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CHARACTERIZATION OF BANACH FUNCTION SPACES THAT PRESERVE THE BURKHOLDER SQUARE-FUNCTION INEQUALITY

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Dedicated to Professor Tamotsu Tsuchikura on his eightieth birthday

ABSTRACT. Let $(X, \|\cdot\|_X)$ be a Banach function space over a nonatomic probability space. We give a necessary and sufficient condition on X for the inequalities $c\|f_\infty\|_X \le \|S(f)\|_X \le C\|f_\infty\|_X$ to hold for all uniformly integrable martingales $f = (f_n)_{n\ge 0}$, where $f_\infty = \lim_n f_n$ a.s. and $S(f) = \left\{f_0^2 + \sum_{n=1}^{\infty}(f_n - f_{n-1})^2\right\}^{1/2}$.

1. Introduction

In 1966 Burkholder [4] proved that if $1 < p < \infty$, then there are positive constants c_p and C_p such that

$$c_p \|f_\infty\|_p \le \|S(f)\|_p \le C_p \|f_\infty\|_p \tag{1}$$

for all uniformly integrable martingales $f = (f_n)_{n\ge 0}$, where $f_\infty = \lim_n f_n$ almost surely (a.s.) and $S(f) = \left\{f_0^2 + \sum_{n=1}^{\infty}(f_n - f_{n-1})^2\right\}^{1/2}$. Recall that (1) holds neither for $p = 1$ nor for $p = \infty$. Here we consider this inequality for Banach function spaces (see Definition 1 below). Our main result is that if such a space X satisfies the inequality

$$c\|f_\infty\|_X \le \|S(f)\|_X \le C\|f_\infty\|_X$$

for all uniformly integrable martingales $f = (f_n)$, then X is rearrangement-invariant and its norm is equivalent to a rearrangement-invariant norm for which the Boyd indices satisfy $0 < \alpha_X \le \beta_X < 1$.

Both the Doob maximal inequality and the Burkholder-Davis-Gundy inequality, in which the maximal function of f replaces the limit function f_∞, have already been studied for rearrangement-invariant spaces (see Antipa [1]

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and the closely related and independent work of Johnson and Schechtman [7], Kikuchi [8], and Novikov [13]). This work shows that the converse of our main result is true (see Proposition 3).

2. Notation and terminology

Let $(\Omega, \Sigma, \mathbb{P})$ be a *nonatomic* probability space*.

2.1. Banach function spaces. If X and Y are Banach spaces of random variables, we write $X \hookrightarrow Y$ to mean that X is continuously embedded in Y, i.e., that $X \subset Y$ and $\|x\|_Y \leq c\,\|x\|_X$ for all $x \in X$ with some constant $c > 0$.

Definition 1. A real Banach space $(X, \|\cdot\|_X)$ of (equivalence classes of) random variables on Ω is called a *Banach function space* if it satisfies the following conditions:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$;

(B2) if $x \in X$ and $|y| \leq |x|$ a.s., then $y \in X$ and $\|y\|_X \leq \|x\|_X$;

(B3) if $x_n \in X,\ 0 \leq x_n \uparrow x$ a.s. and $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_n \|x_n\|_X$.

We adopt the convention that $\|x\|_X = \infty$ unless $x \in X$.

Let x and y be random variables. We write $x \simeq_d y$ if they are *equimeasurable*, or in other words, they are *identically distributed*.

Definition 2. (i) A Banach function space X is said to be *rearrangement-invariant* (or simply *r.i.*) if it satisfies the following condition:

(R1) if $x \in X$ and $x \simeq_d y$, then $y \in X$.

(ii) The norm of a Banach function space X is said to be *rearrangement-invariant* (or simply *r.i.*) if it satisfies the following condition:

(R2) if $x,\, y \in X$ and $x \simeq_d y$, then $\|x\|_X = \|y\|_X$.

Note that if the norm of a Banach function space X is r.i., then the space X is r.i. To see this, suppose that $x \simeq_d y$ and $x \in X$. Then, for all integers $n \geq 1$, we have $|x| \wedge n \simeq_d |y| \wedge n$ and hence $\big\||y| \wedge n\big\|_X = \big\||x| \wedge n\big\|_X \leq \|x\|_X$ by (R2) and (B2). This, together with (B3), implies that $y \in X$. As for the converse, the norm of an r.i. space† X is not always r.i. (see [11, p. 114] or [5, p. 99]). There is, however, an r.i. norm $|||\cdot|||_X$ on X such that $\|\cdot\|_X \approx |||\cdot|||_X$ (see [11, p. 138] or [5, p. 106]). Here we write $\|\cdot\|_X \approx |||\cdot|||_X$ if these norms are equivalent.

Now let $I = (0, 1]$ and let μ be Lebesgue measure on the σ-algebra $\mathfrak{M}$ of Lebesgue measurable subsets of I. The *nonincreasing rearrangement* of a

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* In essence, we may assume that Ω is the unit interval $(0, 1]$ with Lebesgue measure on the σ-algebra of Lebesgue measurable sets.

† By an *r.i. space* X, we mean a rearrangement-invariant Banach function space X.

random variable x on Ω, which is denoted by x^*, is the function on I defined as

$$x^*(t) = \inf\bigl\{\lambda > 0 \bigm| \mathbb{P}(|x| > \lambda) \le t\bigr\} \quad (t \in I),$$

where we follow the convention that $\inf\emptyset = \infty$. Note that x^* and $|x|$ are equimeasurable, i.e.,

$$\mu(x^* > \lambda) = \mathbb{P}\bigl(|x| > \lambda\bigr) \quad \text{for all } \lambda > 0.$$

The nonincreasing rearrangement φ^* of a measurable function $\varphi\colon I \to \mathbb{R}$ is defined in the same way. If φ and ψ are measurable functions on I, then

$$\int_I \bigl|\varphi(s)\psi(s)\bigr|\, ds \le \int_I \varphi^*(s)\psi^*(s)\, ds. \tag{2}$$

This is called the *Hardy-Littlewood inequality* (see, e.g., [2, p. 44]). In particular,

$$\int_E \bigl|\varphi(s)\bigr|\, ds \le \int_0^{\mu(E)} \varphi^*(s)\, ds \quad (E \in \mathfrak{M}). \tag{3}$$

Following [2], we write $\varphi \prec \psi$ to mean that

$$\int_0^t \varphi^*(s)\, ds \le \int_0^t \psi^*(s)\, ds \qquad \text{for all } t \in I.$$

It is then clear that $\varphi \prec \psi$ if and only if $\varphi^* \prec \psi^*$. Moreover, if x and y are random variables on Ω, then we write $x \prec y$ to mean that $x^* \prec y^*$.

