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ON $\mathcal{M}$-PERMUTABLE SYLOW SUBGROUPS OF FINITE GROUPS

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ABSTRACT. A p-subgroup $P \neq 1$ of G is called $\mathcal{M}$-permutable in G if there exists a set $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$ of maximal subgroup P_i of P and a subgroup B of G such that: (1) $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and $|P : \Phi(P)| = p^d$; (2) $G = PB$ and $P_iB = BP_i < G$ for any P_i of $\mathcal{M}_d(P)$. In this paper, we investigate the influence of $\mathcal{M}$-permutability of Sylow subgroups in finite groups. Some new results about supersolvable groups and formations are obtained.

1. Introduction

All the groups in this paper are finite. Let G be a finite group and $\mathcal{M}(G)$ be the set of all maximal subgroups of the Sylow subgroups of G. An interesting question is how the elements in $\mathcal{M}(G)$ influence the structure of finite groups. As a typical example of this aspect Srinivasan [13] states that G is supersolvable provided that each member of $\mathcal{M}(G)$ is normal in G. Later, this result has been widely generalized (see [8], [9], [16], [17]).

Recall that a subgroup H of G is said to be supplemented in G, if there exists a subgroup K of G such that $G = HK$. The relationship between the property of primary subgroups and the supplements of some restricted conditions has been studied extensively by many scholars. For instance, Hall [5] in 1937 proved that a group G is solvable if and only if every Sylow subgroup of G is complemented in G. Later on, Arad and Ward [1] further proved that a group G is solvable if and only if every Sylow 2-subgroup and every Sylow 3-subgroup of G are complemented in G. Recently, Ballester-Bolinches, Wang and Guo ([2], [16]) introduced the concept of c-supplemented subgroup and proved that G is solvable if and only if every Sylow subgroup

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of G is c-supplemented in G. More recently, Miao and Lempken [9] considered $\mathcal{M}$-supplemented subgroups of finite groups G and obtained some new characterization of saturated formations containing all supersolvable groups.

Now, we introduce the following new concept of $\mathcal{M}$-permutable subgroups.

DEFINITION 1.1. Let G be a finite group and p a prime divisor of $|G|$. A p-subgroup $P \neq 1$ of G is called $\mathcal{M}$-permutable in G if there exists a set $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$ of maximal subgroups P_i of P and a subgroup B of G such that

(1) $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and $|P : \Phi(P)| = p^d$ (so d is the smallest generator number of P);

(2) $G = PB$ and $P_i B = BP_i < G$ for any P_i of $\mathcal{M}_d(P)$.

Recall that, a subgroup H is called $\mathcal{M}$-supplemented in a finite group G, if there exists a subgroup B of G such that $G = HB$ and $H_1 B$ is a proper subgroup of G for any maximal subgroup H_1 of H [9, Definition 1.1]. Obviously, if a p-subgroup H is $\mathcal{M}$-supplemented in G, then H is also $\mathcal{M}$-permutable in G. The following example shows that the converse is not true.

EXAMPLE 1.2. $G = \langle s, a\rangle \times \langle t, b\rangle$ where $|a| = |b| = 3, |s| = |t| = 2$ and $\langle s, a\rangle \cong \langle t, b\rangle \cong S_3$. Clearly, $P = \langle a, b\rangle \in \mathrm{Syl}_3(G)$, $d = 2$ and $\mathcal{M}_2(P) = \{\langle a\rangle, \langle b\rangle\}$. Choose $B = \langle s, t\rangle$. $\langle a\rangle B = B\langle a\rangle, \langle b\rangle B = B\langle b\rangle$, but $\langle ab\rangle B \neq B\langle ab\rangle$. Therefore, we conclude that Sylow 3-subgroup of G is $\mathcal{M}$-permutable in G, but is not $\mathcal{M}$-supplemented in G.

Most of the notation is standard and can be found in [4] and [11]. In particular, $H < G$ indicates that H is a proper subgroup of G, $|G|$ denotes the order of G, G_p is a Sylow p-subgroup of G and $\pi(G)$ is the set of all prime divisors of $|G|$. Moreover, $\Phi(G), F(G)$ and $F^*(G)$ denote the Frattini subgroup, the Fitting subgroup and the generalized Fitting subgroup of G, respectively. Furthermore, $\mathcal{U}$ denotes the class of all supersolvable groups.

In this paper, we will investigate the properties of the $\mathcal{M}$-permutable Sylow subgroups in a finite group G. The main goal of this paper is to prove the following theorem.

THEOREM 3.6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every noncyclic Sylow subgroup of $F^*(H)$ is $\mathcal{M}$-permutable in G, then $G \in \mathcal{F}$.*

In order to prove Theorem 3.6, we shall prove the following fact which is one of the main step in the proof of Theorem 3.2 and Theorem 3.4.

