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TRACE ESTIMATION OF COMMUTATORS OF MULTIPLICATION OPERATORS ON FUNCTION SPACES

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ABSTRACT. Let $A = \sum_{k\geq 1} T_{\varphi_k} T^*_{\varphi_k}$ be a bounded linear operator on Bergman space $L^2_a(B_d)$ or Hardy space $H^2(B_d)$, where φ_k is a multiplier for each k. We will show by trace estimation that for such an operator, $[A, T_{z_i}]$ belongs to Schatten class $\mathcal{L}_{2p}$ for $p > d$, and satisfies $\|[A, T_{z_i}]\|_{2p} \leq C\|A\|$ for some constant C depending only on p and d.

1. Introduction

In this paper, we will be concerned with commutators of multiplication operators on Bergman space and Hardy space over the d-dimensional complex unit ball B_d.

From the viewpoint of Hilbert modules, these spaces admit natural $C[z_1, \dots, z_d]$-module structure which comes from multiplication by polynomials. By a submodule M of these modules, we mean that M is closed, and invariant under multiplication by polynomials.

In [Arv3], [Arv4], Arveson conjectured that graded submodules of Drury–Arveson module on B_d, are p-essentially normal for $p > d$. Some affirmative results were obtained. Guo and Wang [GW1] proved p-essential normality of graded principal submodules of such Hilbert modules, and that of graded submodules when dimension $d = 2, 3$. Arveson showed in [Arv3], [Arv4] that, the p-essential normality of submodule M is equivalent to

$$[P_M, M_{z_i}] = P_M M_{z_i} - M_{z_i} P_M \in \mathcal{L}_{2p}, \quad 1 \leq i \leq d.$$

Thus it is of interest to investigate the p-essential commutative of certain operators.

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The famous Berger–Shaw theorem [BS1], [BS2], [Con2] shows that for an m-multicyclic hyponormal operator T, the commutator $[T^*, T]$ is a trace class operator, and the trace of it is not larger than $\frac{m}{\pi}\,\mathrm{Area}(\sigma(T))$.

Recently Q. Fang and J. Xia [FX] estimated the $2p$-norm of $[M_f^*, M_{z_i}]$ for multiplier f in Drury–Arveson space, and showed that when $p > d$, there is a constant C depending only on p and d such that

$$\big\|[M_f^*, M_{z_i}]\big\|_{2p} \le C\|M_f\|.$$

A question naturally arises:

> Let $p > d$. Is there a constant C such that, for any operator A in certain classes, such as the C^*-algebra or von Neumann algebra generated by multiplication operators of analytic symbols on Bergman space, Hardy space, or Drury–Arveson space, it holds that
>
> $$\big\|[A, M_{z_i}]\big\|_{2p} \le C\|A\|? \tag{1.1}$$

This is a refinement of Arveson's conjecture.

On Bergman space and Hardy space, for an analytic function $\varphi \in H^\infty(B_d)$, we will use the Toeplitz operator T_φ in place of M_φ, to prevent from mixing it with the multiplication operator on the corresponding Hilbert space L^2 of square-integrable functions.

When A is a multiplication operator on $L_a^2(B_d)$ or $H^2(B_d)$ of analytic symbol, by a technique based on Hankel operator one can show that the constant $C = \|[T_{z_i}, T_{z_i}^*]\|_p^{\frac{1}{2}}$ makes (1.1) hold. See Section 2 for details.

Let M be a submodule on Drury–Arveson space H_d^2. Arveson [Arv2] showed that there exists a sequence of multipliers $\{\varphi_k : k = 1, 2, \ldots\}$ such that

$$P_M \overset{\mathrm{SOT}}{=} \sum_{k\ge 1} M_{\varphi_k} M_{\varphi_k}^*. \tag{1.2}$$

Then Arveson's conjecture is equivalent to

$$\Big[\sum_{k\ge 1} M_{\varphi_k} M_{\varphi_k}^*, M_{z_i}\Big] \in \mathcal{L}_{2p} \tag{1.3}$$

for $p > d$ and $i = 1, 2, \ldots, d$. There is a counterexample in Section 4.1 of [GRS] that, (1.3) do not hold for some submodule M which is not homogeneous. For homogeneous submodule M of H_d^2, whether or not (1.1) holds is not known. However, on Bergman spaces and Hardy spaces, we will show that (1.1) holds for operators of the form $A \overset{\mathrm{SOT}}{=} \sum_{k\ge 1} T_{\varphi_k} T_{\varphi_k}^*$. Also, we give a sufficient condition by trace estimation. The following is our main result.

THEOREM 1.1. *Let $A \overset{\mathrm{SOT}}{=} \sum_{k=1}^\infty T_{\varphi_k} T_{\varphi_k}^*$ be a bounded operator on the Bergman space or Hardy space over B_d, where $\{\varphi_k \in H^\infty(B_d) : k = 1, 2, \ldots\}$ are multipliers. Then the commutator $[A, T_{z_i}]$ belongs to Schatten class $\mathcal{L}_{2p}$ for $p > d$, and there is a constant C depending only on p and d such that*

$$\big\|[A, T_{z_i}]\big\|_{2p} \le C\|A\|.$$

We will prove Theorem 1.1 first in the case of $L^2_a(B_d)$ and then generalize it to Hardy spaces.

The paper is organized as follows.

In Section 2, we use a technique based on Hankel operator to prove inequality (1.1) for multiplication operators of analytic symbols on Bergman space or Hardy space, and give some other applications of Hankel operator.

In Section 3, we use trace estimation to prove Theorem 1.1 on Bergman space over the unit ball B_d, and generalizes it to Hardy spaces and weighted Bergman spaces.