Note that if $(X, \|\cdot\|_X)$ is endowed with an r.i. norm, then (R2) can be replaced by the following condition (cf. [2, p. 90]):

(R2$'$) if $x \in X$ and $y \prec x$, then $y \in X$ and $\|y\|_X \le \|x\|_X$.

We now recall the *Luxemburg representation theorem*. If $(X, \|\cdot\|_X)$ is an r.i. space over Ω endowed with an r.i. norm, then there exists an r.i. space $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ *over* I endowed with an r.i. norm such that

(L1) $x \in X$ if and only if $x^* \in \widehat{X}$;

(L2) $\|x\|_X = \|x^*\|_{\widehat{X}}$ for all $x \in X$.

See [2, pp. 62–64] for a proof. Such a space $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ is unique; we call $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ the *Luxemburg representation of* X.

In order to state our results, we need the notion of the Boyd indices. For each $s \in (0, \infty)$, the dilation operator D_s, acting on the space of measurable functions on I, is defined by

$$(D_s\,\varphi)(t) = \begin{cases} \varphi(st), & \text{if } st \in I, \\ 0, & \text{otherwise,} \end{cases} \qquad (t \in I).$$

If $(Y, \|\cdot\|_Y)$ is an r.i. space over I, then each D_s is a bounded linear operator from Y into Y, and $\|D_s\|_{B(Y)} \le 1 \vee s^{-1}$, where $\|\cdot\|_{B(Y)}$ denotes the operator norm. The *lower* and *upper Boyd indices* are defined by

$$\alpha_Y = \sup_{0<s<1} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s} = \lim_{s\to 0+} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s}$$

and

$$\beta_Y = \inf_{1<s<\infty} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s} = \lim_{s\to\infty} \frac{\log \|D_{s^{-1}}\|_{B(Y)}}{\log s},$$

respectively. If $(X, \|\cdot\|_X)$ is an r.i. space over Ω endowed with an r.i. norm, then the Boyd indices of $(X, \|\cdot\|_X)$ are defined by $\alpha_X = \alpha_{\widehat{X}}$ and $\beta_X = \beta_{\widehat{X}}$, where $\widehat{X}$ is the Luxemburg representation of X. Moreover, if $(X, \|\cdot\|_X)$ is an arbitrary r.i. space and if $|||\cdot|||_X$ is an r.i. norm on X such that $\|\cdot\|_X \approx |||\cdot|||_X$, then the Boyd indices of $(X, \|\cdot\|_X)$ are defined to be those of $(X, |||\cdot|||_X)$. In any case, we have $0 \le \alpha_X \le \beta_X \le 1$ (see [3] or [2, p. 149]).

2.2. Martingales. By a *filtration* we mean a nondecreasing sequence $\mathcal{F} = (\mathcal{F}_n)_{n\ge 0}$ of sub-σ-algebras of Σ. Given a filtration $\mathcal{F} = (\mathcal{F}_n)$, we denote by $\mathcal{M}_{\mathcal{F}}$, the collection of all uniformly integrable martingales with respect to $\mathcal{F}$. As is well known, every $f \in \mathcal{M}_{\mathcal{F}}$ converges almost surely (a.s.) to some $f_\infty \in L_1(\Omega)$ and $f_n = \mathbb{E}[\, f_\infty \,|\, \mathcal{F}_n\,]$ $(n = 1, 2, \ldots)$ (see, e.g., [6, p. 26]).

In what follows, we will consider martingales with respect to various filtrations, and accordingly we let $\mathcal{M} = \bigcup_{\mathcal{F}} \mathcal{M}_{\mathcal{F}}$, where the union is over all filtrations $\mathcal{F}$. We will use the following notation for $f = (f_n)_{n\ge 0} \in \mathcal{M}$:

- $\Delta_0 f := f_0\,;\ \Delta_n f := f_n - f_{n-1} \qquad (n = 1, 2, \ldots),$
- $S_n(f) := \Big\{ \sum_{j=0}^{n} (\Delta_j f)^2 \Big\}^{1/2} \qquad (n = 0, 1, 2, \ldots),$
- $S(f) := \lim_{n\to\infty} S_n(f),$
- $M_n(f) := \max_{0\le j\le n} |f_j| \qquad (n = 0, 1, 2, \ldots),$
- $M(f) := \lim_{n\to\infty} M_n(f),$
- $f_\infty := \lim_{n\to\infty} f_n$ a.s.

3. Main results

Given a Banach function space $(X, \|\cdot\|_X)$ over Ω, we let

$$\mathcal{M}(X) = \big\{ f = (f_n) \in \mathcal{M} \;\big|\; f_\infty \in X \big\}; \tag{4}$$

$$\mathcal{H}(X) = \big\{ f = (f_n) \in \mathcal{M} \;\big|\; S(f) \in X \big\}. \tag{5}$$

Our main result is as follows:

THEOREM 1. *Let* $(X, \|\cdot\|_X)$ *be a Banach function space over* Ω. *Then the following are equivalent*:

(i) *there are constants* c *and* C, *depending only on* X, *such that*

$$c\,\|f_\infty\|_X \le \|S(f)\|_X \le C\,\|f_\infty\|_X \quad (f \in \mathcal{M}); \tag{4}$$

(ii) $\mathcal{M}(X) = \mathcal{H}(X)$;

(iii) X *is rearrangement-invariant and can be renormed with an equivalent rearrangement-invariant norm for which the Boyd indices satisfy* $0 < \alpha_X \le \beta_X < 1$.

Note that except for possible changes in the constants, inequality (4) holds for a norm if and only if it holds for every equivalent norm.

Recall the convention that $\|x\|_X = \infty$ unless $x \in X$. This shows that (i) implies (ii). That (ii) implies (iii) follows from Propositions 1 and 2 below, and that (iii) implies (i) is just the assertion of Proposition 3 below.

PROPOSITION 1. *Let* $(X, \|\cdot\|_X)$ *be a Banach function space over* Ω. *If* $\mathcal{M}(X) \subset \mathcal{H}(X)$, *then*:

(i) X *is rearrangement-invariant*;

(ii) $\beta_X < 1$.

PROPOSITION 2. *Let* $(X, \|\cdot\|_X)$ *be a rearrangement-invariant space over* Ω. *If* $\beta_X < 1$ *and if* $\mathcal{H}(X) \subset \mathcal{M}(X)$, *then* $\alpha_X > 0$.

