where a potential $q(x) \in \mathcal{P}(1) \cap L^2(\mathbb{R})$ satisfies (D). We denote still by $\{i\mu_1, i\mu_2, \ldots, i\mu_M\}$ ($\mu_1 > \mu_2 > \cdots > \mu_M > 0$) all the zeros of $a(\lambda)$ in $\mathbb{C}^+$.

Proposition 4.1. *The function $R^+(x)$ defined by* (3.15) *can be written in the following way*;

$$R^+(x) = \left(-i\frac{d}{dx} + i\right)\tilde{R}^+(x),$$

where $\tilde{R}^+(x)$ *is continuous and bounded on* $\mathbb{R}$ *and satisfies, with some* $\delta > 0$,

$$|\tilde{R}^+(x)| \le Ce^{-\delta\sqrt{x}} \text{ for } x > 0. \tag{4.2}$$

Furthermore there exists a sequence $\{q_n(x)\}_{n=1,2,\ldots}$ *of real potentials belonging to* $C_0^\infty(\mathbb{R})$ *and converging to* $q(x)$ *in* $L^2(\mathbb{R})$ *such that letting be* $F_n^+(x)$ *the function* (3.17) *derived from the potential* $q_n(x)$, $F_n^+(x)$ *can be written in the following way*;

$$F_n^+(x) = \sum_{j=1}^{M}(m_j^{(n)+})^2 e^{-\mu_j^{(n)}x} + \left(-i\frac{d}{dx} + i\right)\tilde{R}_n^+(x),$$

where $\tilde{R}_n^+(x)$ *is continuous on* $\mathbb{R}$ *and satisfies the estimate uniformly with respect to* n; *with* $C, C_1, C_2 > 0$,

$$|\tilde{R}_n^+(x)| \le \begin{cases} Ce^{-C_1\sqrt{x}}, & x \ge 0, \\ Ce^{C_2|x|}, & x < 0. \end{cases} \tag{4.3}$$

Furthermore as $n \to \infty$,

$$\tilde{R}_n^+(x) \to \tilde{R}^+(x), \tag{4.4}$$

$$\mu_j^{(n)} \to \mu_j \tag{4.5}$$

and

$$(m_j^{(n)+})^2 \to (m_j^+)^2. \tag{4.6}$$

Proof. We define $q_{(n)}(x)$, $q_{(n,m)}(x)$ and $q_{(n,m,\epsilon)}(x)$ with $n, m > 0$ and $0 < \epsilon < 1$ by the followings;

$$q_{(n)}(x) = \begin{cases} 0, & x > 2^n, \\ q(x), & x \le 2^n, \end{cases}$$

$$q_{(n,m)}(x) = \begin{cases} q_{(n)}(x), & x > -m, \\ 0, & x \le -m, \end{cases}$$

$$q_{(n,m,\epsilon)}(x) = \int_{-\infty}^{\infty} \rho(y) q_{(n,m)}(x - \epsilon y)\,dy,$$

where $\rho(y)$ is a function in $C_0^\infty((-1,1))$ satisfying $\rho(y) \ge 0$ and

$$\int_{-\infty}^{\infty} \rho(y)\,dy = 1.$$

Then the condition (D) implies that

$$\int_0^\infty |q_{(n)}(x)|^2 e^{\delta|x|^{\frac{1}{2}}}\,dx \le C$$

and

$$\int_{-\infty}^{\infty} |q(x) - q_{(n)}(x)|^2 d\,x \leq C \exp(-\delta 2^{\frac{n}{2}}).$$

On the other hand for any integer $n > 0$ there are $m(n)$ and $\epsilon(n)$ such that

$$\int_{-\infty}^{\infty} |q_{(n)}(y) - q_{(n,m(n),\epsilon(n))}(y)|^2 d\,y \leq \exp(-\delta 2^{\frac{n}{2}})$$

and

$$\int_{-\infty}^{\infty} (1 + |x|)|q_{(n)}(y) - q_{(n,m(n),\epsilon(n))}(y)| d\,y \leq \frac{1}{n}.$$

Hence, letting be

$$q_n(x) = q_{(n,m(n),\epsilon(n))}(x), \tag{4.7}$$

we see that

$$\int_{0}^{\infty} |q_n(x)|^2 e^{\delta |x|^{\frac{1}{2}}} d\,x \leq C, \tag{4.8}$$

$$\int_{-\infty}^{\infty} |q(x) - q_n(x)|^2 d\,x \leq C \exp(-\delta 2^{\frac{n}{2}}), \tag{4.9}$$

$$\int_{-\infty}^{\infty} (1 + |x|)|q(y) - q_n(y)| d\,y \to 0 \quad \text{as } n \to \infty \tag{4.10}$$

and

$$q_n(x) = 0 \qquad \text{for } x \geq 2^n + 1. \tag{4.11}$$

We show that the sequence $\{q_n(x)\}_{n=1,2,\ldots}$ of potentials given by (4.7) satisfies the desired properties. We denote by $a_n(\lambda)$ and $c_n(\lambda)$ the coefficients of (3.1) associated with the operator $L_{q_n(x)}$ and by $r_n^+(\lambda)$, $R_n^+(x)$ and $F_n^+(x)$ (3.15), (3.16) and (3.17) derived from $a_n(\lambda)$ and $c_n(\lambda)$.