THEOREM 3.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let H be a normal subgroup of G such that $G/H \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup of H is $\mathcal{M}$-permutable in G, then $G \in \mathcal{F}$.*

THEOREM 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every noncyclic Sylow subgroup of $F(H)$ is $\mathcal{M}$-permutable in G, then $G \in \mathcal{F}$.*

Recall that a class $\mathcal{F}$ of groups is said to be a formation if $G/H \in \mathcal{F}$ whenever $G \in \mathcal{F}$ and $H \trianglelefteq G$ and if $G/(M \cap N) \in \mathcal{F}$ whenever G/M and G/N are in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. Note that for a formation $\mathcal{F}$ every group G has a uniquely determined smallest normal subgroup $G^{\mathcal{F}}$ such that $G/G^{\mathcal{F}} \in \mathcal{F}$. It is also well known that the class of all supersolvable groups and the class of all p-nilpotent groups are saturated formations (e.g., see [4]).

2. Preliminaries

For the sake of convenience, we first list here some results which will be used in the sequel.

LEMMA 2.1. *Let G be a finite group and $P \neq 1$ a p-subgroup of G for some $p \in \pi(G)$. Assume that P is $\mathcal{M}$-permutable in G with respect to $\mathcal{M}_d(P)$ and that L is a normal subgroup of G contained in P. Then the following hold*:

(1) *There exists a subgroup B of G such that $G = PB$ and $|G : P_iB| = p$ for any $P_i \in \mathcal{M}_d(P)$; moreover, $P \cap B = P_i \cap B = \Phi(P) \cap B$.*

(2) *If $P \leq H \leq G$, then P is $\mathcal{M}$-permutable in H.*

(3) *If $L \leq \Phi(G)$, then $L \leq \Phi(P)$.*

(4) *If $L \leq \Phi(P)$, then P/L is $\mathcal{M}$-permutable in G/L.*

(5) *If L is a minimal normal subgroup of G and $L \nleq \Phi(P)$, then $|L| = p$.*

Proof. (1) By definition, there exists a subgroup B of G with $G = PB$ and $P_iB = BP_i < G$ for $P_i \in \mathcal{M}_d(P)$. Since $|P : P_i| = p$, order considerations show that $|G : P_iB| = p$ and $P \cap B = P_i \cap B$ for any $P_i \in \mathcal{M}_d(P)$. Hence, $P \cap B = \bigcap_{i=1}^{d}(P_i \cap B) = \Phi(P) \cap B$.

(2) Now we have $H = H \cap PB = P(H \cap B)$ and $H \geq H \cap P_iB = P_i(H \cap B)$ for any $P_i \in \mathcal{M}_d(P)$. Since $P \cap (H \cap B) = P \cap B = P_i \cap B = P_i \cap (H \cap B)$ and $P_i < P$, we have $P_i(H \cap B) < P(H \cap B) = H$. Therefore, P is $\mathcal{M}$-permutable in H.

(3) If $L \leq \Phi(G)$, then $L \leq \bigcap_{i=1}^{d} P_iB = \Phi(P)B$ and thus $L \leq P \cap \Phi(P)B = \Phi(P)(P \cap B) = \Phi(P)$.

(4) If $L \leq \Phi(P)$, then we may set $\mathcal{M}_d(P/L) = \{P_i/L \mid P_i \in \mathcal{M}_d(P)\}$. Then we have $L \leq BL \leq \Phi(P)B \leq P_iB < G$ and $BL/L < G/L$ as well as $G/L = (P/L)(BL/L)$ and $(P_i/L)(BL/L) = P_iB/L < G/L$; so P/L is $\mathcal{M}$-permutable in G/L.

(5) If L is a minimal normal subgroup of G and $L \nleq \Phi(P)$, then there exists $P_i \in \mathcal{M}_d(P)$ with $L \nleq P_iB$ by the proof of part (3). Then $G = LP_iB$ and so $L \cap P_iB \trianglelefteq G$. As L is minimal normal in G, $L \cap P_iB = 1$ and hence $|L| = p$. □

LEMMA 2.2 ([17, Theorem 3.1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, G a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

LEMMA 2.3 ([4, Theorem 1.8.17]). *Let N be a solvable normal subgroup of a group G $(N \neq 1)$. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

LEMMA 2.4 ([10, Lemma 2.6]). *If H is a subgroup of G with $|G : H| = p$, where p is the prime divisor of $|G|$ such that $(|G|, p-1) = 1$, then $H \trianglelefteq G$.*

LEMMA 2.5. *Let $p \in \pi(G)$ and $P \in \mathrm{Syl}_p(G)$. Then the following hold:*

(1) *If $N_G(P) = C_G(P)$, then G is p-nilpotent. In particular, G is p-nilpotent whenever P is cyclic and p is the smallest prime in $\pi(G)$.*

(2) *If $N \trianglelefteq G$ with $P \cap N \leq \Phi(P)$, then N is p-nilpotent.*

Proof. (1) This is a result of W. Burnside; see [6, Theorem IV.2.6 and IV.2.8].