PROPOSITION 3. *Let* $(X, \|\cdot\|_X)$ *be as in Proposition 2. If* $0 < \alpha_X \le \beta_X < 1$, *then there are constants* c *and* C, *depending only on* X, *such that* (4) *holds.*

Proposition 3 follows from the results of Antipa [1]; however, we will give an alternative proof of Proposition 3 via Shimogaki's Theorem.

In order to prove Propositions 1 and 2, we need the following lemmas.

LEMMA 1. *Let* $(X, \|\cdot\|_X)$ *be a Banach function space. Then* X *is rearrangement-invariant if and only if it satisfies the following condition*:

(R1′) *if* $x, y \ge 0$ *a.s.*, $\{x > 0\} \cap \{y > 0\} = \emptyset$, $x \simeq_d y$, *and* $x \in X$, *then* $y \in X$.

Proof. It suffices to show that (R1′) implies (R1). Suppose that $x \simeq_d y$ and $x \in X$. To prove that $y \in X$, we may assume $y \notin L_\infty$ (cf. (B1)). Choose $\lambda > 0$ so large that $\mathbb{P}(|x| > \lambda) < 1/3$. Clearly $x' := |x| 1_{\{|x| > \lambda\}}$ and $y' := |y| 1_{\{|y| > \lambda\}}$ are equimeasurable and $x' \in X$. Since the set $\{x' = 0,\, y' = 0\}$ contains no atom and $\mathbb{P}(x' = 0,\, y' = 0) > 1/3$, there is a random variable $z \ge 0$ such that $\{z > 0\} \subset \{x' = 0,\, y' = 0\}$ and $z \simeq_d x'$ (cf. [5, p. 44]). Then condition (R1′) yields that $z \in X$, since $\{z > 0\} \cap \{x' > 0\} = \emptyset$ and $x' \in X$. Now the same reasoning shows that $y' \in X$. Therefore $|y| \le y' + \lambda \in X$, and thus $y \in X$ as desired. $\square$

LEMMA 2 ([2, p. 46]). *Suppose that $x \in L_1(\Omega)$ is nonnegative. Then there is a family $\{A(t) \mid t \in I\}$ of measurable subsets of Ω satisfying the following conditions*:

(i) $A(s) \subset A(t)$ *whenever* $0 < s \le t \le 1$;

(ii) $\mathbb{P}\big(A(t)\big) = t$ *for all* $t \in I$;

(iii) $\int_{A(t)} x \, d\mathbb{P} = \int_0^t x^*(s)\, ds$ *for all* $t \in I$;

(iv) $\{\omega \in \Omega \mid x(\omega) > x^*(t)\} \subset A(t) \subset \{\omega \in \Omega \mid x(\omega) \ge x^*(t)\}$ *for all* $t \in I$.

In particular, if $\mathbb{P}(x = s) = 0$ *for all* $s > 0$ *and if* $t_0 = \mathbb{P}(x > 0)$, *then* $A(t)$ *may be taken to be the set* $\{\omega \in \Omega \mid x(\omega) > x^*(t)\}$ *for each* $t \in (0, t_0]$.

We now consider an averaging operator $\mathcal{P}$ and its adjoint $\mathcal{Q}$: for $\varphi \in L_1(I)$ define

$$(\mathcal{P}\varphi)(t) = \frac{1}{t}\int_0^t \varphi(s)\, ds \qquad (t \in I),$$

and for $\varphi \in \bigcap_{0<t<1} L_1(t, 1)$ define

$$(\mathcal{Q}\varphi)(t) = \int_t^1 \frac{\varphi(s)}{s}\, ds \qquad (t \in I).$$

Then it is easy to derive the following formulae:

$$\mathcal{P}\mathcal{Q}\varphi = \mathcal{P}\varphi + \mathcal{Q}\varphi \qquad \big(\varphi \in L_1(I)\big); \tag{6a}$$

$$\mathcal{Q}\mathcal{P}\varphi = \mathcal{P}\varphi + \mathcal{Q}\varphi - \int_I \varphi\, d\mu \qquad \big(\varphi \in L_1(I)\big). \tag{6b}$$

We recall Shimogaki's Theorem on the boundedness of $\mathcal{P}$ and $\mathcal{Q}$. In terms of Boyd indices, it can be expressed as follows:

SHIMOGAKI'S THEOREM ([14]). *Let $(Y, \|\cdot\|_Y)$ be a rearrangement-invariant space over I endowed with a rearrangement-invariant norm. Then*:

(i) $\beta_Y < 1$ *if and only if $\mathcal{P}$ is a bounded operator from Y into Y*;

(ii) $\alpha_Y > 0$ *if and only if $\mathcal{Q}$ is a bounded operator from Y into Y.*

For a proof of (an extension of) this theorem see [2, p. 150] or [3]. Note that $\mathcal{P}$ (resp. $\mathcal{Q}$) is a bounded linear operator from Y into Y if and only if $\mathcal{P}(Y) \subset Y$ (resp. $\mathcal{Q}(Y) \subset Y$). This is an immediate consequence of the closed graph theorem, since $Y \hookrightarrow L_1(I)$. Thus:

- $\beta_Y < 1$ if and only if $\mathcal{P}(Y) \subset Y$;
- $\alpha_Y > 0$ if and only if $\mathcal{Q}(Y) \subset Y$.

The next lemma is a variant of Shimogaki's Theorem. Before stating it, we must introduce some notation.

NOTATION. Let $(Y, \|\cdot\|_Y)$ be a Banach function space over I.

(i) We denote by $\mathcal{D}_Y$ the collection of all nonnegative nonincreasing functions in Y.

(ii) We denote by $\mathcal{D}'_Y$ the collection of functions $\varphi \in \mathcal{D}_Y$ such that $\mu(\varphi > 0) \le 1/2$.

LEMMA 3. *Let $(Y, \|\cdot\|_Y)$ be as in Shimogaki's Theorem. Then*:

(i) $\beta_Y < 1$ *if and only if* $\mathcal{P}(\mathcal{D}'_Y) \subset Y$;

(ii) $\alpha_Y > 0$ *if and only if* $\mathcal{Q}(\mathcal{D}_Y) \subset Y$.

Furthermore $\mathcal{D}'_Y$ may be replaced by $\mathcal{D}'_Y \setminus L_\infty(I)$ in (i).