2. Trace estimation of commutators of multiplication operators

First, we give some notations on Bergman space and Hardy space. The readers are suggested to see [Rud] for details.

For a multi-index $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_d)\in\mathbb{Z}^d_+$, denote $\alpha!=\alpha_1!\cdots\alpha_d!$, $|\alpha|=\alpha_1+\cdots+\alpha_d$, and $z^\alpha=z_1^{\alpha_1}\cdots z_d^{\alpha_d}$ for $z\in\mathbb{C}^d$.

Let $d\nu$ be the normalized Lebesgue measure on B_d. Bergman space $L^2_a(B_d)$ is the Hilbert space of all square-integrable analytic functions over B_d, with inner product $\langle f,g\rangle=\int_{B_d} f(z)\bar{g}(z)\,d\nu(z)$. $L^2_a(B_d)$ has a canonical orthonormal basis

$$\left\{\sqrt{\frac{(d+|\alpha|)!}{d!\alpha!}}z^\alpha : \alpha\in\mathbb{Z}^d_+\right\}.$$

For $\lambda\in B_d$, the reproducing kernel of $L^2_a(B_d)$ at λ is

$$K_\lambda(z)=\big(1-\langle z,\lambda\rangle\big)^{-(d+1)},$$

which has the property that $f(\lambda)=\langle f,K_\lambda\rangle$ for every $f\in L^2_a(B_d)$.

If f is any function over B_d, and $0<r<1$, then f_r denotes the dilated function defined for $\|z\|<1/r$ by $f_r(z)=f(rz)$.

Let $d\sigma$ denote the rotation-invariant positive Borel measure on ∂B_d for which $\sigma(S)=1$. Then Hardy space $H^2(B_d)$ consists of all analytic functions f on B_d satisfying

$$\sup_{r\in(0,1)}\int_{\partial B_d}\big|f_r(z)\big|^2\,d\sigma(z)<\infty.$$

For $f\in H^2(B_d)$, define its norm

$$\|f\|=\left(\sup_{r\in(0,1)}\int_{\partial B_d}\big|f_r(z)\big|^2\,d\sigma(z)\right)^{\frac{1}{2}}=\lim_{r\to1-}\left(\int_{\partial B_d}\big|f_r(z)\big|^2\,d\sigma(z)\right)^{\frac{1}{2}}$$

and then $H^2(B_d)$ comes to be a Hilbert space, with inner product

$$\langle f,g\rangle=\lim_{r\to1-}\int_{\partial B_d} f_r(z)\bar{g}_r(z)\,d\sigma(z).$$

$H^2(B_d)$ has a canonical orthonormal basis

$$\left\{\sqrt{\frac{(d-1+|\alpha|)!}{(d-1)!\alpha!}}z^\alpha : \alpha \in \mathbb{N}^d\right\},$$

and reproducing kernel $K_\lambda(z) = (1-\langle z,\lambda\rangle)^{-d}$ at $\lambda \in B_d$.

Let P denote the orthogonal projection from $L^2(B_d)$ to $L^2_a(B_d)$. For $\varphi \in L^\infty(B_d)$, let $M_\varphi \in B(L^2(B_d))$ be the multiplication operator of symbol φ, $T_\varphi = PM_\varphi P$ the Toeplitz operator, and $H_\varphi = P^\perp M_\varphi P$ the Hankel operator. Multiplication operator, Toeplitz operator and Hankel operator on $H^2(B_d)$ are similarly defined.

Note that for $f \in L^\infty(B_d)$ and $g \in H^\infty(B_d)$, we have

$$[T_f, T_g] = P_H M_f M_g P_H - P_H M_g P_H M_f P_H = P_H M_g P_H^\perp M_f P_H = H^*_{\bar{g}} H_f.$$

That is

$$[T_f, T_g] = H^*_{\bar{g}} H_f. \tag{2.1}$$

Using this equality, one can prove the following result.

Lemma 2.1. *If $f \in L^\infty(B_d)$ and $p > d$, then on $L^2_a(B_d)$, we have $[T_f, T_{z_i}] \in \mathcal{L}_{2p}$ and*

$$\big\|[T_f, T_{z_i}]\big\|_{2p} \le \|M_f\| \cdot \big\|\big[T^*_{z_j}, T_{z_j}\big]\big\|_p^{\frac{1}{2}}.$$

Proof. By equality (2.1), we have that for $g \in H^\infty(B_d)$,

$$[T_f, T_g][T_f, T_g]^* = H^*_{\bar{g}} H_f H^*_f H_{\bar{g}}.$$

Since

$$H_f H^*_f \le \|H_f\|^2 \le \|M_f\|^2,$$

we have

$$[T_f, T_g][T_f, T_g]^* \le \|M_f\|^2 \cdot H^*_{\bar{g}} H_{\bar{g}} = \|M_f\|^2 \cdot \big[T^*_g, T_g\big].$$

Especially if we take $g = z_i$, then $[T^*_{z_i}, T_{z_i}] \in \mathcal{L}_p$ as is well known, and

$$\big\|[T_f, T_{z_i}][T_f, T_{z_i}]^*\big\|_p \le \|M_f\|^2 \cdot \big\|\big[T^*_{z_i}, T_{z_i}\big]\big\|_p$$

which implies that

$$\big\|[T_f, T_{z_i}]\big\|_{2p} \le \|M_f\| \cdot \big\|\big[T^*_{z_i}, T_{z_i}\big]\big\|_p^{\frac{1}{2}}.$$

This completes the proof. □

Similar result holds on $H^2(B_d)$. The same proof shows the following corollary, which is well known.