(2) This is a result of Tate [14]; also see [6, Theorem IV.4.7]. □

LEMMA 2.6. *Let G be a finite group and P a Sylow p-subgroup of G where p is the prime divisor of $|G|$ such that $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if P is $\mathcal{M}$-permutable in G.*

Proof. If G is p-nilpotent, then G has a normal p-complement D. For the Sylow p-subgroup P of G and every maximal subgroup P_1 of P, we may easily get $G = PD$ and $P_1D < G$. Therefore P is $\mathcal{M}$-permutable in G.

Conversely, if P is $\mathcal{M}$-permutable in G, there exists a subgroup B of G such that $G = PB$ and $P_iB < G$ for any P_i of $\mathcal{M}_d(P)$. By Lemma 2.1, we have $|G : P_iB| = p$ and hence $P_iB \trianglelefteq G$ by Lemma 2.4. Since $|G : P_iB| = |PB : P_iB| = p$, we have $P \cap B = P_i \cap B$ for any P_i of $\mathcal{M}_d(P)$. On the other hand $\bigcap_{i=1}^d P_i = \Phi(P)$ and hence $P \cap B = \bigcap_{i=1}^d (P_i \cap B) = \Phi(P) \cap B$. Next we will prove $\bigcap_{P_i \in \mathcal{M}_d(P)} (P_iB) = (\bigcap_{P_i \in \mathcal{M}_d(P)} P_i)B$. In fact, we only need to prove $P_iB \cap P_jB = (P_i \cap P_j)B$ for any two maximal subgroups P_i and P_j of $\mathcal{M}_d(P)$. Clearly, $P_iB \cap P_jB \geq (P_i \cap P_j)B$. On the other hand, we may choose $xb_1 = yb_2 \in P_iB \cap P_jB$, where $x \in P_i$, $y \in P_j$ and $b_1, b_2 \in B$. Hence, $y^{-1}x = b_2b_1^{-1} \in P \cap B = P_i \cap B = P_j \cap B$. Therefore, $x \in P_i \cap P_j$ and we get $P_iB \cap P_jB = (P_i \cap P_j)B$. Therefore, we have that $\bigcap_{i=1}^d (P_iB) = (\bigcap_{i=1}^d P_i)B = \Phi(P)B$ and $\Phi(P)B \trianglelefteq G$. It follows from $P \cap \Phi(P)B = \Phi(P)(P \cap B) \leq \Phi(P)$ that we have $\Phi(P)B$ is p-nilpotent by Lemma 2.5. Let H be a normal Hall p'- subgroup of $\Phi(P)B$. Clearly, H is also the normal Hall p'-subgroup of G and hence G is p-nilpotent. The proof is over. □

LEMMA 2.7 ([8, Lemma 2.7]). *Let P be an elementary Abelian p-group of order p^d, $d \geq 2$, p a prime and let $\mathcal{M}_d(P) = \{M_1, \dots, M_d\}$. Then*

(a) $X_i = \bigcap_{i \neq j} M_j$ *is cyclic of order* p;

(b) $P = \langle X_1, \dots, X_d \rangle$.

LEMMA 2.8 ([7]). *Let G be a group and N a subgroup of G. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Then*

(1) *If N is normal in G, then $F(N) = N \cap F(G)$ and $F^*(N) = N \cap F^*(G)$;*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G)$;*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$;*

(4) *$C_G(F^*(G)) \leq F(G)$;*

(5) *Let $P \trianglelefteq G$ and $P \leq O_p(G)$; then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$;*

(6) *If K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

LEMMA 2.9. *Let H and L be normal subgroups of G and let $p \in \pi(G)$. Then the following hold:*

(1) $\Phi(H) \leq \Phi(G)$;

(2) *If $L \leq \Phi(G)$, then $F(G/L) = F(G)/L$;*

(3) *If $L \leq H \cap \Phi(G)$, then $F(H/L) = F(H)/L$;*

(4) *If H is a p-group and $L \leq \Phi(H)$, then $F^*(G/L) = F^*(G)/L$.*

Proof. (1) See [6, Lemma III.3.3].

(2) Note that $F(G/\Phi(G)) = F(G)/\Phi(G)$ and $\Phi(G/L) = \Phi(G)/L$. With this we obtain $(F(G)/L)/\Phi(G/L) = (F(G)/L)/(\Phi(G)/L) \cong F(G)/\Phi(G) = F(G/\Phi(G)) \cong F((G/L)/(\Phi(G)/L)) = F((G/L)/\Phi(G/L)) = F(G/L)/\Phi(G/L)$ and then $F(G)/L = F(G/L)$.

(3) Note that $F(H/L) = H/L \cap F(G/L) = H/L \cap F(G)/L = (H \cap F(G))/L = F(H)/L$.