Proof. The last statement is clear, since $\mathcal{P}\varphi \in L_\infty$ for any $\varphi \in L_\infty$.

To prove (i) and (ii), it suffices to show that:

(i′) if $\mathcal{P}(\mathcal{D}'_Y) \subset Y$, then $\mathcal{P}(Y) \subset Y$;

(ii′) if $\mathcal{Q}(\mathcal{D}_Y) \subset Y$, then $\mathcal{Q}(Y) \subset Y$.

To prove (i′), assume that $\mathcal{P}\varphi \in Y$ whenever $\varphi \in \mathcal{D}'_Y$. Let $\psi \in Y$ and choose $\lambda > 0$ so that $\mu(|\psi| > \lambda) \le 1/2$. If we let $\varphi = \psi^* 1_{\{\psi^* > \lambda\}}$, then $\varphi \in \mathcal{D}'_Y$ and hence $\mathcal{P}\varphi \in Y$. By inequality (3) and the inequality $\psi^* \le \varphi + \lambda$ we have

$$\big|(\mathcal{P}\psi)(t)\big| \le (\mathcal{P}\psi^*)(t) \le (\mathcal{P}\varphi)(t) + \lambda \qquad (t \in I).$$

Hence $\mathcal{P}\psi \in Y$, as desired.

To prove (ii′), assume that $\mathcal{Q}\varphi \in Y$ whenever $\varphi \in \mathcal{D}_Y$. Let $\psi \in Y$, or equivalently, let $\psi^* \in \mathcal{D}_Y$; then $\mathcal{Q}\psi^* \in Y$. It suffices to show that $\mathcal{Q}|\psi| \in Y$, since $|\mathcal{Q}\psi| \le \mathcal{Q}|\psi|$ (cf. (B2)). Using inequality (2), we find that

$$\begin{aligned}\int_0^t (\mathcal{Q}|\psi|)(s)\,ds &= \int_0^1 (1 \wedge s^{-1}t)\,|\psi(s)|\,ds \\ &\le \int_0^1 (1 \wedge s^{-1}t)\,\psi^*(s)\,ds \\ &= \int_0^t (\mathcal{Q}\psi^*)(s)\,ds \qquad (t \in I).\end{aligned}$$

This shows that $\mathcal{Q}|\psi| \prec \mathcal{Q}\psi^*$ (since these two functions are nonincreasing). Hence $\mathcal{Q}|\psi| \in Y$ (cf. (R2′)), as desired. □

We are now ready to prove Propositions 1 and 2.

Proof of Proposition 1. Suppose $\mathcal{M}(X) \subset \mathcal{H}(X)$.

(i) To prove that X is r.i., it suffices to show that X satisfies (R1′) (see Lemma 1). Assume that $x, y \ge 0$ a.s., $\{x > 0\} \cap \{y > 0\} = \emptyset$, $x \simeq_d y$, and $x \in X$. We must prove that $y \in X$. To this end, we may assume $y \notin L_\infty$.

There are two cases to consider:

Case 1: $\mathbb{P}(y = s) = 0$ for any $s > 0$;
Case 2: $\mathbb{P}(y = s) > 0$ for some $s > 0$.

In Case 1, we define $\tilde{y} \in L_1$ and $\alpha \in \mathbb{R}$ by letting $\tilde{y} = y$ and $\alpha = 0$.

In Case 2, we define $\tilde{y}$ and α as follows. Let $\Omega_0 = \bigcup_{s\in\Gamma}\{y = s\}$ and let $\alpha = \mathbb{P}(\Omega_0)$, where Γ is the set of $s > 0$ such that $\mathbb{P}(y = s) > 0$. Since Ω is nonatomic, we can find a nonnegative random variable r such that $\{r > 0\} = \Omega_0$ and $r^*(t) = (\alpha - t)^+$ for all $t \in I$ (see [5, p. 44]). We then define $\tilde{y} = y + r$.

In any case, we have:

- $\mathbb{P}(\tilde{y} = s) = 0$ for all $s > 0$;
- $\{y > 0\} = \{\tilde{y} > 0\}$;
- $y \le \tilde{y} \le y + \alpha$ on Ω, and hence $y^* \le \tilde{y}^* \le y^* + \alpha$ on I.

In the rest of the proof of (i), we do not have to distinguish the two cases.

Define a sequence $\{t_n\}_{n=1}^{\infty}$ in I by setting

$$t_0 = \mathbb{P}(\tilde{y} > 0);$$
$$t_n = \sup\bigl\{s \in I \bigm| (\mathcal{P}\tilde{y}^*)(s) > 2(\mathcal{P}\tilde{y}^*)(t_{n-1})\bigr\} \quad (n = 1, 2, \ldots).$$

Then, since $y \notin L_\infty$ and $\mathcal{P}\tilde{y}^*$ is continuous, it is easy to verify that $0 < t_n < t_{n-1}$ for all $n \ge 1$, and

$$(\mathcal{P}\tilde{y}^*)(t_n) = 2(\mathcal{P}\tilde{y}^*)(t_{n-1}) \quad (n = 1, 2, \ldots). \tag{7}$$

From (7) it follows that $t_n \downarrow 0$ as $n \to \infty$. Let $\{A(t) \,|\, t \in I\}$ be a family of sets in Σ satisfying the four conditions of Lemma 2 (relative to x). Let

$$A_n = A(t_n), \;\; B_n = \bigl\{\omega \,|\, \tilde{y}(\omega) > \tilde{y}^*(t_n)\bigr\}, \;\text{ and }\; \Lambda_n = A_n \cup B_n$$

for each $n = 0, 1, 2 \ldots$. We define a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\ge 0}$ and a martingale as follows:

$$\begin{aligned} \mathcal{F}_n &= \sigma\{\Lambda \setminus \Lambda_n \,|\, \Lambda \in \Sigma\}, \\ f_n &= \mathbb{E}[\,x \,|\, \mathcal{F}_n\,], \end{aligned} \qquad (n = 0, 1, 2, \ldots). \tag{8}$$

Because $\mathbb{P}(A_n) = \mathbb{P}(B_n) = t_n$ and $A_n \cap B_n = \emptyset$, we see from (iii) of Lemma 2 that

$$f_n = \frac{1_{\Lambda_n}}{\mathbb{P}(\Lambda_n)}\,\mathbb{E}\bigl[x 1_{\Lambda_n}\bigr] + x 1_{\Omega\setminus\Lambda_n} = \frac{1}{2}(\mathcal{P}y^*)(t_n) 1_{\Lambda_n} + x 1_{\Omega\setminus\Lambda_n}.$$