From now on until the end of proof of Proposition 4.1, we use constants without the suffix n in order to express constants independent of n. Let $\{i\mu_1^{(n)}, \ldots, i\mu_{M_n}^{(n)}\}$ $(\mu_1^{(n)} > \cdots > \mu_{M_n}^{(n)})$ be the set of $a_n(\lambda)$'s zeros in $\mathbb{C}^+$. Since we see that

$$\|L_{q_n(x)}u - L_{q(x)}u\|_2 \leq \|q_n - q\|_2 \|u\|_\infty \tag{4.12}$$

and from Sobolev's imbedding theorem that for any $\epsilon > 0$

$$\|u\|_\infty \leq \epsilon \|L_{q(x)}u\|_2 + C_{\epsilon,q}\|u\|_2, \tag{4.13}$$

we get from (4.9) the following estimates of $\mu_k^{(n)}$.

Lemma 4.1. *For large n we have*

$$M_n \geq M, \tag{4.14}$$

$$|\mu_k^{(n)} - \mu_k| < C_0 \exp(-\delta 2^{\frac{n}{2}-2}) \quad (1 \leq k \leq M) \tag{4.15}$$

and

$$|\mu_k^{(n)}| < C_0 \exp(-\delta 2^{\frac{n}{2}-2}) \quad (k > M). \tag{4.16}$$

The proof of Lemma 4.1 is given in the next section. Now we assume that n is large enough and that the estimates above (4.14), (4.15), (4.16) and $C_0 \exp(-\delta 2^{\frac{n}{2}-2}) < \frac{1}{3}\mu_M$ hold. We remark that $\frac{1}{a_n(\lambda)}$ is analytic in $\{\lambda \in \mathbb{C}\ ;\ C_0 \exp(-\delta 2^{\frac{n}{2}-2}) < \Im\lambda < \frac{2}{3}\mu_M\}$.

Since $q_n(x) \in C_0^\infty(\mathbb{R})$, we see from (4.1) that

$$F_n^+(x) = \sum_{j=1}^{M} (m_j^{(n)+})^2 e^{-\mu_j^{(n)} x} + \frac{1}{2\pi} \int_{-\infty+i\sigma}^{\infty+i\sigma} r_n^+(\lambda) e^{ix\lambda} d\,\lambda,$$

where

$$\sigma \in \left[C_0 \exp(-\delta 2^{\frac{n}{2}-2}), \frac{1}{2}\mu_M \right].$$

We define $\tilde{R}_n^+(x)$ by

$$\tilde{R}_n^+(x) = \frac{1}{2\pi} \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{r_n^+(\lambda)}{\lambda + i} e^{ix\lambda} d\,\lambda. \tag{4.17}$$

We have

$$F_n^+(x) = \sum_{j=1}^{M} (m_j^{(n)+})^2 e^{-\mu_j^{(n)} x} + \left(-i\frac{d}{dx} + i \right) \tilde{R}_n^+(x).$$

It follows from (2.21), (2.22) and (4.10) that, for $\lambda \in \overline{\mathbb{C}^+}$, $e_n^+(x,\lambda)$, $\frac{d}{dx}e_n^+(x,\lambda)$, $e_n^-(x,-\lambda)$ and $\frac{d}{dx}e_n^-(x,-\lambda)$ converge to $e^+(x,\lambda)$, $\frac{d}{dx}e^+(x,\lambda)$, $e^-(x,-\lambda)$ and $\frac{d}{dx}e^-(x,-\lambda)$ respectively. Hence we obtain that on $\overline{\mathbb{C}^+}$

$$\lambda a_n(\lambda) \to \lambda a(\lambda) \text{ compact uniformly as } n \to \infty \tag{4.18}$$

and

$$|a_n(\lambda) - 1| \leq \frac{C}{|\lambda|}. \tag{4.19}$$

Similarly we see from (4.10) that on $\mathbb{R}$

$$\lambda c_n(\lambda) \to \lambda c(\lambda) \ (n \to \infty). \tag{4.20}$$

Furthermore noting that (4.8) and Schwarz's inequality imply

$$\int_0^\infty (|q_n(x)| + \int_x^\infty |q_n(y)|d\,y) e^{\frac{1}{3}\delta|x|^{\frac{1}{2}}} d\,x \leq C,$$

we obtain from (2.19)

$$\int_0^{\infty}(|K^+_{q_n}(0,y)|+|\partial_x K^+_{q_n}(0,y)|+|\partial_y K^+_{q_n}(0,y)|)e^{\frac{1}{6}\delta|y|^{\frac{1}{2}}}d\,y\le C \tag{4.21}$$

and from (4.10) and (2.23)

$$\int_{-\infty}^{0}(|K^-_{q_n}(0,y)|+|\partial_x K^-_{q_n}(0,y)|+|\partial_y K^-_{q_n}(0,y)|)d\,y\le C. \tag{4.22}$$

Then we have the "almost analytic extension" of $c_n(\lambda)$ in the following way;

Lemma 4.2. *There exists a smooth function $\tilde c_n(\lambda)$ on $\overline{\mathbb{C}^+}\setminus\{0\}$ satisfying the followings*;

$$\tilde c_n(\lambda)=c_n(\lambda) \qquad (0\le \Im\lambda< C_1 2^{-\frac{n}{2}}), \tag{4.23}$$

$$|\lambda\tilde c_n(\lambda)|\le C \qquad (0\le\Im\lambda), \tag{4.24}$$

$$|\lambda\bar\partial_\lambda\tilde c_n(\lambda)|\le Ce^{-\delta_1\frac{1}{|\Im\lambda|}} \qquad (0<\Im\lambda). \tag{4.25}$$

The proof of Lemma 4.2 is given in the next section.