(4) Since $L \leq \Phi(H)$, we have $\Phi(H/L) = \Phi(H)/L$. By Lemma 2.8, we obtain that $F^*((G/L)/\Phi(H/L)) = F^*(G/L)/\Phi(H/L) \cong F^*(G/\Phi(H)) = F^*(G)/\Phi(H)$ and hence $(F^*(G)/L)/(\Phi(H)/L) = F^*(G/L)/\Phi(H/L)$. Therefore, $F^*(G/L) = F^*(G)/L$. □

LEMMA 2.10 ([12, Lemma 1.9]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathcal{F}$. If E is cyclic, then $G \in \mathcal{F}$.*

LEMMA 2.11 ([4, Lemma 3.6.10]). *Let K be a normal subgroup of G and P a p-subgroup of G where p is a prime divisor of $|G|$. Then $N_{G/K}(PK/K) = N_G(P_1)K/K$, here P_1 is a Sylow p-subgroup of PK.*

LEMMA 2.12. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every Sylow subgroup of $F(H)$ is cyclic, then $G \in \mathcal{F}$.*

Proof. Assume that the assertion is false and choose G to be a counterexample of smallest order.

Let p be a prime of $\pi(H)$ and assume that $\Phi(O_p(H)) \neq 1$. Then we have $F(H/\Phi(O_p(H))) = F(H)/\Phi(O_p(H))$ by Lemma 2.9. Now, we easily verify that the pair $(G/\Phi(O_p(H)), H/\Phi(O_p(H)))$ satisfies the hypotheses of the lemma. Therefore, by the minimal choice of G, $G/\Phi(O_p(H)) \in \mathcal{F}$. As $O_p(H) \trianglelefteq G$, $\Phi(O_p(H)) \leq \Phi(G)$. As $\mathcal{F}$ is a saturated formation, we now get $G/\Phi(G) \in \mathcal{F}$ and hence $G \in \mathcal{F}$, a contradiction.

So we have $\Phi(O_p(H)) = 1$. We have shown that every Sylow subgroup of $F(H)$ is cyclic group of prime order.

Assume now that $\pi(F(H)) = \{p_1, \ldots, p_r\}$ and that $R_i := O_{p_i}(F(H))$ is cyclic of order p_i for $i \in \{1, \ldots, r\}$. So $C_H(F(H)) = F(H) = R_1 \times \cdots \times R_r$ and $H/F(H) \lesssim \mathrm{Aut}(F(H)) \cong \bigtimes_{i=1}^{r} \mathrm{Aut}(R_i)$ where $\mathrm{Aut}(R_i)$ is cyclic of order $p_i - 1$.

Set $F_i = R_1 \times \cdots \times R_i$ and $H_i = C_H(F_i)$ for $i \in \{1, \ldots, r\}$; clearly, $F_i \trianglelefteq G$ and $H_i \trianglelefteq G$ with $R_1 = F_1 < F_2 < \cdots < F_r = H_r \leq H_{r-1} \leq \cdots \leq H_1 \leq H$ such that $H/H_1, H_1/H_2, \ldots, H_{r-1}/H_r, F_r/F_{r-1}, \ldots, F_2/F_1$ and F_1 are cyclic. Since $G/H \in \mathcal{F}$, iterated application of Lemma 2.10 yields $G \in \mathcal{F}$, a contradiction.

The final contradiction completes our proof. □

3. Main results

THEOREM 3.1. *Let p be an odd prime divisor of $|G|$ and P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and P is $\mathcal{M}$-permutable in G.*

Proof. As the necessity part is obvious, we only need to prove the sufficiency part. Assume that the assertion is false and choose G to be a counterexample of minimal order. We will divide the following steps.

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$. By Lemmas 2.1 and 2.11, $G/O_{p'}(G)$ satisfies the condition of the theorem, the minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent, and hence G is p-nilpotent, a contradiction.

(2) If S is a proper subgroup of G containing P, then S is p-nilpotent.

Clearly, $N_S(P) \leq N_G(P)$ and hence $N_S(P)$ is p-nilpotent. Applying Lemma 2.1, we find that S satisfies the hypotheses of our theorem. Then the minimal choice of G implies that S is p-nilpotent.

(3) $G = PQ$, where Q is the Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by Thompson ([15], Corollary), there exists a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P with $H < K \leq P$. Since $N_G(P) \leq N_G(H)$ and $N_G(H)$ is not p-nilpotent, we have $N_G(P) < N_G(H)$. Then by (2), we have $N_G(H) = G$. This leads to $O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P such that $O_p(G) < K \leq P$. Now by Thompson ([15], Corollary), again, we see that $G/O_p(G)$ is p-nilpotent and therefore, G is p-solvable. Since G is p-solvable, for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G such that $PQ = QP$ is a subgroup of G by Gorenstein ([3], Theorem 6.3.5). If $PQ < G$, then PQ is p-nilpotent by (2). This leads to $Q \leq C_G(O_p(G)) \leq O_p(G)$ by Robinson ([11], Theorem 9.3.1) since $O_{p'}(G) = 1$, a contradiction. Thus, we have proven that $G = PQ$.

(4) Final contradiction.

If $O_p(G) \cap \Phi(G) \neq 1$, then we pick a minimal normal subgroup L of G with $L \leq O_p(G) \cap \Phi(G)$. By Lemma 2.1(3), we have $L \leq \Phi(P)$ and, furthermore, G/L satisfies the condition of the theorem by Lemma 2.1(4), the minimal choice of G implies that G/L is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, we obtain that G is p-nilpotent, a contradiction.