Therefore

$$\Delta_n f = \begin{cases} \frac{1}{2}(\mathcal{P}y^*)(t_n) - \frac{1}{2}(\mathcal{P}y^*)(t_{n-1}) & \text{on } \Lambda_n, \\ x - \frac{1}{2}(\mathcal{P}y^*)(t_{n-1}) & \text{on } \Lambda_{n-1} \setminus \Lambda_n, \\ 0 & \text{on } \Omega \setminus \Lambda_{n-1}, \end{cases} \quad (n = 1, 2, \ldots).$$

Since $x = 0$ on B_n, it follows that

$$\Delta_n f = -\frac{1}{2}(\mathcal{P}y^*)(t_{n-1}) \quad \text{on } B_{n-1} \setminus B_n. \tag{9}$$

Using (7), (9), the continuity of $\mathcal{P}\tilde{y}^*$, and the nonincreasing property of $\tilde{y}^*$, we see that on $B_{n-1} \setminus B_n$

$$\begin{aligned} y \le \tilde{y} &\le \tilde{y}^*(t_n) \\ &\le (\mathcal{P}\tilde{y}^*)(t_n) = 2(\mathcal{P}\tilde{y}^*)(t_{n-1}) \\ &\le 2(\mathcal{P}y^*)(t_{n-1}) + 2\alpha = 4|\Delta_n f| + 2\alpha. \end{aligned}$$

Because $\{y > 0\} = \{\tilde{y} > 0\} = B_0$ and $B_n \downarrow \emptyset$ a.s., we deduce that a.s.

$$y = \sum_{n=1}^{\infty} y\,1_{B_{n-1}\setminus B_n} = 4\bigg(\sum_{n=1}^{\infty} |\Delta_n f|^2 1_{B_{n-1}\setminus B_n}\bigg)^{1/2} + 2\alpha \le 4S(f) + 2\alpha. \tag{10}$$

On the other hand, since $\mathbb{P}(\Lambda_n) = 2t_n \to 0$ as $n \to \infty$, we see that $f_\infty = x \in X$ or equivalently $f = (f_n) \in \mathcal{M}(X)$. Hence $S(f) \in X$ by hypothesis. Combining this with (10), we conclude that $y \in X$. This completes the proof of (i).

(ii) As shown above, X is r.i. Hence we may assume (see the discussion following Definition 2) that X is endowed with an r.i. norm. Let $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ be the Luxemburg representation of X. To prove that $\beta_X < 1$, it suffices to show that $\mathcal{P}\varphi \in \widehat{X}$ whenever $\varphi \in \mathcal{D}'_{\widehat{X}} \setminus L_\infty(I)$ (see Lemma 3).

Since $\mu(\varphi > 0) \le 1/2$ (and Ω is nonatomic), we can find nonnegative random variables x and y such that $x^* = y^* = \varphi$ on I and $\{x > 0\} \cap \{y > 0\} = \emptyset$. We then define $\tilde{y}$, α, $\{t_n\}$, $\{A_n\}$, $\{B_n\}$, $\{\Lambda_n\}$, $\mathcal{F} = (\mathcal{F}_n)$, and $f = (f_n)$ as in the proof of (i). Then

$$(\mathcal{P}\tilde{y}^*)(t_n) \le 4|\Delta_n f| + 2\alpha \quad \text{on} \quad B_{n-1} \setminus B_n,$$

as shown above. Therefore

$$\sum_{n=1}^{\infty} (\mathcal{P}\tilde{y}^*)(t_n) 1_{B_{n-1}\setminus B_n} \le 4S(f) + 2\alpha.$$

Observe that the nonincreasing rearrangement of the left-hand side is the function $s \mapsto \sum_{n=1}^{\infty} (\mathcal{P}\tilde{y}^*)(t_n) 1_{[t_n,\, t_{n-1})}(s)$. It is greater than or equal to $\mathcal{P}\tilde{y}^*$. Thus we find that

$$\mathcal{P}\varphi = \mathcal{P}y^* \le \mathcal{P}\tilde{y}^* \le \sum_{n=1}^{\infty} (\mathcal{P}\tilde{y}^*)(t_n) 1_{[t_n,\, t_{n-1})} \le 4S(f)^* + 2\alpha.$$

Since $x^* = \varphi \in \mathcal{D}_{\widehat{X}} \subset \widehat{X}$, we see that $f_\infty = x \in X$ (cf. (L1)). Hence $S(f) \in X$ by hypothesis. As a consequence, $\mathcal{P}\varphi \le 4S(f)^* + 2\alpha \in \widehat{X}$. This completes the proof of Proposition 1. □

Proof of Proposition 2. Assume that $\beta_X < 1$ and $\mathcal{H}(X) \subset \mathcal{M}(X)$. To prove Proposition 2, we may assume that X is endowed with an r.i. norm. According to Lemma 3, it suffices to show that $\mathcal{Q}\varphi \in \widehat{X}$ whenever $\varphi \in \mathcal{D}_{\widehat{X}}$. To this end, we may assume $\varphi \not\equiv 0$; hence $(\mathcal{Q}\varphi)(t) \to \infty$ as $t \to 0+$. Choose a random variable x so that $x^* = \mathcal{Q}\varphi$ on I and define a sequence $\{t_n\}$ in I by setting

$$t_0 = 1;$$
$$t_n = \sup\{s \in I \mid (\mathcal{P}x^*)(s) > (\mathcal{P}x^*)(t_{n-1}) + n^{-1}\} \quad (n = 1, 2, \ldots).$$

Then it is easy to verify that $0 < t_n < t_{n-1}$ and

$$(\mathcal{P}x^*)(t_n) = (\mathcal{P}x^*)(t_{n-1}) + n^{-1} \quad (n = 1, 2, \ldots). \tag{11}$$

Since $(\mathcal{P}x^*)(t_n) = (\mathcal{P}x^*)(t_0) + \sum_{j=1}^{n} j^{-1} \to \infty$, we see that $t_n \downarrow 0$ as $n \to \infty$. Let $\{A(t) \,|\, t \in I\}$ be a family of sets in Σ satisfying the four conditions of Lemma 2, and let $\Lambda_n = A(t_n)$ for each $n \geq 0$. Then, by (iv) of Lemma 2,

$$x^*(t_{n-1}) \leq x \leq x^*(t_n) \qquad \text{on } \Lambda_{n-1} \setminus \Lambda_n. \tag{12}$$