Let be

$$\tilde r^+_n(\lambda)=\frac{\tilde c^+_n(\lambda)}{a_n(\lambda)}.$$

Noting that (4.23) implies that

$$\tilde r^+_n(\lambda)=r^+_n(\lambda) \qquad \left(C_0\exp(-\delta 2^{\frac{n}{2}-2})\le\Im\lambda\le C_1 2^{-\frac{n}{2}}\right),$$

$$\bar\partial_\lambda\tilde r^+_n(\lambda)=\frac{\bar\partial_\lambda\tilde c_n(\lambda)}{a_n(\lambda)} \qquad \left(C_0\exp(-\delta 2^{\frac{n}{2}-2})\le\Im\lambda\le \frac{1}{2}\mu_M\right),$$

we obtain by Cauchy's integral theorem

(4.26)

$$\int_{-\infty+i\sigma}^{\infty+i\sigma}\frac{\tilde r^+_n(\lambda)}{\lambda+i}e^{ix\lambda}d\,\lambda=\int_{D_\sigma}\frac{\bar\partial_\lambda\tilde r^+_n(\lambda)}{\lambda+i}e^{ix\lambda}d\,\bar\lambda\wedge d\,\lambda+\int_{-\infty+i\frac{1}{2}\mu_M}^{\infty+i\frac{1}{2}\mu_M}\frac{\tilde r^+_n(\lambda)}{\lambda+i}e^{ix\lambda}d\,\lambda,$$

where

$$D_\sigma=\left\{\lambda\in\mathbb{C}\ ;\ \sigma\le\Im\lambda\le\frac{1}{2}\mu_M\right\}$$

and $\sigma\in[C_0\exp(-\delta 2^{\frac{n}{2}-2}),C_1 2^{-\frac{n}{2}}]$.

In order to obtain the uniform estimate of the right hand side of (4.26), we remark the followings. Since the number of zeros M_n is that of eigenvalues of $L_{q_n(x)}$, we obtain

$$M_n\le 2+\int_{-\infty}^{\infty}|xq_n(x)|d\,x$$

(see for example Theorem 4.3.II of [4] or Prob. 22 in Ch. XIII of [15]). Then (4.10) implies that $\{M_n\}$ is bounded, say

$$M_n \leq M_+. \tag{4.27}$$

Let be

$$\Gamma_n(\lambda) = \prod_{k>M} \frac{\lambda - i\mu_k^{(n)}}{\lambda + i\mu_k^{(n)}}.$$

Then $\frac{\Gamma_n(\lambda)}{a_n(\lambda)}$ is analytic for $0 < \Im\lambda < \frac{2\mu_M}{3}$. Since

$$|\Gamma_n(\lambda)| \leq 1 \ (\Im\lambda \geq 0),$$

we see from (3.11), (4.18) and (4.19) that

$$\left|\frac{\Gamma_n(\lambda)}{a_n(\lambda)}\right| \leq 1 \quad \text{on } \mathbb{R},$$
$$\left|\frac{\Gamma_n(\lambda)}{a_n(\lambda)}\right| \leq C \quad \text{if } \Im\lambda = \frac{\mu_M}{2}$$

and

$$\frac{\Gamma_n(\lambda)}{a_n(\lambda)} \to 1 \quad \text{as } \lambda \in \overline{\mathbb{C}^+} \to \infty.$$

Then the maximum principle implies that

$$\left|\frac{\Gamma_n(\lambda)}{a_n(\lambda)}\right| \leq C \quad \text{on } D_0.$$

Since it follows from (4.16) and (4.27) that

$$|\Gamma_n(\lambda)| \geq C \quad \text{if } C_1 2^{-\frac{n}{2}} \leq \Im\lambda,$$

we obtain

$$\left|\frac{1}{a_n(\lambda)}\right| \leq C \quad \text{if } C_1 2^{-\frac{n}{2}} \leq \Im\lambda \leq \frac{\mu_M}{2}$$

from which, noting (4.24) and (4.25), we draw

$$|\tilde{r}_n^+(\lambda)| \leq C\frac{1}{|\lambda|+1} \quad \text{if } \Im\lambda = \frac{\mu_M}{2} \tag{4.28}$$

and

$$|\bar{\partial}_\lambda \tilde{r}_n^+(\lambda)| \leq C\frac{1}{|\lambda|+1}e^{-\delta_2\frac{1}{|\Im\lambda|}} \text{ if } C_1 2^{-\frac{n}{2}} \leq \Im\lambda \leq \frac{\mu_M}{2} \tag{4.29}$$

where we used the estimate; for $\delta_1 > \delta_2 > 0$

$$\frac{1}{|\lambda|}e^{-\delta_1\frac{1}{|\Im\lambda|}} \leq C\frac{1}{|\lambda|+1}e^{-\delta_2\frac{1}{|\Im\lambda|}}.$$

Since $\bar{\partial}_\lambda \tilde{r}_n^+(\lambda) = 0$ if $C_1 2^{-\frac{n}{2}} \geq \Im\lambda > C_0 \exp(-\delta^{\frac{n}{2}-2})$, we see that on D_σ

$$
\begin{aligned}
|\bar{\partial}_\lambda \tilde{r}_n^+(\lambda) e^{ix\lambda}| &\leq \frac{C}{1+|\lambda|} \exp\left(-\frac{\delta_2}{|\Im\lambda|} - x\Im\lambda\right) \\
&\leq \begin{cases} \dfrac{C}{1+|\lambda|} \exp(-2\sqrt{\delta_2 x}) & \text{if } x \geq 0, \\ \dfrac{C}{1+|\lambda|} \exp\left(\dfrac{\mu_M}{2}|x|\right) & \text{if } x \leq 0. \end{cases}
\end{aligned}
\tag{4.30}
$$

It follows from (4.17), (4.26), (4.28) and (4.30) that

$$
|\tilde{R}_n^+(x)| \leq \begin{cases} C\exp(-2\sqrt{\delta_2 x}) & \text{if } x \geq 0, \\ C\exp\left(\dfrac{\mu_M}{2}|x|\right) & \text{if } x \leq 0, \end{cases}
\tag{4.31}
$$

which is just (4.3).