So we may assume $O_p(G) \cap \Phi(G) = 1$. Let L be any minimal normal subgroup of G contained in $O_p(G)$. Clearly, $L \nleq \Phi(P)$. By Lemma 2.1(5), we have $|L| = p$. Thus, $O_p(G)$ is the direct product of some minimal normal subgroups of order p of G by Lemma 2.3. If $p < q$, then LQ is p-nilpotent by Lemma 2.5 and therefore $Q \leq C_G(O_p(G))$, which contradicts to $C_G(O_p(G)) = O_p(G)$. On the other hand, if $q < p$, since $O_p(G)$ is the direct product of some minimal normal subgroup of order p, we have $G/C_G(O_p(G))$ is supersolvable by [6, Lemma 6.9.8] and hence $G/O_p(G)$ is supersolvable.

Since $G/O_p(G)$ is supersolvable and $q < p$, we know that $G/O_p(G)$ is q-nilpotent and then $P/O_p(G)$ is normal in $G/O_p(G)$. Therefore, P is normal in G. Hence, $N_G(P) = G$ is p-nilpotent, a contradiction.

The final contradiction completes our proof. □

THEOREM 3.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, H a normal subgroup of G such that $G/H \in \mathcal{F}$. Suppose that every noncyclic Sylow subgroup of H is $\mathcal{M}$-permutable in G, then $G \in \mathcal{F}$.*

Proof. Assume that the assertion is false and choose G to be a counterexample of minimal order.

By hypotheses and Lemma 2.1, we know that every noncyclic Sylow subgroup of H is $\mathcal{M}$-permutable in H, and hence H has a supersolvable type Sylow tower by Lemma 2.6. Let P be a Sylow p-subgroup of H where p is the largest prime divisor of $|H|$. Then P char H and hence $P \trianglelefteq G$. Moreover, we have the following.

CLAIM 1. $G/P \in \mathcal{F}$ and $P \nleq \Phi(G)$, furthermore, P is not cyclic.

First, we check that $(G/P, H/P)$ satisfies the hypotheses for (G, H). We know that $H/P \trianglelefteq G/P$ and $(G/P)/(H/P) \cong G/H \in \mathcal{F}$. We may assume that H_1/P is the noncyclic Sylow q-subgroup of H/P where $p \neq q$, clearly, $H_1 = PQ$ and Q is a noncyclic Sylow q-subgroup of H. By hypotheses, Q is $\mathcal{M}$-permutable in G, there exists a subgroup B of G such that $G = QB$ and $Q_iB < G$ for any Q_i of $\mathcal{M}_l(Q)$ where l is the smallest generator number of Q. Therefore, $G/P = (QP/P)(B/P)$ and $(Q_iP/P)(B/P) = Q_iB/P < G/P$ for any Q_iP/P of $\mathcal{M}_l(QP/P)$. So G/P satisfies the condition of the theorem. The minimal choice of G implies that $G/P \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, we know that $P \nleq \Phi(G)$. If P is cyclic, then $G \in \mathcal{F}$ by Lemma 2.10, a contradiction.

CLAIM 2. $P \cap \Phi(G) = 1$, in particular, $P = R_1 \times \cdots \times R_t$ with minimal normal subgroups $R_1, \ldots, R_t$ of G.

If $P \cap \Phi(G) \neq 1$, then we may choose a minimal normal subgroup L of G contained in $P \cap \Phi(G)$. On the other hand, by hypotheses, P is $\mathcal{M}$-permutable in G, i.e., there exists a subgroup B of G such that $G = PB$ and $P_iB < G$ for any P_i of $\mathcal{M}_d(P)$. By Lemma 2.1, $|G : P_iB| = p$ and $P \cap B = P_i \cap B \leq \Phi(P)$ for any P_i of $\mathcal{M}_d(P)$. Clearly, P_iB is the maximal subgroup of G for any P_i of $\mathcal{M}_d(P)$. Since L is a minimal normal subgroup of G, we have $G = LP_iB$ or $L \leq P_iB$. If $G = LP_iB$ for some P_i of $\mathcal{M}_d(P)$, we have $G = P_iB$ since L is contained in $P \cap \Phi(G)$, a contradiction. Therefore, $L \leq P_iB$ for any P_i of $\mathcal{M}_d(P)$. Moreover, if $L \nleq P_i$ for some P_i of $\mathcal{M}_d(P)$, then $P = LP_i$ and hence $P_iB = LP_iB = PB = G$, a contradiction. Therefore, we have $L \leq P_i$ for any P_i of $\mathcal{M}_d(P)$. According to the choice of $\mathcal{M}_d(P)$, we have $L \leq \bigcap_{i=1}^{d} P_i = \Phi(P)$. Hence, G/L satisfies the condition of the theorem by Lemma 2.1. The minimal choice of G implies that $G/L \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, it follows from $G/L \in \mathcal{F}$ that we have $G \in \mathcal{F}$, a contradiction.