Corollary 2.2. *Let H denote the Bergman space $L^2_a(B_d)$ or Hardy space $H^2(B_d)$. If $\varphi \in H^\infty(B_d)$, then for $p > d$,*

$$\big\|\big[T^*_\varphi, T_{z_i}\big]\big\|_{2p} \le \|T_\varphi\| \cdot \big\|\big[T^*_{z_i}, T_{z_i}\big]\big\|_p^{\frac{1}{2}}.$$

In Section 3, we will use a generalization of this idea to estimate p-norm of some operators.

Drury–Arveson space H_d^2 is defined to be the completion of the space of polynomials with the inner product defined such that $\|z^\alpha\|^2=\alpha!/|\alpha|!$ for $\alpha\in\mathbb{Z}_+^d$, and $\langle z^\alpha,z^\beta\rangle=0$ for $\alpha\neq\beta$. From a general reproducing kernel theory [Aro], Drury–Arveson space is the analytic function space over B_d determined by reproducing kernel $K_\lambda(z)=(1-\langle z,\lambda\rangle)^{-1}$.

As an application of Corollary 2.2, we have the following corollary.

Corollary 2.3. *Let H denote Bergman space $L_a^2(B_d)$, Hardy space $H^2(B_d)$ or Drury–Arveson space H_d^2. Let $p>d$, $\varphi\in H^\infty(B_d)$ and M be a p-essentially normal submodule of H. Then there is a constant C depending only on p and d such that, $\|[\dot{M}_\varphi,\dot{M}_{z_i}^*]\|_{2p}\le C\|M_\varphi\|$, where $\dot{M}_f$ denotes the multiplication operator on M of symbol f.*

Proof. Let $M^\perp$ denote the orthogonal complement of M in H, and define

$$A: M\to M^\perp,\qquad f\mapsto P_M^\perp M_{z_i}^* f$$

and

$$B: M\to M^\perp,\qquad f\mapsto P_M^\perp M_\varphi^* f.$$

Then

$$B^*=P_M M_\varphi|_{M^\perp}$$

and

$$\begin{aligned}B^*A&=P_M M_\varphi P_M^\perp M_{z_i}^* P_M\\&=P_M M_\varphi M_{z_i}^* P_M-P_M M_{z_i}^* M_\varphi P_M+P_M M_{z_i}^* M_\varphi P_M-P_M M_\varphi P_M M_{z_i}^* P_M\\&=P_M\big[M_\varphi,M_{z_i}^*\big]P_M-\big[\dot{M}_\varphi,\dot{M}_{z_i}^*\big].\end{aligned}$$

Similarly, we have

$$A^*A=P_M\big[M_{z_i},M_{z_i}^*\big]P_M-\big[\dot{M}_{z_i},\dot{M}_{z_i}^*\big].$$

Therefore, there is a constant C depending only on p and d such that

$$\begin{aligned}\big\|\big[\dot{M}_\varphi,\dot{M}_{z_i}^*\big]\big\|_{2p}&\le\big\|B^*A\big\|_{2p}+\big\|\big[M_{z_i}^*,M_\varphi\big]\big\|_{2p}\\&\le\|B\|\cdot\|A\|_{2p}+\|M_\varphi\|\cdot C\quad\text{(Corollary 2.2, [FX])}\\&\le\|M_\varphi\|\big(\big\|P_M\big[M_{z_i},M_{z_i}^*\big]P_M-\big[\dot{M}_{z_i},\dot{M}_{z_i}^*\big]\big\|_p^{\frac12}+C\big)\\&\le\|M_\varphi\|\big[\big(\big\|\big[M_{z_i},M_{z_i}^*\big]\big\|_p+\big\|\big[\dot{M}_{z_i},\dot{M}_{z_i}^*\big]\big\|_p\big)^{\frac12}+C\big],\end{aligned}$$

which completes the proof. ∎

3. Trace estimation on Bergman space $L_a^2(B_d)$

In this section, we will prove the Bergman space version of Theorem 1.1. The proof of Hardy space version of Theorem 1.1 is nearly the same, so we do not give a complete proof, but only exhibit some differences in remark.

For $A \in B(L_a^2(B_d))$, define

$$\tilde{A}(\lambda) = \langle AK_\lambda, K_\lambda \rangle, \quad \lambda \in B_d.$$

By Proposition 1 in [Eng], $A = 0$ whenever $\tilde{A} = 0$, and therefore the mapping $A \to \tilde{A}$ is injective.

First, we introduce a known result that will be used later.

Corollary 3.1 ([AFP]). *For $A \in B(L_a^2(B_d))$ and $A \geq 0$, then $A \in \mathcal{L}_1$ if and only if*

$$\int_{B_d} \tilde{A}(\lambda)\, dv(\lambda) < \infty. \tag{3.1}$$

In this case, $\operatorname{Tr} A = \int_{B_d} \tilde{A}(\lambda)\, dv(\lambda)$.

In [AFP] they only proved inequality (3.1) for $d = 1$, and their proof can be applied for $d > 1$.