Next we show the convergence (4.4). Noting that, as $n \to \infty$,

$$\Gamma_n(\lambda) \to 1 \quad \text{for } \lambda \in \mathbb{R} \setminus \{0\},$$

we see from (4.18), (4.19) and (4.20) that

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_n(\lambda) \frac{r_n^+(\lambda)}{\lambda + i} e^{ix\lambda} d\,\lambda \to \tilde{R}^+(x) \quad \text{as } n \to \infty.
\tag{4.32}
$$

Since in the strip $0 < \Im\lambda < C_1 2^{-\frac{n}{2}}$ we have

$$\Gamma_n(\lambda)\tilde{r}_n^+(\lambda) = \Gamma_n(\lambda) r_n^+(\lambda)$$

that is analytic there, we see

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \Gamma_n(\lambda) \frac{r_n^+(\lambda)}{\lambda + i} e^{ix\lambda} d\,\lambda \\
&= \int_{D_{C_1 2^{-\frac{n}{2}}}} \Gamma_n(\lambda) \frac{\bar{\partial}_\lambda \tilde{r}_n^+(\lambda)}{\lambda + i} e^{ix\lambda} d\,\bar{\lambda} \wedge d\,\lambda + \int_{-\infty + i\frac{1}{2}\mu_M}^{\infty + i\frac{1}{2}\mu_M} \Gamma_n(\lambda) \frac{\tilde{r}_n^+(\lambda)}{\lambda + i} e^{ix\lambda} d\,\lambda.
\end{aligned}
\tag{4.33}
$$

Since $|\prod_{1\leq k\leq l}(1+b_k) - 1| \leq e^{\sum_{1\leq k\leq l}|b_k|}(\sum_{1\leq k\leq l}|b_k|)$, we get from (4.16) and (4.27),

$$|\Gamma_n(\lambda) - 1| \leq \exp(CM_+ e^{-\delta 2^{\frac{n}{2}-2}} 2^{\frac{n}{2}}) CM_+ e^{-\delta 2^{\frac{n}{2}-2}} 2^{\frac{n}{2}} \quad \text{on } D_{C_1 2^{-\frac{n}{2}}}.$$

Thus we see that the difference between the right hand side of (4.33) and

$$\int_{D_{C_1 2^{-\frac{n}{2}}}} \frac{\bar{\partial}_\lambda \tilde{r}_n^+(\lambda)}{\lambda + i} e^{ix\lambda} d\,\bar{\lambda} \wedge d\,\lambda + \int_{-\infty + i\frac{1}{2}\mu_M}^{\infty + i\frac{1}{2}\mu_M} \frac{\tilde{r}_n^+(\lambda)}{\lambda + i} e^{ix\lambda} d\,\lambda$$

converges to zero. Therefore from (4.26) and (4.32) we obtain (4.4). This and (4.31) imply (4.2). Taking into account (3.20), we see from (4.15) that (4.5) and (4.6) hold. Indeed, since $e_n^+(x, i\mu_j^{(n)})$ and $e_n^-(x, -i\mu_j^{(n)})$ are linearly dependent, we have

$$\begin{pmatrix} e_n^+(0, i\mu_j^{(n)}) \\ e_n^{+\prime}(0, i\mu_j^{(n)}) \end{pmatrix} = c_{\mu_j^{(n)}} \begin{pmatrix} e_n^-(0, -i\mu_j^{(n)}) \\ e_n^{-\prime}(0, -i\mu_j^{(n)}) \end{pmatrix}$$

from which we obtain for $1 \leq j \leq M$

$$c_{\mu_j^{(n)}} \to c_{\mu_j} \quad \text{as } n \to \infty.$$

While (4.18) and (4.15) imply that for $1 \leq j \leq M$

$$a_n'(i\mu_j^{(n)}) \to a'(i\mu_j).$$

Then we get (4.6). Thus the proof of Proposition 4.1 is completed except the proof of two lemmas. □

5. Proof of Lemmas

In this section we give the proof of Lemmas 4.1 and 4.2.
First we prove Lemma 4.1, following T. Kato [8, Ch. V §4.3].

Lemma 5.1. *Let A_1 and A_2 be self-adjoint operators on a Hilbert space X with the common domain X_1. We assume that we have on X_1*

$$\|A_2 u - A_1 u\| \leq \nu \|A_1 u\| + C_1 \|u\|$$

with $0 < \nu < 1$. If $(\lambda - d, \lambda + d)$ belongs to the resolvent set of A_1, where we assume $\nu|\lambda| + C_1 < d(1-\nu)$, then λ belongs to the resolvent set of A_2.