So we may assume that $P \cap \Phi(G) = 1$ and then P is the direct product of minimal normal subgroups of G contained in P by Lemma 2.3. We denote that $P = R_1 \times \cdots \times R_t$, where R_j is a minimal normal subgroup of G, $j = 1, 2, \ldots, t$. By hypotheses, P is $\mathcal{M}$-permutable in G, i.e., there exists a subgroup B of G such that $G = PB$ and $P_iB < G$ for any P_i of $\mathcal{M}_d(P)$. By Lemma 2.1, we have $|G : P_iB| = p$ and $P \cap B = P_i \cap B \leq \Phi(P)$ for any P_i of $\mathcal{M}_d(P)$. Without loss of generality, choose any minimal normal subgroup L of G contained in P. Since P_iB is the maximal subgroup of G for any P_i of $\mathcal{M}_d(P)$, we know that there exists some P_i of $\mathcal{M}_d(P)$ such that $L \nleq P_iB$. Otherwise, if $L \leq P_iB$ for any P_i of $\mathcal{M}_d(P)$, then $L \leq P_i$ and hence $L \leq \bigcap_{i=1}^{d} P_i = \Phi(P)$. If not so, there exists P_i of $\mathcal{M}_d(P)$ such that $P = LP_i$, so we have $P_iB = LP_iB = PB = G$, a contradiction. Therefore, $L \leq \Phi(P)$. With the similar discussion as above, we have that G/L satisfies the condition of the theorem. The minimal choice of G implies that $G/L \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation and $L \leq \Phi(G)$, we have $G \in \mathcal{F}$, a contradiction. Consequently, there exist at least a P_i of $\mathcal{M}_d(P)$ such that $L \nleq P_iB$. Since $|G : P_iB| = p$, we know that $|L| = p$. Thus, P is the direct product of some minimal normal subgroup of order p of G.

Then for any maximal subgroup M of G, if $P \leq M$, then $P \leq \Phi(G)$, a contradiction. If $P \nleq M$, then there exist at least a minimal normal subgroup R_j of G contained in P such that $R_j \nleq M$. Since $G = R_jM$ and $|R_j| = p$, we get that M have a prime index in G, and hence $G \in \mathcal{F}$ by Lemma 2.2, a contradiction.

The final contradiction completes our proof. □

Corollary 3.3. *Let G be a finite group. If every noncyclic Sylow subgroup of G is $\mathcal{M}$-permutable in G, then G is supersolvable.*

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every noncyclic Sylow subgroup of $F(H)$ is $\mathcal{M}$-permutable in G, then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is false and choose G to be a counterexample of minimal order. The proof is divided into two cases.

Case 1. Suppose that $\Phi(G) \cap H \neq 1$.

Since $\Phi(G) \cap H \neq 1$, there exists a minimal normal subgroup L of G contained in $\Phi(G) \cap H$. Clearly, $L \leq O_p(H)$. Note that $F(H/L) = F(H)/L$ by Lemma 2.8. If $O_p(H)$ is cyclic, then G/L satisfies the hypotheses of the theorem; therefore $G/L \in \mathcal{F}$ by the minimal choice of G. Now Lemma 2.10 implies $G \in \mathcal{F}$, a contradiction. We have shown that $O_p(H)$ is not cyclic. By hypotheses, $O_p(H)$ is $\mathcal{M}$-permutable in G. There exists a subgroup B of G such that $G = O_p(H)B$ and $P_iB < G$ for any P_i of $\mathcal{M}_d(O_p(H))$. Firstly, we have that $L \leq P_iB$ for any P_i of $\mathcal{M}_d(O_p(H))$. Otherwise, there exists some P_i of $\mathcal{M}_d(O_p(H))$ such that $L \nleq P_iB$. By Lemma 2.1, $|G : P_iB| = p$ and $O_p(H) \cap B = P_i \cap B \leq \Phi(O_p(H))$ for any P_i of $\mathcal{M}_d(O_p(H))$. Obviously, P_iB is the maximal subgroup of G and $L \leq \Phi(G)$, so $L \leq P_iB$, a contradiction. Moreover, next we will prove $L \leq P_i$ for any P_i of $\mathcal{M}_d(O_p(H))$. If not so, there exist some P_i such that $L \nleq P_i$. Since P_i is the maximal subgroup of $O_p(H)$, we have $O_p(H) = LP_i$. Furthermore, $P_iB = LP_iB = O_p(H)B = G$, a contradiction. Therefore, $1 \neq L \leq \bigcap_{i=1}^{d} P_i = \Phi(O_p(H))$. Clearly, $G/\Phi(O_p(H))$ satisfies the hypotheses of the theorem by Lemma 2.8. The minimal choice of G implies that $G/\Phi(O_p(H)) \in \mathcal{F}$ and hence $G \in \mathcal{F}$ since $\mathcal{F}$ is a saturated formation, a contradiction.

Case 2. Suppose that $\Phi(G) \cap H = 1$.