Now define

$$G(A) = A - \sum_{i=1}^{d} T_{z_i} A T_{z_i}^*,$$

and defect operator (see [Guo])

$$\begin{aligned} D(A) &= G^{(d+1)}(A) \ (d+1 \text{ composition of } G) \\ &= A + \sum_{k=1}^{d+1} (-1)^k \frac{(d+1)!}{k!(d+1-k)!} \\ &\quad \times \sum_{i_1,\dots,i_k \in \{1,2,\dots,d\}} T_{z_{i_1} z_{i_2} \cdots z_{i_k}} A T^*_{z_{i_1} z_{i_2} \cdots z_{i_k}}. \end{aligned} \tag{3.2}$$

An easy calculation shows that

$$\big\langle G(A)K_\lambda, K_\mu \big\rangle = \big(1 - \langle \mu, \lambda \rangle\big) \langle AK_\lambda, K_\mu \rangle, \quad \forall \lambda, \mu \in B_d. \tag{3.3}$$

The key property of $D(A)$ is

$$\big\langle D(A)K_\lambda, K_\lambda \big\rangle = \big(1 - \|\lambda\|^2\big)^{d+1} \langle AK_\lambda, K_\lambda \rangle = \langle Ak_\lambda, k_\lambda \rangle, \tag{3.4}$$

where

$$k_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$$

is the normalized reproducing kernel. Hence

$$\big\langle D(A)K_\lambda, K_\lambda \big\rangle \in L^\infty(B_d).$$

Therefore if $D(A) \geq 0$, we can apply Proposition 3.1 to $D(A)$ and get $D(A) \in \mathcal{L}_1$. Moreover, we have the following lemma.

Lemma 3.2. *Let $A\in B(L_a^2(B_d))$. Then $D(A)\geq 0$ if and only if there exists a sequence $\{\varphi_k\in H^\infty(B_d): k=1,2,\ldots\}$, such that*

$$A \overset{\text{SOT}}{=} \sum_{k\in\mathbb{N}} T_{\varphi_k}T_{\varphi_k}^*.$$

In this case $D(A)\in\mathcal{L}_1$.

Proof. If $D(A)\geq 0$, then by the argument preceding the lemma, $D(A)$ is in trace class and therefore there exist orthogonal vectors $\{\varphi_k : k=1,2,\ldots\}$ in $L_a^2(B_d)$ such that

$$D(A)=\sum_k \varphi_k\otimes\varphi_k.$$

Then we have

$$\sum_{k\in\mathbb{N}}\left|\varphi_k(\lambda)\right|^2=\left\langle D(A)K_\lambda,K_\lambda\right\rangle=\langle Ak_\lambda,k_\lambda\rangle\leq\|A\|\quad\text{(by equality (3.4))}$$

which implies

$$\varphi_k\in H^\infty(B_d),\quad k=1,2,\ldots.$$

Thus,

$$\langle Ak_\lambda,k_\lambda\rangle=\sum_{k\in\mathbb{N}}\left|\varphi_k(\lambda)\right|^2=\sum_{k\in\mathbb{N}}\left\langle T_{\varphi_k}T_{\varphi_k}^*k_\lambda,k_\lambda\right\rangle,\quad\forall\lambda\in B_d. \tag{3.5}$$

By subnormality

$$T_{\varphi_k}T_{\varphi_k}^*\leq T_{\varphi_k}^*T_{\varphi_k},$$

and since

$$\sum_{k\in\mathbb{N}}T_{\varphi_k}^*T_{\varphi_k}\overset{\text{SOT}}{=}T_{\sum_k|\varphi_k|^2},$$

apply Theorem 43.1 in [Con1] and we find that $\sum_{k\in\mathbb{N}}T_{\varphi_k}T_{\varphi_k}^*$ converges to a bounded operator in strong operator topology. Then by equality (3.5) and the injectivity of the mapping $A\to\tilde{A}$, we have

$$A \overset{\text{SOT}}{=} \sum_{k\in\mathbb{N}} T_{\varphi_k}T_{\varphi_k}^*$$

as desired.

Conversely, if $A \overset{\text{SOT}}{=} \sum_{k\in\mathbb{N}} T_{\varphi_k}T_{\varphi_k}^*$, then for any $\lambda\in B_d$ we have

$$\begin{aligned}\left\langle D(A)K_\lambda,K_\lambda\right\rangle&=\langle Ak_\lambda,k_\lambda\rangle\quad\text{(by equality (3.4))}\\&=\left\langle\sum_{k\in\mathbb{N}}T_{\varphi_k}T_{\varphi_k}^*k_\lambda,k_\lambda\right\rangle\\&=\sum_{k\in\mathbb{N}}\left|\varphi_k(\lambda)\right|^2\\&=\sum_{k\in\mathbb{N}}\langle\varphi_k\otimes\varphi_kK_\lambda,K_\lambda\rangle.\end{aligned} \tag{3.6}$$

Since

$$\begin{aligned}\sum_{k\in\mathbb{N}}\|\varphi_k\otimes\varphi_k\| = \sum_{k\in\mathbb{N}}\|\varphi_k\|^2 &= \sum_{k\in\mathbb{N}}\int_{B_d}\left|\varphi_k(z)\right|^2 d\nu(z)\\ &= \sum_{k\in\mathbb{N}}\int_{B_d}\left\langle T_{\varphi_k}T_{\varphi_k}^* k_\lambda, k_\lambda\right\rangle d\nu(z)\\ &= \int_{B_d}\langle Ak_\lambda, k_\lambda\rangle\, d\nu(z)\\ &\le \|A\|,\end{aligned}$$

$\sum_{k\in\mathbb{N}}\varphi_k\otimes\varphi_k$ converges to a bounded operator. Then by equality (3.6)

$$D(A) = \sum_{k\in\mathbb{N}}\varphi_k\otimes\varphi_k \ge 0.$$

□

Now consider the defect operator $D(A)$ on the unit disc.