Proof. Indeed we see from the assumption that

$$\|(\lambda - A_1)^{-1}\| \leq \frac{1}{d}$$

and

$$\|A_1(\lambda - A_1)^{-1}\| \leq 1 + \frac{|\lambda|}{d}.$$

Then

$$\begin{aligned} \|(A_2 - A_1)(\lambda - A_1)^{-1}\| &\leq \nu \left(1 + \frac{|\lambda|}{d}\right) + C_1 \frac{1}{d} \\ &< 1. \end{aligned}$$

Hence

$$(\lambda - A_2)(\lambda - A_1)^{-1} = I - (A_2 - A_1)(\lambda - A_1)^{-1}$$

is an automorphism in X, which implies the assertion of Lemma 5.1. □

We see from Ch. V, Thm. 4.3 in Kato [8] that the operator $L_{q(x)}$ with a potential $q(x) \in \mathcal{P}(1) \cap L^2(\mathbb{R})$ is a self-adjoint operator on $L^2(\mathbb{R})$ with its domain

$$H^2(\mathbb{R}) = \{u(x) \in L^2(\mathbb{R}) \ ; \ u'(x), u''(x) \in L^2(\mathbb{R})\}.$$

Since we obtain from Sobolev's imbedding theorem

$$(L_{q(x)}u, u) + C_q \|u\|_2^2 \geq 0$$

with a constant C_q depending only on the norm $\|q(\cdot)\|_2$, we see that all the eigenvalues of $L_{q_n(x)}$ have a common lower bound, say, $-\Lambda$. Since only $-{\mu_j}^2$ $(j = 1, \ldots, M)$ are eigenvalues of $L_{q(x)}$, (4.9), (4.12), (4.13) with $\epsilon = \frac{1}{2}$ and Lemma 5.1 imply that for large n the following set

$$(-\Lambda - 1, -d) \setminus \cup_{1 \leq j \leq M}(-d - {\mu_j}^2, d - {\mu_j}^2)$$

with $d = C\exp(-\delta 2^{\frac{n}{2}-1})$ belongs to the resolvent set of $L_{q_n(x)}$. On the other hand by (4.18) and Hurwitz theorem, on any neighborhood of $i\mu_j$ we can find a zero of $a_n(\lambda)$ for large n. Hence we see the assertion of Lemma 4.1 is valid. q.e.d.

Next we prove Lemma 4.2. Taking into account the relation (3.6), Remark 6, (4.11), (4.21) and (4.22), we have only to prove the following lemma in order to prove Lemma 4.2.

Lemma 5.2. *We assume that a function $g(y) \in L^1([0, +\infty))$ satisfies the following estimate*:

$$\int_0^{+\infty} |g(y)| e^{\delta |y|^{\frac{1}{2}}} d\,y < +\infty \tag{5.1}$$

with a positive constant δ. Then there exists a function $\tilde{g}(\lambda + i\mu) \in C^\infty(\mathbb{R}^2)$ that satisfies

$$\tilde{g}(\lambda) = \hat{g}(\lambda) \quad \text{on } \mathbb{R}, \tag{5.2}$$

$$|(\partial_\lambda + i\partial_\mu))\tilde{g}(\lambda + i\mu)| \leq Ce^{-\delta_1|\mu|^{-1}} \int_0^{+\infty} |g(y)| e^{\delta |y|^{\frac{1}{2}}} d\,y \tag{5.3}$$

and

$$|\partial_\lambda^k \partial_\mu^l \tilde{g}(\lambda + i\mu)| \leq C_{k,l} \int_0^{\infty} |g(y)| e^{\delta |y|^{\frac{1}{2}}} d\,y, \tag{5.4}$$

where

$$\hat{g}(\lambda) = \int_0^\infty g(y) e^{-iy\lambda} d\,y$$

and the positive constants above C, δ_1 and C_{kl} depend on the constant δ of (5.1). *Furthermore if $g(y) = 0$ for $y \geq 2^{n_o}$, then $\hat{g}(\lambda + i\mu) = \tilde{g}(\lambda + i\mu)$ for $|\mu| \leq 2^{\frac{-n_o-5}{2}}\delta$.*

Proof. Let $\chi(y)$ be a function in $C_0^\infty(\mathbb{R})$ satisfying $0 \le \chi(y) \le 1$ and

$$\chi(y) = \begin{cases} 1, & |y| \le 1, \\ 0, & |y| \ge 2. \end{cases}$$

We define $\tilde{g}(\lambda + i\mu)$ by

$$\tilde{g}(\lambda + i\mu) = \sum_{n=0}^{\infty} \chi(2^{\frac{n+5}{2}}\mu\delta^{-1})\tilde{g}_n(\lambda + i\mu),$$

where

$$\tilde{g}_0(\lambda + i\mu) = \int_0^1 g(y)e^{-iy(\lambda+i\mu)}dy,$$

$$\tilde{g}_n(\lambda + i\mu) = \int_{2^{n-1}}^{2^n} g(y)e^{-iy(\lambda+i\mu)}dy \quad \text{for } n \ge 1.$$

First we remark that $\tilde{g}_n(\lambda + i\mu)$ is an entier function.

Since for $n \ge 1$

$$|\partial_\lambda^k \partial_\mu^l \tilde{g}_n(\lambda + i\mu)| \le 2^{n(k+l)} \exp\left(|\mu|2^n - \delta 2^{\frac{n-1}{2}}\right) \int_{2^{n-1}}^{2^n} |g(y)|e^{\delta|y|^{\frac{1}{2}}}dy \tag{5.5}$$

and

$$\chi(2^{\frac{n+5}{2}}\mu\delta^{-1}) = 0 \quad \text{when } |\mu|2^n \ge \frac{1}{2}\delta 2^{\frac{n-1}{2}}, \tag{5.6}$$

then we see that $\tilde{g}(\lambda + i\mu)$ is in $C^\infty(\mathbb{R}^2)$ and that the estimate (5.4) is valid. Furthermore we obtain

$$(\partial_\lambda + i\partial_\mu)\tilde{g}(\lambda + i\mu) = \sum_{n=0}^{\infty} i2^{\frac{n+5}{2}}\delta^{-1}\chi'(2^{\frac{n+5}{2}}\mu\delta^{-1})\tilde{g}_n(\lambda + i\mu).$$