If $H = 1$, nothing need to prove, so we may assume that $H \neq 1$. The solvability of H implies that $F(H) \neq 1$. By Lemma 2.3, $F(H)$ is the direct product of minimal normal subgroups of G contained in H. There exists a noncyclic Sylow p-subgroup of $F(H)$ by Lemma 2.12 for some prime $p \in \pi(G)$. Denote $P = O_p(H)$. Then P is the direct product of some minimal normal subgroup of G. Denote $P = R_1 \times \cdots \times R_t$, where $R_1, \ldots, R_t$ is minimal normal subgroup of G contained in P. By hypotheses, P is $\mathcal{M}$-permutable in G. There exists a subgroup B of G such that $G = PB$ and $P_iB < G$ for any P_i of

$\mathcal{M}_d(P)$. By Lemma 2.1, we have $|G:P_iB|=p$ and $P\cap B=P_i\cap B\leq \Phi(P)$. Let L be any minimal normal subgroup of G contained in P. Next, we will prove that there exist at least some P_i such that $L\nleq P_iB$. Otherwise, if $L\leq P_iB$ for any P_i of $\mathcal{M}_d(P)$, then we claim that $L\leq P_i$ and hence $L\leq \bigcap_{i=1}^d P_i=\Phi(P)$. If not so, there exists P_i of $\mathcal{M}_d(P)$ such that $L\neq P_i$, so we have $P_iB=LP_iB=PB=G$, a contradiction. Therefore, $L\leq\Phi(P)$. With the similar discussion, we have that G/L satisfies the condition of the theorem. The minimal choice of G implies that $G/L\in\mathcal{F}$. Since $\mathcal{F}$ is a saturated formation and $L\leq\Phi(G)$, we have $G\in\mathcal{F}$, a contradiction. Consequently, there exist at least a P_i of $\mathcal{M}_d(P)$ such that $L\nleq P_iB$. Since $|G:P_iB|=p$, we know that $|L|=p$. Thus, P is the direct product of some minimal normal subgroup of order p of G, so is $F(H)$.

Denote $F(H)=H_1\times H_2\times\cdots\times H_r$, where H_i is a minimal normal subgroup of prime order of G, then $G/C_G(H_i)$ is Abelian, $i=1,2,\ldots,r$. Since $C_G(F(H))=\bigcap_{i=1}^r C_G(H_i)$, $\mathcal{F}$ is a saturated formation, $G/C_G(F(H))\in\mathcal{F}$. By assumption, $G/H\in\mathcal{F}$ and hence $G/(H\cap C_G(F(H))=G/C_H(F(H))\in\mathcal{F}$. Since H is solvable, we have $C_H(F(H))\leq F(H)$. Then $G/F(H)$ is an epimorphic image of $G/C_H(F(H))$, thus $G/F(H)\in\mathcal{F}$. Now applying Theorem 3.2 for $(G,F(H))$, we get $G\in\mathcal{F}$, a contradiction.

The final contradiction completes our proof. □

Corollary 3.5. *Let H be a solvable normal subgroup of G such that $G/H\in\mathcal{U}$. If every noncyclic Sylow subgroup of $F(H)$ is $\mathcal{M}$-permutable in G, then $G\in\mathcal{U}$.*

Theorem 3.6. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups. Suppose that G is a finite group with a normal subgroup H such that $G/H\in\mathcal{F}$. If every noncyclic Sylow subgroup of $F^*(H)$ is $\mathcal{M}$-permutable in G, then $G\in\mathcal{F}$.*

Proof. Suppose that the theorem is false and choose G to be a counterexample of minimal order. We consider the following two cases.

Case 1. $\mathcal{F}=\mathcal{U}$.

(1) $F^*(H)=F(H)\neq 1$.

By hypotheses and Lemma 2.1, every noncyclic Sylow subgroup of $F^*(H)$ is $\mathcal{M}$-permutable in G and hence is $\mathcal{M}$-permutable in $F^*(H)$. By Corollary 3.3, $F^*(H)$ is supersolvable. In particular, $F^*(H)$ is solvable and hence $F^*(H)=F(H)\neq 1$ by Lemma 2.8.

(2) $H=G$, $F^*(G)=F(G)\neq 1$.

Since H satisfies the hypotheses of the theorem, the minimal choice of G implies that H is supersolvable if $H<G$. It follows that $G\in\mathcal{F}$ by Corollary 3.5.

(3) Every proper normal subgroup N of G containing $F^*(G)$ is supersolvable.

By Lemma 2.8, $F^*(G) = F^*(F^*(G)) \le F^*(N) \le F^*(G)$, so $F^*(N) = F^*(G)$. And every noncyclic Sylow subgroup of $F^*(N)$ is $\mathcal{M}$-permutable in N by Lemma 2.1. Hence, N is supersolvable by the minimal choice of G.

(4) $\Phi(G) < F(G)$.