Recall that in this case

$$G(A) = A - T_z A T_z^*$$

and

$$D(A) = A - 2T_z A T_z^* + T_z^2 A T_z^{*2}.$$

Let $D(A) \ge 0$. Then successively we have a sequence of compact operators:

$$\begin{aligned} D(A) &= G(A) - T_z G(A) T_z^*,\\ T_z D(A) T_z^* &= T_z G(A) T_z^* - T_z^2 G(A) T_z^{*2},\\ &\ \vdots\\ T_z^l D(A) T_z^{*l} &= T_z^l G(A) T_z^{*l} - T_z^{l+1} G(A) T_z^{*l+1},\\ &\ \vdots\end{aligned}$$

Since

$$T_z^l \xrightarrow{\text{SOT}} 0 \quad (l\to\infty),$$

sum up the above equalities and we have

$$G(A) \overset{\text{SOT}}{=} \sum_{l=0}^{\infty} T_z^l D(A) T_z^{*l}.$$

We claim that $G(A)$ is compact. Then $[A, T_z]$ is compact since T_z is an essentially unitary operator and $A - T_z A T_z^* = G(A)$ which is compact.

LEMMA 3.3. *If $A \in B(L_a^2(\mathbb{D}))$ satisfies $D(A) \ge 0$, then $G(A)$ is compact, and $\|G(A)\| \le \|A\|$.*

Proof. As in the proof of Lemma 3.2, we have a sequence of orthogonal vectors

$$\{\varphi_k \in H^\infty(\mathbb{D}) : k = 1, 2, \ldots\}$$

such that

$$D(A) = \sum_{k\in\mathbb{N}} \varphi_k \otimes \varphi_k \in \mathcal{L}_1.$$

We will prove that the sum

$$G(A) = \sum_{l=0}^{\infty} T_z^l D(A) T_z^{*l}$$

converges in norm. For any finite subset L of $\mathbb{Z}_+$ and $f \in L_a^2(\mathbb{D})$, we have

$$\begin{aligned}
\left\langle \sum_{l\in L} T_z^l D(A) T_z^{*l} f, f \right\rangle &= \sum_{l\in L} \sum_{k\in\mathbb{N}} \langle T_z^l (\varphi_k \otimes \varphi_k) T_z^{*l} f, f \rangle \\
&= \sum_{l\in L} \sum_{k\in\mathbb{N}} \langle (z^l \otimes z^l) T_{\varphi_k}^* f, T_{\varphi_k}^* f \rangle \\
&= \sum_{k\in\mathbb{N}} \left\langle \sum_{l\in L} (z^l \otimes z^l) T_{\varphi_k}^* f, T_{\varphi_k}^* f \right\rangle \\
&\le \left\| \sum_{l\in L} z^l \otimes z^l \right\| \left\langle \sum_{k\in\mathbb{N}} T_{\varphi_k} T_{\varphi_k}^* f, f \right\rangle \\
&\le \left\| \sum_{l\in L} z^l \otimes z^l \right\| \cdot \left\| \sum_{k\in\mathbb{N}} T_{\varphi_k} T_{\varphi_k}^* \right\| \cdot \|f\|^2 \\
&= \|A\| \cdot \left\| \sum_{l\in L} z^l \otimes z^l \right\| \cdot \|f\|^2.
\end{aligned}$$

Therefore

$$\left\| \sum_{l\in L} T_z^l D(A) T_z^{*l} \right\| \le \|A\| \cdot \left\| \sum_{l\in L} z^l \otimes z^l \right\|.$$

Since

$$\left\| \sum_{l\in L} z^l \otimes z^l \right\| \le \frac{1}{\min L + 1},$$

we have that

$$\sum_{l=0}^{\infty} T_z^l D(A) T_z^{*l}$$

converges in norm to a compact operator, with norm at most $\|A\|$. This completes the proof. $\square$

In fact, the same result holds in higher dimensional cases, but the proof is slightly different.

LEMMA 3.4. *If $A \in B(L_a^2(B_d))$ satisfies $D(A) \geq 0$, then $G(A)$ is compact and $\|G(A)\| \leq \|A\|$. Moreover,*

$$G(A) = \sum_{\alpha \in \mathbb{Z}_+^d} \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} T_{z^\alpha} D(A) T_{z^\alpha}^*.$$

Proof. Define operator

$$F_n(A) = \sum_{|\alpha|=n} \frac{(d+n-1)!}{(d-1)!\alpha!} T_{z^\alpha} D(A) T_{z^\alpha}^*, \quad n \in \mathbb{Z}_+, \alpha \in \mathbb{Z}_+^d.$$

Since $D(A)$ is positive, so is $F_n(A)$. Then similar to the proof of Lemma 3.3, we can prove that $\tilde{G}(A) = \sum_{n=0}^\infty F_n(A)$ is a compact operator with norm at most $\|A\|$.

For the details, as in the proof of Lemma 3.2, we have a sequence or orthogonal vectors

$$\{\varphi_k \in H^\infty(B_d) : k = 1, 2, \ldots\}$$

such that

$$D(A) = \sum_{k \in \mathbb{N}} \varphi_k \otimes \varphi_k.$$

For $\alpha \in \mathbb{Z}_+^d$ we have

$$\left\| \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} z^\alpha \otimes z^\alpha \right\| = \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} \cdot \frac{d!\alpha!}{(d+|\alpha|)!} = \frac{d}{d+|\alpha|},$$

then for any finite subset L of $\mathbb{Z}_+$,

$$\left\| \sum_{|\alpha| \in L} \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} z^\alpha \otimes z^\alpha \right\| = \frac{d}{d+\min L}.$$