Since $\chi'(2^{\frac{n+5}{2}}\mu\delta^{-1}) \ne 0$ implies $\delta 2^{\frac{n-5}{2}} \le |\mu|2^n \le \frac{1}{2}\delta 2^{\frac{n-1}{2}}$ from which we obtain

$$\frac{1}{4}\delta 2^{\frac{n-1}{2}} \ge \frac{\delta^2}{2^5|\mu|},$$

it follows from (5.5) and (5.6) that for $n \ge 1$

$$|2^{\frac{n+5}{2}}\delta^{-1}\chi'(2^{\frac{n+5}{2}}\mu\delta^{-1})\tilde{g}_n(\lambda + i\mu)|$$
$$\le 2^{\frac{n+5}{2}}\delta^{-1}e^{-\delta 2^{\frac{n-5}{2}}}e^{-\frac{\delta^2}{2^5|\mu|}}\|\chi'(\cdot)\|_{L^\infty}\int_{2^{n-1}}^{2^n}|g(y)|e^{\delta|y|^{\frac{1}{2}}}dy.$$

Hence we obtain (5.3).

If $g(y) = 0$ for $y \ge 2^{n_o}$, then

$$\tilde{g}(\lambda + i\mu) = \sum_{n=0}^{n_o} \chi(2^{\frac{n+5}{2}}\mu\delta^{-1})\tilde{g}_n(\lambda + i\mu).$$

Since $\chi(2^{\frac{n+5}{2}}\mu\delta^{-1}) = 1$ for $|\mu| \le 2^{\frac{-n_o-5}{2}}\delta$ and $0 \le n \le n_0$, we see $\hat{g}(\lambda + i\mu) = \tilde{g}(\lambda + i\mu)$ for $|\mu| \le 2^{\frac{-n_o-5}{2}}\delta$. $\square$

6. Proof of Theorem 1.2.

In this section we complete the proof of the main theorem. For any initial data $u_0(x) \in \mathcal{P}(1) \cap L^2(\mathbb{R})$ that satisfies the decay condition (D), according to Proposition 4.1 we can find a sequence $\{u_{0,n}(x)\}$ in $C_0^\infty(\mathbb{R})$ so that

$$u_{0,n}(x) \to u_0(x) \quad \text{in } L^2(\mathbb{R})$$

and letting be $F^+(x)$ and $F_n^+(x)$ functions given by (3.17) derived from the potential $u_0(x)$ and $u_{0,n}(x)$ respectively we have

$$\begin{aligned} F^+(x) &= \sum_{j=1}^{M} (m_j^+)^2 e^{-\mu_j x} + \left(-i\frac{d}{dx} + i\right) \tilde{R}^+(x), \\ F_n^+(x) &= \sum_{j=1}^{M} (m_j^{(n)+})^2 e^{-\mu_j^{(n)} x} + \left(-i\frac{d}{dx} + i\right) \tilde{R}_n^+(x), \end{aligned}$$

where $\tilde{R}^+(x)$ and $\tilde{R}_n^+(x)$ satisfy (4.2) and (4.3) respectively and the convergence (4.4), (4.5) and (4.6) are valid. Since the Cauchy problem (1.8) is well-posed in the space $\mathcal{S}$, the solution $u_n(t,x)$ to the problem (1.8) with the initial data $u_{0,n}(x)$ belongs to $C^1([0,+\infty), \mathcal{S})$ (see Kato [9]). Then for $t > 0$ the solution $u_n(t,x)$ can be expressed by the following way;

$$u_n(t,x) = -2\frac{d}{dx} K_n^+(x,x,t), \tag{6.1}$$

where $K_n^+(x,y,t)$ satisfies the following Gelfand-Levitan-Marchenko equation

$$F_n^+(x+y,t) + K_n^+(x,y,t) + \int_x^\infty K_n^+(x,w,t) F_n^+(w+y,t)\, dw = 0 \tag{6.2}$$

with the function $F_n^+(x,t)$ that is given by

$$F_n^+(x,t) = \sum_{j=1}^{M} (m_j^{(n)+})^2 e^{-\mu_j^{(n)} x + 8(\mu_j^{(n)})^3 t} + R_n^+(x,t), \tag{6.3}$$

where $R_n^+(x,t)$ is given by

$$R_n^+(x,t) = E_8(t,x) * \left(-i\frac{d}{dx} + i\right) \tilde{R}_n^+(x). \tag{6.4}$$

Indeed, from (3.28) and (3.29), we obtain

$$F_n^+(x,t) = \sum_{j=1}^{M_n} (m_j^{(n)+})^2 e^{-\mu_j^{(n)} x + 8(\mu_j^{(n)})^3 t} + E_8(t,x) * R_n^+(x)$$

with

$$R_n^+(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{c_n(\lambda)}{a_n(\lambda)} e^{i\lambda x}\, d\lambda.$$

Noting $E_8(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x\omega+8t\omega^3)}\,d\omega$, we see

$$E_8(t,x) * R_n^+(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\omega+8t\omega^3}\frac{c_n(\omega)}{a_n(\omega)}\,d\omega.$$

Just as the derivation of (4.1), we see the right hand side is equal to

$$\sum_{j>M}\frac{ic_n(i\mu_j^{(n)})}{a_n'(i\mu_j^{(n)})}e^{-\mu_j^{(n)}x+8t(\mu_j^{(n)})^3} + \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} e^{ix\omega+8t\omega^3}\frac{c_n(\omega)}{a_n(\omega)}\,d\omega$$

with $\sigma \in [C_0\exp(-\delta 2^{\frac{n}{2}-2}), \frac{1}{2}\mu_M]$. Then it follows from (3.20) that $E_8(t,x) * R_n^+(x)$ is equal to

$$-\sum_{j>M}(m_j^{(n)+})^2 e^{-\mu_j^{(n)}x+8t(\mu_j^{(n)})^3} + E_8(t,x) * \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma}\frac{c_n(\lambda)}{a_n(\lambda)}e^{i\lambda x}\,d\lambda.$$

Hence we obtain (6.3).