If every Sylow subgroup of $F(G)$ is cyclic, then we denote that $F(G) = H_1 \times \cdots \times H_r$ and hence $G/C_G(H_i)$ is Abelian for any $i \in \{1 \cdots r\}$. Moreover, we have $G/\bigcap_{i=1}^r C_G(H_i) = G/F(G)$ is Abelian. Therefore, G is supersolvable by Lemma 2.12, a contradiction. Let $O_p(G)$ be a noncyclic Sylow subgroup of $F(G)$. By hypotheses, $O_p(G)$ is $\mathcal{M}$-permutable in G, and there exists a subgroup B of G such that $G = O_p(G)B$ and $P_iB < G$ for any P_i of $\mathcal{M}_d(O_p(G))$. If $\Phi(G) = F(G)$, then $O_p(G) \le \Phi(G)$ and hence $G = O_p(G)B = B$, a contradiction.

(5) Final contradiction.

By (4), there exists some Sylow p-subgroup $O_p(G)$ of $F(G)$ and the maximal subgroup M of G with $O_p(G) \nleq M$ and $G = O_p(G)M$.

If $|O_p(G)| = p$, then set $C = C_G(O_p(G))$. Clearly, $F(G) \le C \trianglelefteq G$. If $C < G$, then C is solvable by (3). On the other hand, since G/C is cyclic, we have G is solvable and hence G is supersolvable by Corollary 3.5, a contradiction. So we may assume $C = G$. Now we have $O_p(G) \le Z(G)$. Then we consider factor group $G/O_p(G)$. By Lemma 2.8, we have $F^*(G/O_p(G)) = F^*(G)/O_p(G) = F(G)/O_p(G)$. In fact, every noncyclic Sylow subgroup of $F^*(G/O_p(G))$ are $\mathcal{M}$-permutable in $G/O_p(G)$. Therefore, the minimal choice of G implies that $G/O_p(G) \in \mathcal{U}$ and hence G is supersolvable by Lemma 2.10, a contradiction.

So we may assume that $|O_p(G)| > p$. If $\Phi(O_p(G)) \ne 1$, then it is easy to obtain that factor group $G/\Phi(O_p(G))$ satisfies the condition of the theorem by Lemma 2.8. The minimal choice of G implies that $G/\Phi(O_p(G))$ is supersolvable and hence G is supersolvable since the class of all supersolvable groups is a saturated formation, a contradiction. Therefore, $\Phi(O_p(G)) = 1$ and $O_p(G)$ is an elementary Abelian p-group. By hypotheses, $O_p(G)$ is $\mathcal{M}$-permutable in G, there exists a subgroup B of G such that $G = O_p(G)B$ and $P_iB < G$ for any P_i of $\mathcal{M}_d(O_p(G))$. By Lemma 2.1, $|G : P_iB| = p$ and $O_p(G) \cap B = P_i \cap B \le \Phi(O_p(G)) = 1$ for any P_i of $\mathcal{M}_d(O_p(G))$. In this case, $O_p(G) \cap P_iB = P_i(O_p(G) \cap B) = P_i$ is normal in G since $G = O_p(G)B$ and $O_p(G)$ is an elementary Abelian p-group. Therefore, we have that any P_i of $\mathcal{M}_d(O_p(G))$ is normal in G. By Lemma 2.7, there exist minimal normal subgroup X_i of G of order p where $X_i = \bigcap_{i \ne j} P_i$ and $i = 1, \ldots, d$, such that $O_p(G) = \langle X_1, \ldots, X_d \rangle$. For any X_i of $O_p(G)$, with the similar discussion, we may consider $C_G(X_i)$. Clearly, $F(G) \le C_G(X_i) \trianglelefteq G$. If $C_G(X_i) < G$, then $C_G(X_i)$ is solvable by (3). On the other hand, since $G/C_G(X_i)$ is cyclic, then we have G is solvable, a contradiction. So we may assume $C_G(X_i) = G$. Since $X_i \leq Z(G)$ for any minimal normal subgroup X_i in $O_p(G)$, we have $O_p(G) \leq Z(G)$. Then we consider factor group $G/O_p(G)$. By Lemma 2.8, we have $F^*(G/O_p(G)) = F^*(G)/O_p(G) = F(G)/O_p(G)$. In fact, every noncyclic Sylow subgroup of $F^*(G/O_p(G))$ are $\mathcal{M}$-permutable in $G/O_p(G)$ by Lemma 2.1. Therefore, the minimal choice of G implies that $G/O_p(G) \in \mathcal{U}$ and hence G is supersolvable, a contradiction.

Case 2. $\mathcal{F} \neq \mathcal{U}$.

By case 1, H is supersolvable. Particularly, H is solvable and $F(H) = F^*(H)$. By Lemma 2.2 and Theorem 3.4, we may get $G \in \mathcal{F}$, a contradiction. The final contradiction completes our proof. □

Corollary 3.7. *Let H be a normal subgroup of G such that $G/H \in \mathcal{U}$. If every noncyclic Sylow subgroup of $F^*(H)$ is $\mathcal{M}$-permutable in G, then $G \in \mathcal{U}$.*

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