Let $f \in L_a^2(B_d)$. Then

$$\begin{aligned}
\left\langle \sum_{l \in L} F_l(A) f, f \right\rangle &= \left\langle \sum_{l \in L} \sum_{|\alpha|=l} \frac{(d+l-1)!}{(d-1)!\alpha!} T_{z^\alpha} D(A) T_{z^\alpha}^* f, f \right\rangle \\
&= \sum_{|\alpha| \in L} \sum_{k \in \mathbb{N}} \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} \langle T_{z^\alpha} (\varphi_k \otimes \varphi_k) T_{z^\alpha}^* f, f \rangle \\
&= \sum_{|\alpha| \in L} \sum_{k \in \mathbb{N}} \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} \langle (z^\alpha \otimes z^\alpha) T_{\varphi_k}^* f, T_{\varphi_k}^* f \rangle \\
&= \sum_{k \in \mathbb{N}} \left\langle \sum_{|\alpha| \in L} \left(\frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} z^\alpha \otimes z^\alpha \right) T_{\varphi_k}^* f, T_{\varphi_k}^* f \right\rangle \\
&\leq \left\| \sum_{|\alpha| \in L} \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} z^\alpha \otimes z^\alpha \right\| \left\langle \sum_{k \in \mathbb{N}} T_{\varphi_k} T_{\varphi_k}^* f, f \right\rangle
\end{aligned}$$

$$\leq \frac{d}{d+\min L} \cdot \left\| \sum_{k\in\mathbb{N}} T_{\varphi_k} T_{\varphi_k}^* \right\| \cdot \|f\|^2$$
$$= \|A\| \cdot \frac{d}{d+\min L} \cdot \|f\|^2.$$

Therefore,

$$\left\| \sum_{l\in L} F_l(A) \right\| \leq \|A\| \cdot \frac{d}{d+\min L}.$$

Thus $\sum_{l\in\mathbb{Z}_+} F_l(A)$ converges in norm to a compact operator, denoted by $\tilde{G}(A)$, with norm at most $\|A\|$.

Actually, we have

$$\begin{aligned}
\langle \tilde{G}(A)K_\lambda, K_\lambda \rangle &= \left\langle \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{(d+n-1)!}{(d-1)!\alpha!} T_{z^\alpha} D(A) T_{z^\alpha}^* K_\lambda, K_\lambda \right\rangle \\
&= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{(d+n-1)!}{(d-1)!\alpha!} \langle T_{z^\alpha} D(A) T_{z^\alpha}^* K_\lambda, K_\lambda \rangle \\
&= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{(d+n-1)!}{(d-1)!\alpha!} |\lambda^\alpha|^2 \langle D(A) K_\lambda, K_\lambda \rangle \\
&= \left(1-\|\lambda\|^2\right)^{-d} \langle D(A)K_\lambda, K_\lambda \rangle \\
&= \left(1-\|\lambda\|^2\right)^{-d} \cdot \left(1-\|\lambda\|^2\right)^{d+1} \langle AK_\lambda, K_\lambda \rangle \\
&= \langle G(A)K_\lambda, K_\lambda \rangle.
\end{aligned}$$

This implies

$$G(A) = \tilde{G}(A) = \sum_{\alpha\in\mathbb{Z}_+^d}^{\infty} \frac{(d+n-1)!}{(d-1)!\alpha!} T_{z^\alpha} D(A) T_{z^\alpha}^*$$

and the proof is complete. □

The following lemma provides a key estimate.

LEMMA 3.5 ([AFP], [Zhu]). *Let $p>1$, A be a normal operator on Hilbert space H, and $h\in H$ with norm $\|h\|\leq 1$. Then we have*

$$\left|\langle Ah, h\rangle\right|^p \leq \langle |A|^p h, h \rangle. \tag{3.7}$$

REMARK 3.6. Inequality (3.7) for $\|h\|=1$ appeared in [AFP] and [Zhu]. Then it is easy to see that (3.7) holds for all $\|h\|\leq 1$.

By Lemma 3.4, we have

$$\begin{aligned} G(A) &= \sum_{\alpha\in\mathbb{Z}_+^d} \frac{(d+n-1)!}{(d-1)!\alpha!} T_{z^\alpha}\Big(\sum_{k\in\mathbb{N}} \varphi_k\otimes\varphi_k\Big)T^*_{z^\alpha} \\ &\overset{\mathrm{SOT}}{=} \sum_{k\in\mathbb{N}} T_{\varphi_k}\Big(\sum_{\alpha\in\mathbb{Z}_+^d} \frac{(d+n-1)!}{(d-1)!\alpha!} z^\alpha\otimes z^\alpha\Big)T^*_{\varphi_k}. \end{aligned}$$

Write

$$R \overset{\mathrm{SOT}}{=} \sum_{\alpha\in\mathbb{Z}_+^d}^{\infty} \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} z^\alpha\otimes z^\alpha$$

on $L^2_a(B_d)$, and then $G(A) \overset{\mathrm{SOT}}{=} \sum_{k\in\mathbb{N}} T_{\varphi_k} R T^*_{\varphi_k}$. We have the following lemma.

Lemma 3.7. *The operator R defined above belongs to $\mathcal{L}_p$ for $p > d$.*

Proof. For $\alpha\in\mathbb{Z}_+^d$, we have

$$\left\|\frac{(d+|\alpha|-1)!}{(d-1)!\alpha!} z^\alpha\otimes z^\alpha\right\| = \frac{(d+|\alpha|-1)!}{(d-1)!\alpha!}\,\frac{d!\alpha!}{(|\alpha|+d)!} = \frac{d}{d+|\alpha|}.$$

Then

$$\|R\|_p^p = \sum_{\alpha\in\mathbb{Z}_+^d}\left(\frac{d}{d+|\alpha|}\right)^p = \sum_{n\in\mathbb{Z}_+}\binom{n+d-1}{d-1}\left(\frac{d}{d+n}\right)^p < \infty$$

since for each $n\in\mathbb{Z}_+$, $\operatorname{card}\{\alpha\in\mathbb{Z}_+^d : |\alpha| = n\} = \binom{n+d-1}{d-1}$. □