It follows from (4.3), (4.4), (4.5) and (4.6) that $F_n^+(x,t)$ converges to

$$\sum_{j=1}^{M}(m_j^+)^2 e^{-\mu_j x+8{\mu_j}^3 t} + R^+(x,t),$$

which we denote by $F^+(x,t)$, where

$$R^+(x,t) = E_8(t,x) * \left(-i\frac{d}{dx}+i\right)\tilde{R}^+(x).$$

We remark that the Fourier transformation of $R^+(x,t)$ with respect to the variable x is $r^+(\lambda)e^{i8\lambda^3 t}$. It follows from Proposition 1.1 and the estimates (4.2) and (4.3) that for $t > 0$ $F^+(x,t)$ and $F_n^+(x,t)$ are real analytic on $\mathbb{R}$ and that for any $t_1 > t_0 > 0$ and $x_o \in \mathbb{R}$ there exist positive constants δ_1, δ_2 and C such that for $t_1 \geq t \geq t_0$, $F^+(x,t)$ and $F_n^+(x,t)$ are holomorphic on $B_{x_0,\delta_1} = \{z \in \mathbb{C}\ ;\ \Re z > x_0 - \delta_1 \text{ and } |\Im z| < \delta_1\ \}$ and they satisfy the following decay estimate on B_{x_0,δ_1};

$$|F_n^+(z,t)| \leq C\exp(-\delta_2|z|^{\frac{1}{2}}) \quad (z \in B_{x_0,\delta_1}) \tag{6.5}$$

and

$$|F^+(z,t)| \leq C\exp(-\delta_2|z|^{\frac{1}{2}}) \quad (z \in B_{x_0,\delta_1}). \tag{6.6}$$

Furthermore we see from (4.4), (4.5) and (4.6) that

$$F_n^+(z,t) \to F^+(z,t) \quad \text{on } B_{x_0,\delta_1}. \tag{6.7}$$

Now we consider the Gelfand-Levitan-Marchenko equation.

Lemma 6.1. *For $t > 0$, the Gelfand-Levitan-Marchenko equation*

$$F^+(x+y,t) + K^+(x,y,t) + \int_x^\infty K^+(x,w,t)F^+(w+y,t)d\,w = 0 \tag{6.8}$$

has a solution $K^+(x,y,t)$. For any $t > 0$ and $x_0 \in \mathbb{R}$ there exist a complex neighborhood U of x_0 such that $K_n^+(x,x,t)$ and $K^+(x,x,t)$ have analytic continuations in U and $K_n^+(z,z,t) \to K^+(z,z,t)$ uniformly on any compact in U.

Proof. Let be

$$\tilde{K}_n^+(x,y,t) = K_n^+(x,x+y,t),$$
$$\tilde{K}^+(x,y,t) = K^+(x,x+y,t).$$

Then (6.2) and (6.8) are equivalent to

$$\tilde{K}_n^+(x,y,t) + \int_0^\infty F_n^+(2x+w+y,t)\tilde{K}_n^+(x,w,t)d\,w + F_n^+(2x+y,t) = 0 \tag{6.9}$$

and

$$\tilde{K}^+(x,y,t) + \int_0^\infty F^+(2x+w+y,t)\tilde{K}^+(x,w,t)d\,w + F^+(2x+y,t) = 0 \tag{6.10}$$

respectively.

For any $x \in \mathbb{R}$ and $t > 0$ we define the operator $F_{x,t}$ on $L^2([0,\infty))$ by

$$F_{x,t}(k(\cdot))(y) = \int_0^\infty F^+(2x+w+y,t)k(w)d\,w.$$

Hence (6.10) is

$$\tilde{K}^+(x,y,t) + F_{x,t}(\tilde{K}^+(x,\cdot,t))(y) + F^+(2x+y,t) = 0.$$

Since (6.6) implies that $F^+(2x+w+y,t) \in L^2([0,+\infty)\times[0,+\infty))$ for any real x and $t > 0$, $F_{x,t}$ is a compact operator on $L^2([0,+\infty))$. Furthermore $I + F_{x,t}$ is injective. Indeed if a real-valued $k(y) \in L^2([0,+\infty))$ satisfies

$$k(y) + F_{x,t}(k(\cdot))(y) = 0,$$

then

$$\int_0^\infty |k(y)|^2 d\,y + \int_0^\infty k(y)\int_0^\infty F^+(2x+y+w,t)k(w)d\,w d\,y = 0. \tag{6.11}$$