Define $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ and $f = (f_n)_{n\geq 0}$ by (8). Then it is easy to see that

$$f_n = (\mathcal{P}x^*)(t_n)1_{\Lambda_n} + x1_{\Omega\setminus\Lambda_n} \quad (n = 0, 1, 2, \ldots).$$

Therefore

$$|\Delta_n f| = \begin{cases} n^{-1} & \text{on } \Lambda_n, \\ \big|x - (\mathcal{P}x^*)(t_{n-1})\big| & \text{on } \Lambda_{n-1} \setminus \Lambda_n, \\ 0 & \text{on } \Omega \setminus \Lambda_{n-1}, \end{cases} \quad (n = 1, 2, \ldots). \tag{13}$$

Using (11) and (12) we find that, on $\Lambda_{n-1} \setminus \Lambda_n$,

$$\begin{aligned} -n^{-1} &\leq (\mathcal{P}x^*)(t_n) - x^*(t_n) - n^{-1} \\ &= (\mathcal{P}x^*)(t_{n-1}) - x^*(t_n) \\ &\leq (\mathcal{P}x^*)(t_{n-1}) - x \\ &\leq (\mathcal{P}x^*)(t_{n-1}) - x^*(t_{n-1}). \end{aligned}$$

Hence it follows that

$$\big|(\mathcal{P}x^*)(t_{n-1}) - x\,\big| \leq (\mathcal{P}x^*)(t_{n-1}) - x^*(t_{n-1}) + n^{-1} \qquad \text{on } \Lambda_{n-1} \setminus \Lambda_n.$$

Since $\mathcal{P}x^* - x^* = \mathcal{P}\mathcal{Q}\varphi - \mathcal{Q}\varphi = \mathcal{P}\varphi$ by (6a), we have

$$\big|(\mathcal{P}x^*)(t_{n-1}) - x\,\big| \leq (\mathcal{P}\varphi)(t_{n-1}) + n^{-1} \qquad \text{on } \Lambda_{n-1} \setminus \Lambda_n.$$

This, together with (13), implies that

$$\begin{aligned} |\Delta_n f| &\leq n^{-1}\, 1_{\Lambda_n} + \big\{(\mathcal{P}\varphi)(t_{n-1}) + n^{-1}\big\}1_{\Lambda_{n-1}\setminus\Lambda_n} \\ &= n^{-1}\, 1_{\Lambda_{n-1}} + (\mathcal{P}\varphi)(t_{n-1})1_{\Lambda_{n-1}\setminus\Lambda_n} \end{aligned}$$

for each $n \geq 1$. Since $|f_0| = \big|\mathbb{E}[\,x \,|\, \mathcal{F}_0\,]\big| = \|x\|_1 = \|\varphi\|_1$, it follows that

$$S(f) \leq \|\varphi\|_1 + K + \sum_{n=1}^{\infty} (\mathcal{P}\varphi)(t_{n-1})1_{\Lambda_{n-1}\setminus\Lambda_n}, \tag{14}$$

where $K = \left\{\sum_{n=1}^{\infty} n^{-2}\right\}^{1/2}$. Note that the nonincreasing rearrangement of the last sum in (14) is the function $s \mapsto \sum_{n=1}^{\infty}(\mathcal{P}\varphi)(t_{n-1})1_{[t_n,\, t_{n-1})}(s)$. Hence by (14),

$$S(f)^* \le \|\varphi\|_1 + K + \sum_{n=1}^{\infty}(\mathcal{P}\varphi)(t_{n-1})1_{[t_n,\, t_{n-1})} \le \|\varphi\|_1 + K + \mathcal{P}\varphi.$$

Note that right-hand side belongs to $\widehat{X}$. Indeed, since $\beta_{\widehat{X}} = \beta_X < 1$ and $\varphi \in \widehat{X}$, Shimogaki's Theorem shows that $\mathcal{P}\varphi \in \widehat{X}$. Therefore $S(f) \in X$. Since $\mathcal{H}(X) \subset \mathcal{M}(X)$, we conclude that $x = f_\infty \in X$, or equivalently that $\mathcal{Q}\varphi = x^* \in \widehat{X}$. This completes the proof. □

We now turn to the proof of Proposition 3. As mentioned before, Proposition 3 follows from the results of [1]. We give here another proof. We begin with a lemma which extends Garsia's lemma. For notation and terminology see, e.g., [6].

LEMMA 4 ([8]). *Let $(x_n)_{n\ge 0}$ be a nondecreasing sequence of nonnegative random variables adapted to a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\ge 0}$, let $x_\infty = \lim_{n\to\infty} x_n$, and let y be a nonnegative integrable random variable. If the inequality*

$$\mathbb{E}[\, x_\infty - x_{\tau-1} \,|\, \mathcal{F}_\tau \,] \le \mathbb{E}[\, y \,|\, \mathcal{F}_\tau \,] \tag{15}$$

holds a.s. for every $\mathcal{F}$-stopping time τ, with the convention that $x_{-1} = 0$, then $x_\infty^* \prec \mathcal{Q}y^*$.

Proof. Let $t \in I$ and $t' = \inf\left\{s \in I \,\middle|\, x_\infty^*(s) = x_\infty^*(t)\right\}$. Then $0 \le t' \le t$, $(0, t') = \left\{s \in I \,\middle|\, x_\infty^*(s) > x_\infty^*(t)\right\}$, and $x_\infty^*(s) = x_\infty^*(t)$ whenever $t' \le s \le t$. Applying (15) to the stopping time $\tau = \inf\left\{n \ge 0 \,\middle|\, x_n > x_\infty^*(t)\right\}$ and using the Hardy-Littlewood inequality (cf. (3)), we have

$$\begin{aligned}
\int_0^t x_\infty^*(s)\, ds - t\, x_\infty^*(t) &= \int_0^{t'} \big(x_\infty^*(s) - x_\infty^*(t)\big)\, ds \\
&= \mathbb{E}\big[\big(x_\infty - x_\infty^*(t)\big)1_{\{x_\infty > x_\infty^*(t)\}}\big] \\
&\le \mathbb{E}\big[(x_\infty - x_{\tau-1})1_{\{\tau<\infty\}}\big] \\
&\le \mathbb{E}\big[\, y\, 1_{\{x_\infty > x_\infty^*(t)\}}\big] \\
&\le \int_0^t y^*(s)\, ds.
\end{aligned}$$