Corollary 3.8. *If $A\in B(L^2_a(B_d))$ satisfies $D(A)\ge 0$, then $G(A)$ belongs to Schatten class $\mathcal{L}_p$ for $p > d$ and*

$$\big\|G(A)\big\|_p \le \|A\|\cdot\|R\|_p.$$

Proof. Recall that for $\phi\in L^\infty(B_d)$, $M_\phi\in B(L^2(B_d))$ is the multiplication operator of symbol ϕ. Let $\{\varphi_k\in H^\infty(B_d) : k = 1,2,\dots\}$ be the orthogonal vectors as in the proof of Lemma 3.2. Then for $f\in L^2(B_d)$, we have

$$\sum_{k\in\mathbb{N}}\big\langle M^*_{\varphi_k}f, M^*_{\varphi_k}f\big\rangle = \langle M_{\sum|\varphi_k|^2}f, f\rangle \le \|A\|\cdot\|f\|^2.$$

Therefore, the matrix

$$\begin{pmatrix} M^*_{\varphi_1} & 0 & \cdots \\ M^*_{\varphi_2} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

defines a bounded linear operator on $\bigoplus_{n=1}^{\infty} L^2(B_d)$, with norm $a = \|A\|^{\frac{1}{2}}$. Denote

$$M_\Phi = \begin{pmatrix} M_{\varphi_1} & M_{\varphi_2} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Then M_Φ is a bounded operator with norm a. It is easy to see that

$$P^\infty = \operatorname{diag}\{P, P, \ldots\}$$

is the orthogonal projection from $\bigoplus_{n=1}^{\infty} L^2(B_d)$ onto $\bigoplus_{n=1}^{\infty} L_a^2(B_d)$. As in the 1-dimensional case, define $T_\Phi = P^\infty M_\Phi P^\infty$ and $H_{\bar{\Phi}} = (P^\infty)^\perp M_\Phi^* P^\infty$.

By Lemma 3.4, we have

$$G(A) = \big(T_\Phi R^\infty T_\Phi^*\big)_{1,1}$$

and that the other entries of the matrix of $T_\Phi R^\infty T_\Phi^*$ are 0. By Lemma 3.4, $G(A)$ is compact and so is $T_\Phi R^\infty T_\Phi^*$.

Let $\{e_m : m = 1, 2, \ldots\}$ be an orthonormal basis for $\bigoplus_{n=1}^{\infty} L_a^2(B_d)$ that diagonalizes $T_\Phi R^{(\infty)} T_\Phi^*$. Then

$$\begin{aligned}
\sum_m \big\langle T_\Phi R^\infty T_\Phi^* e_m, e_m\big\rangle^p &= a^{2p} \sum_m \left\langle R^\infty \frac{T_\Phi^* e_m}{a}, \frac{T_\Phi^* e_m}{a}\right\rangle^p \\
&\le a^{2p} \sum_m \left\langle \big(R^p\big)^\infty \frac{T_\Phi^* e_m}{a}, \frac{T_\Phi^* e_m}{a}\right\rangle \quad \text{(by Lemma 3.5)} \\
&= a^{2p-2} \operatorname{Tr}\big(T_\Phi \big(R^p\big)^\infty T_\Phi^*\big) \\
&= a^{2p-2} \operatorname{Tr}\big(\big(R^{\frac{p}{2}}\big)^\infty T_\Phi^* T_\Phi \big(R^{\frac{p}{2}}\big)^\infty\big) \\
&\quad \big(\text{since } \operatorname{Tr}\big(BB^*\big) = \operatorname{Tr}\big(B^*B\big) \text{ for any operator } B\big) \\
&= a^{2p-2} \operatorname{Tr}\left(R^{\frac{p}{2}} \sum_k T_{\varphi_k}^* T_{\varphi_k} R^{\frac{p}{2}}\right) \\
&\le a^{2p-2} \operatorname{Tr}\big(R^{\frac{p}{2}} \cdot \|T_{\sum |\varphi_k|^2}\| \cdot R^{\frac{p}{2}}\big) \\
&= a^{2p} \operatorname{Tr} R^p.
\end{aligned}$$

Thus, $T_\Phi R^\infty T_\Phi^* \in \mathcal{L}_p$ and so is $G(A)$. Moreover,

$$\big\|G(A)\big\|_p = \big\|T_\Phi R^\infty T_\Phi^*\big\|_p \le \|A\| \cdot \|R\|_p$$

which completes the proof. $\square$

Next, we prove the Bergman space version of Theorem 1.1.

THEOREM 3.9. *If $A \in B(L_a^2(B_d))$ satisfies $D(A) \ge 0$, and $p > d$, then the commutator $[A, T_{z_i}] \in \mathcal{L}_{2p}$ and*

$$\big\|[A, T_{z_i}]\big\|_{2p} \le \|A\| \cdot \|R\|_p^{\frac{1}{2}}.$$

Proof. Define the operator $\frac{1}{N+d} \in B(L_a^2(B_d))$ such that $\frac{1}{N+d}h = \frac{1}{n+d}h$ if h is a homogeneous polynomial of degree n. We know that on $L_a^2(B_d)$

$$\sum_{i=1}^{d} T_{z_i}^* T_{z_i} + \frac{1}{N+d+1} = I.$$

Let $\{\varphi_k \in H^\infty(B_d) : k = 1, 2, \ldots\}$ be the orthogonal vectors as in the proof of Lemma 3.2, such that $A = \sum_k T_{\varphi_k} T_{\varphi_k}^*$. By Proposition 3.8,

$$\begin{aligned}
G(A) &= A - \sum_i T_{z_i} A T_{z_i}^* \\
&= \sum_k T_{\varphi_k} T_{\varphi_k}^* - \sum_i \sum_k T_{\varphi_k} T_{z_i} T_{z_i}^* T_{\varphi_k}^* \\
&= \sum_k T_{\varphi_k} \left(\sum_{i=1}^{d} T_{z_i}^* T_{z_i} + \frac{1}{N+d+1} \right) T_{\varphi_k}^* - \sum_i \sum_k T_{\varphi_k} T_{z_i} T_{z_i}^* T_{\varphi_k}^* \\
&= \sum_{i=1}^{d} \sum_k T_{\varphi_k} \left(T_{z_i}^* T_{z_i} - T_{z_i} T_{z_i}^* \right) T_{\varphi_k}^* + \sum_k T_{\varphi_k} \frac{1}{N+d+1} T_{\varphi_k}^*.
\end{aligned}$$