Since $(m_j^+)^2 > 0$, we obtain

$$
\begin{aligned}
&\int_0^\infty k(y)\int_0^\infty F^+(2x+y+w,t)k(w)d\,w d\,y\\
&\qquad\geq \int_0^\infty k(y)\int_0^\infty R^+(2x+y+w,t)k(w)d\,w d\,y\\
&\qquad= \frac{1}{2\pi}\int_{-\infty}^\infty e^{i(2x\lambda+8\lambda^3 t)}r^+(\lambda)\hat{k}^2(-\lambda)d\,\lambda\\
&\qquad\geq -\frac{1}{2\pi}\int_{-\infty}^\infty |r^+(\lambda)||\hat{k}^2(-\lambda)|^2 d\,\lambda,
\end{aligned}
\tag{6.12}
$$

where

$$\hat{k}(\lambda) = \int_0^\infty e^{-i\lambda y}k(y)d\,y.$$

Since

$$\int_0^\infty |k(y)|^2 d\,y = \frac{1}{2\pi}\int_{-\infty}^\infty |\hat{k}^2(-\lambda)|^2 d\,\lambda,$$

we see from (6.11) and (6.12) that

$$\frac{1}{2\pi}\int_{-\infty}^\infty (1-|r^+(\lambda)|)|\hat{k}^2(-\lambda)|^2 d\,\lambda \leq 0.$$

From (3.19) we see $k(y)=0$, which implies that $I+F_{x,t}$ is injective. Therefore $I+F_{x,t}$ is invertible (see Marchenko [12, Lemma 3.5.3 in Ch. 3, Sec. 5]). Then $\tilde{K}^+(x,y,t)$ given by

$$\tilde{K}^+(x,y,t) = (I+F_{x,t})^{-1}(-F^+(2x+\cdot,t))(y)$$

is a solution of (6.10). Since $\tilde{K}_n^+(x,y,t) \in L^1([0,+\infty)) \cap L^\infty([0,+\infty))$ for any $x\in\mathbb{R}$ and $t>0$, we see $\tilde{K}_n^+(x,y,t)\in L^2([0,+\infty))$. Therefore the argument above implies that

$$\tilde{K}_n^+(x,y,t) = (I+F_{n,x,t})^{-1}(-F_n^+(2x+\cdot,t))(y),$$

where

$$F_{n,x,t}(k(\cdot))(y) = \int_0^\infty F_n^+(2x+w+y,t)k(w)d\,w.$$

Since for any $t>0$ and any $x_0\in\mathbb{R}$ there exists a complex neighborhood V of x_0 such that $F^+(2z+w+y,t)$ is well-defined on $V\times[0,+\infty)\times[0,+\infty)$ and holomorphic with respect to the variable z, taking into account (6.6) we see that the operator $F_{z,t}$ is $\mathcal{L}(L^2([0,+\infty)),L^2([0,+\infty)))$-valued holomorphic function on V. Similarly we see that $\{F_{n,z,t}\}$ is a bounded family of $\mathcal{L}(L^2([0,+\infty)),L^2([0,+\infty)))$-valued holomorphic functions on V. We obtain from (6.5) and (6.7)

$$\|F_n^+(2z+w+y,t) - F^+(2z+w+y,t)\|_{L^2_{w,y}([0,+\infty)\times[0,+\infty))} \to 0$$

uniformly on any compact in V. Then we see that there exist a complex neighborhood U of x_0 and a large positive integer N such that, if $n \geq N$ and $z \in U$, $I + F_{z,t}$ and $I + F_{n,z,t}$ are invertible and their inverses form a bounded family of $\mathcal{L}(L^2([0, +\infty)), L^2([0, +\infty)))$-valued holomorphic functions on U. Therefore $(I+F_{z,t})^{-1}(F^+(2z+\cdot, t))(y)$ and $(I+F_{n,z,t})^{-1}(F_n^+(2z+\cdot, t))(y)$ are $L^2([0, +\infty))$-valued holomorphic function of z. Hence $\tilde{K}_n^+(x, y, t)$ and $\tilde{K}^+(x, y, t)$ have analytic continuations in U as $L^2([0, +\infty))$-valued functions and for $z \in U$

$$\|\tilde{K}_n^+(z, \cdot, t) - \tilde{K}^+(z, \cdot, t)\|_2 \to 0$$

uniformly on any compact in U.

Then $F_{z,t}(\tilde{K}^+(z, \cdot, t))(0) + F^+(2z, t)$ and $F_{n,z,t}(\tilde{K}_n^+(z, \cdot, t))(0) + F_n^+(2z, t)$ are holomorphic functions and

$$F_{n,z,t}(\tilde{K}_n^+(z, \cdot, t))(0) + F_n^+(2z, t) \to F_{z,t}(\tilde{K}^+(z, \cdot, t))(0) + F^+(2z, t)$$

uniformly on any compact in U. Hence $\tilde{K}_n^+(z, 0, t)$ and $\tilde{K}^+(z, 0, t)$ are holomorphic and $\tilde{K}_n^+(z, 0, t) \to \tilde{K}^+(z, 0, t)$ uniformly on any compact in U. $\square$

Lemma above and (6.1) imply that for $t > 0$ solutions $u_n(t, x)$ are real analytic on $\mathbb{R}$ and converge to $-\frac{1}{2}\frac{d}{dx}K^+(x, x, t)$, which is also real analytic on $\mathbb{R}$, uniformly on any compact in $\mathbb{R}$. On the other hand, the Cauchy problem (1.8) is L^2-wellposed. Then $u_n(t, x)$ converges to the solution $u(t, x)$ to the problem (1.8) with the initial data $u_0(x)$ in L^2. Therefore we obtain

$$u(t, x) = -\frac{1}{2}\frac{d}{dx}K^+(x, x, t).$$

Hence we see that for $t > 0$ $u(t, x)$ is real analytic with respect to the variable x. The proof of Theorem 1.1 is completed.

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