Thus $\mathcal{P}x_\infty^* - x_\infty^* \le \mathcal{P}y^*$ on I. Therefore it follows from (6a) and (6b) that

$$\mathcal{P}x_\infty^* - \|x_\infty^*\|_1 = \mathcal{Q}(\mathcal{P}x_\infty^* - x_\infty^*) \le \mathcal{Q}\mathcal{P}y^* = \mathcal{P}\mathcal{Q}y^* - \|y^*\|_1 \,. \tag{16}$$

On the other hand, setting $\tau \equiv 0$ in (15) yields that

$$\|x_\infty^*\|_1 = \|x\|_1 \le \|y\|_1 = \|y^*\|_1 \,.$$

Combining this with (16), we conclude that $\mathcal{P}x_\infty^* \le \mathcal{P}\mathcal{Q}y^*$ on I, or equivalently that $x_\infty^* \prec \mathcal{Q}y^*$. $\square$

Lemma 5. *Let x and y be nonnegative integrable random variables. If the inequality*

$$\lambda\mathbb{P}(x \ge \lambda) \le \int_{\{x\ge\lambda\}} y\,d\mathbb{P} \tag{17}$$

holds for any $\lambda > 0$, then $x^* \le \mathcal{P}y^*$ on I.

Proof. Let $t \in I$ and $t' = \mathbb{P}\big(x \ge x^*(t)\big)$; then $t' \ge t$. Setting $\lambda = x^*(t)$ in (17) and using the Hardy-Littlewood inequality (cf. (3)), we obtain

$$x^*(t) \le \frac{1}{t'}\int_{\{x\ge x^*(t)\}} y\,d\mathbb{P} \le (\mathcal{P}y^*)(t') \le (\mathcal{P}y^*)(t)$$

as desired. $\square$

Proof of Proposition 3. Suppose $0 < \alpha_X \le \beta_X < 1$. Then both $\mathcal{P}$ and $\mathcal{Q}$ are bounded operators from $\widehat{X}$ into $\widehat{X}$. To prove (4), we may assume that X is endowed with an r.i. norm. Recall (the conditional form of) Davis' inequality: there are constants $k > 0$ and $k' > 0$ such that

$$\mathbb{E}\big[\,M(f) - M_{\tau-1}(f) \,\big|\, \mathcal{F}_\tau\,\big] \le k\,\mathbb{E}\big[\,S(f) \,\big|\, \mathcal{F}_\tau\,\big] \quad \text{a.s.,} \quad \text{and}$$
$$\mathbb{E}\big[\,S(f) - S_{\tau-1}(f) \,\big|\, \mathcal{F}_\tau\,\big] \le k'\,\mathbb{E}\big[\,M(f) \,\big|\, \mathcal{F}_\tau\,\big] \quad \text{a.s.}$$

for all $f \in \mathcal{M}_\mathcal{F}$ and for all $\mathcal{F}$-stopping times τ (see, e.g., [6, p. 286] or [10, p. 89]). It then follows from Lemma 4 that $M(f)^* \prec k\,\mathcal{Q}S(f)^*$ and $S(f)^* \prec k'\mathcal{Q}M(f)^*$.[‡] Therefore, by (L2) and (R2′), we have

$$\big\|M(f)\big\|_X = \big\|M(f)^*\big\|_{\widehat{X}} \le k\big\|\mathcal{Q}S(f)^*\big\|_{\widehat{X}} \le k\,\|\mathcal{Q}\|_{B(\widehat{X})}\,\big\|S(f)\big\|_X \tag{18}$$

and

$$\big\|S(f)\big\|_X = \big\|S(f)^*\big\|_{\widehat{X}} \le k'\big\|\mathcal{Q}M(f)^*\big\|_{\widehat{X}} \le k'\,\|\mathcal{Q}\|_{B(\widehat{X})}\,\big\|M(f)\big\|_X. \tag{19}$$

Now we recall Doob's inequality (see, e.g., [10, p. 34]): for any $f \in \mathcal{M}$,

$$\lambda\mathbb{P}\big(M(f) > \lambda\big) \le \int_{\{M(f)>\lambda\}} |f_\infty|\,d\mathbb{P} \qquad (\lambda > 0).$$

It then follows from Lemma 5 that $M(f)^* \le \mathcal{P}f_\infty^*$ on I. Therefore

$$\big\|M(f)\big\|_X = \big\|M(f)^*\big\|_{\widehat{X}} \le \|\mathcal{P}\|_{B(\widehat{X})}\,\|f_\infty\|_X \quad (f \in \mathcal{M}). \tag{20}$$

Combining (18), (19), and (20), we obtain (4) with $c = \big(k\,\|\mathcal{Q}\|_{B(\widehat{X})}\big)^{-1}$ and $C = k'\,\|\mathcal{Q}\|_{B(\widehat{X})}\,\|\mathcal{P}\|_{B(\widehat{X})}$. $\square$

[‡] To prove (4), we can assume that $f \in H_1$; hence $\mathcal{Q}S(f)^*$ and $\mathcal{Q}M(f)^*$ can be defined.

4. Application to weighted norm inequalities

Let $\Phi : [0, \infty) \to [0, \infty)$ be an *N-function*, namely, an increasing convex function such that:

- $\Phi(u) = 0$ if and only if $u = 0$;
- $\displaystyle\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$;
- $\displaystyle\lim_{u\to 0+} \frac{\Phi(u)}{u} = 0$.

Then the *complementary function* Ψ, which is given by

$$\Psi(u) = \sup\bigl\{uv - \Phi(v) \bigm| v \geq 0\bigr\} \quad (v \geq 0),$$

is also an N-function. We say that Φ satisfies the Δ_2*-condition* and write $\Phi \in \Delta_2$ if there exist constants $k > 0$ and $u_0 \geq 0$ such that $\Phi(2u) \leq k\,\Phi(u)$ for $u \geq u_0$. We say that Φ satisfies the ∇_2*-condition* and write $\Phi \in \nabla_2$ if $\Psi \in \Delta_2$. Then $\Phi \in \nabla_2$ if and only if there exist constants $l > 1$ and $v_0 \geq 0$ such that $\Phi(v) \leq (2l)^{-1}\,\Phi(lv)$ for $v \geq v_0$ (see [9, p. 25]).