Since $T_{z_i}^* T_{z_i} - T_{z_i} T_{z_i}^* \geq 0$ for $i = 1, 2, \ldots, d$, we have

$$0 \leq S_i = \sum_k T_{\varphi_k} \left(T_{z_i}^* T_{z_i} - T_{z_i} T_{z_i}^* \right) T_{\varphi_k}^* \leq G(A).$$

By Proposition 3.8,

$$\|S_i\|_p \leq \left\| G(A) \right\|_p \leq \|A\| \cdot \|R\|_p.$$

Using the same notations as in the proof of Proposition 3.8,

$$S_i = T_\Phi \left[T_{z_i}^*, T_{z_i} \right]^\infty T_\Phi^* = T_\Phi \left(H_{\bar{z}_i}^* H_{\bar{z}_i} \right)^\infty T_\Phi^* \in \mathcal{L}_p.$$

Then

$$T_\Phi \left(H_{\bar{z}_i}^* \right)^\infty \in \mathcal{L}_{2p}$$

and

$$\left\| T_\Phi \left(H_{\bar{z}_i}^* \right)^\infty \right\|_{2p} \leq \|A\|^{\frac{1}{2}} \|R\|_p^{\frac{1}{2}}$$

which implies

$$\left\| T_\Phi \left[T_\Phi^*, T_{z_i}^\infty \right] \right\|_{2p} = \left\| T_\Phi \left(H_{\bar{z}_i}^* \right)^\infty H_{\bar{\Phi}} \right\|_{2p} \leq \|A\| \cdot \|R\|_p^{\frac{1}{2}}. \tag{3.8}$$

Since

$$\begin{aligned}
[A, T_{z_i}] &= \left[\sum_{k \in \mathbb{N}} T_{\varphi_k} T_{\varphi_k}^*, T_{z_i} \right] \\
&= \sum_{k \in \mathbb{N}} T_{\varphi_k} \left[T_{\varphi_k}^*, T_{z_i} \right] \\
&= \left(T_\Phi \left[T_\Phi^*, T_{z_i}^\infty \right] \right)_{(1,1)}
\end{aligned}$$

and the other entries of $(T_\Phi[T_\Phi^*, T_{z_i}^\infty])$ are 0, by (3.8),

$$[A, T_{z_i}] \in \mathcal{L}_{2p}$$

and

$$\big\|[A, T_{z_i}]\big\|_{2p} \le \|A\| \cdot \|R\|_p^{\frac{1}{2}}$$

as desired. □

REMARK 3.10. By the same method, we can prove the version of Theorem 1.1 on Hardy space $H^2(B_d)$ for $d \ge 2$. Difference lies in that, on $H^2(B_d)$, defect operator $D(A)$ is defined to be $G^{(d)}(A)$.

By the reproducing kernel theory [Aro], for $v > 0$, the function

$$K_\lambda^{(v)} = \frac{1}{(1 - \langle z, \lambda \rangle)^v}$$

defined on $B_d \times B_d$ is a reproducing kernel, which induces a reproducing kernel function space on B_d, denoted by $\mathscr{H}_v^d$. Then $\mathscr{H}_1^d$ is the Drury–Arveson space H_d^2, $\mathscr{H}_d^d$ is the Hardy space $H^2(B_d)$, and $\mathscr{H}_{d+1}^d$ is the Bergman space $L_a^2(B_d)$. When $v > d$, $\mathscr{H}_v^d$ is the weighted Bergman space $L_a^2[(1 - |z|^2)^{v-d-1} dV]$ on B_d, where dV is the volume measure. The space $\mathscr{H}_v^d$ also admits a natural $C[z_1, \dots, z_d]$-module structure. $\mathscr{H}_v^d$ has a canonical orthonormal basis

$$\left\{ \left[\frac{v(v+1)\cdots(v+|\alpha|-1)}{\alpha!} \right]^{\frac{1}{2}} z^\alpha : \alpha \in \mathbb{Z}_+^d \right\}.$$

Our proof of Theorem 1.1 can be generalized to weighted Bergman spaces $\mathscr{H}_v^d$ $(v > d)$ as well.

On Drury–Arveson space, our method do not apply for a few reasons. The major difficulty is that, there do not exist a measure μ on $\mathbb{C}^d$ such that

$$\|f\|^2 = \int_{\mathbb{C}^d} \big|f(z)\big|^2 \, d\mu(z) \quad \text{(see [Arv1])}.$$

Then we did not find a trace estimation similar to Proposition 3.1, and we cannot speak of the compactness of $\sum_k \varphi_k \otimes \varphi_k$ by this method.

However, we conjecture that (1.1) holds on spaces $\mathscr{H}_v^d$ for $v > 0$, and A has the form $\sum_k M_{\varphi_k} M_{\varphi_k}^*$ where $\{\varphi_k : k = 1, 2, \dots\}$ are homogeneous polynomials.

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