Let L_Φ be the Orlicz space over $(\Omega, \Sigma, \mathbb{P})$ endowed with the *Luxemburg norm* $\|\cdot\|_\Phi$ (see [9, p. 78]), and denote by α_Φ and β_Φ the lower and upper Boyd indices of L_Φ. It is known that $\alpha_\Phi > 0$ if and only if $\Phi \in \Delta_2$ (see [12, Theorems 3.2 and 4.2.]). Moreover, since $\alpha_\Psi + \beta_\Phi = 1$, it follows that $\beta_\Phi < 1$ if and only if $\Phi \in \nabla_2$.

Now let w be a *weight random variable,* i.e., let w be a (strictly) positive and integrable random variable. We assume that $\mathbb{E}[w] = 1$ and consider the probability measure

$$\mathbb{P}_w(\Lambda) = \mathbb{E}[w1_\Lambda] \qquad (\Lambda \in \Sigma).$$

Let $(L_{\Phi,w}, \|\cdot\|_{\Phi,w})$ be the Orlicz space over $(\Omega, \Sigma, \mathbb{P}_w)$ endowed with the Luxemburg norm relative to $\mathbb{P}_w$. Denoting by ψ the right-derivative of Ψ, we claim that if $\psi(w^{-1}) \in L_1$, then $L_\infty \hookrightarrow L_{\Phi,w} \hookrightarrow L_1$, where L_1 and L_∞ are Lebesgue spaces with respect to $\mathbb{P}$. The first embedding is evident. To see the second embedding, suppose that $x \in L_{\Phi,w}$ and $\|x\|_{\Phi,w} \leq 1$. Then

$$\mathbb{E}\bigl[|x|\bigr] \leq \mathbb{E}\bigl[\,\Phi(|x|)\,w\,\bigr] + \mathbb{E}\bigl[\,\Psi(w^{-1})\,w\,\bigr] \leq 1 + \mathbb{E}\bigl[\,\psi(w^{-1})\,\bigr] =: M < \infty.$$

Here we have used the Young inequality $uv \leq \Phi(u) + \Psi(v)$ and the inequality $\Psi(v) \leq v\psi(v)$. Thus $\|\cdot\|_1 \leq M\,\|\cdot\|_{\Phi,w}$ as claimed.

With the notation above, we have:

Theorem 2. *Suppose that $\Phi \in \Delta_2$ and $\psi(w^{-1}) \in L_1$. Then the following are equivalent*:

(i) *there are constants c and C such that*

$$c\,\|f_\infty\|_{\Phi,w} \le \|S(f)\|_{\Phi,w} \le C\,\|f_\infty\|_{\Phi,w} \qquad (f \in \mathcal{M}); \tag{21}$$

(ii) (a) *there are constants c_1 and c_2 such that $c_1 \le w \le c_2$ a.s., and*
(b) $\Phi \in \nabla_2$.

Before proving Theorem 2, we recall that

$$\begin{aligned} \int_0^t w^*(s)\,ds &= \max\left\{ \int_\Lambda w\,d\mathbb{P} \;\middle|\; \Lambda \in \Sigma,\ \mathbb{P}(\Lambda) = t \right\} \\ \int_0^t w_*(s)\,ds &= \min\left\{ \int_\Lambda w\,d\mathbb{P} \;\middle|\; \Lambda \in \Sigma,\ \mathbb{P}(\Lambda) = t \right\} \end{aligned} \qquad (t \in I), \tag{22}$$

where $w_*(s) = w^*(1-s)$ (see [5, p. 47]).

Proof. (ii) $\Rightarrow$ (i). Condition (a) shows that $\|\cdot\|_\Phi \approx \|\cdot\|_{\Phi,w}$ and condition (b) shows that $\beta_\Phi < 1$. Furthermore $\alpha_X > 0$, since $\Phi \in \Delta_2$ by hypothesis. Hence we obtain (21) from Proposition 3.

(i) $\Rightarrow$ (ii). Suppose that (i) holds. Then $L_{\Phi,w}$ is r.i. with respect to $\mathbb{P}$ (or briefly, "$\mathbb{P}$-r.i.") by Theorem 1. Hence there exists a $\mathbb{P}$-r.i. norm $|||\cdot|||_{\Phi,w}$ on $L_{\Phi,w}$ such that $k_1\,\|\cdot\|_{\Phi,w} \le |||\cdot|||_{\Phi,w} \le k_2\,\|\cdot\|_{\Phi,w}$ with some constants $k_1 > 0$ and $k_2 > 0$. By hypothesis, there exists $u_0 \ge 0$ and $K \ge 1$ such that

$$\Phi\left(\frac{k_2\,u}{k_1}\right) \le K\,\Phi(u) \quad \text{for all} \ \ u \ge u_0. \tag{23}$$

Since $w \in L_1$, we can find a positive number δ such that $\mathbb{P}_w(\Lambda) \le 1/\Phi(u_0)$ whenever $\mathbb{P}(\Lambda) < \delta$. Suppose now that $\Lambda, \Lambda' \in \Sigma$ and $0 < \mathbb{P}(\Lambda) = \mathbb{P}(\Lambda') = t < \delta$. Then $1_\Lambda^* = 1_{\Lambda'}^*$ and $\mathbb{P}_w(\Lambda) \le 1/\Phi(u_0)$. Furthermore,

$$\begin{aligned} k_1\left\{\Phi^{-1}\left(\frac{1}{\mathbb{P}_w(\Lambda)}\right)\right\}^{-1} &= k_1\,\|1_\Lambda\|_{\Phi,w} \\ &\le |||1_\Lambda|||_{\Phi,w} = |||1_{\Lambda'}|||_{\Phi,w} \\ &\le k_2\,\|1_{\Lambda'}\|_{\Phi,w} = k_2\left\{\Phi^{-1}\left(\frac{1}{\mathbb{P}_w(\Lambda')}\right)\right\}^{-1}, \end{aligned}$$

or equivalently

$$\Phi^{-1}\left(\frac{1}{\mathbb{P}_w(\Lambda')}\right) \le \frac{k_2}{k_1}\,\Phi^{-1}\left(\frac{1}{\mathbb{P}_w(\Lambda)}\right). \tag{24}$$

Using (23) and (24), we obtain that

$$\int_\Lambda w\,d\mathbb{P} = \mathbb{P}_w(\Lambda) \le K\,\mathbb{P}_w(\Lambda') = K\int_{\Lambda'} w\,d\mathbb{P}.$$

Hence we may use (22) to deduce that

$$\frac{1}{t}\int_0^t w^*(s)\,ds \leq \frac{K}{t}\int_0^t w_*(s) \qquad (0 < t < \delta).$$

Letting $t \to 0+$, we conclude that $\operatorname{ess\,sup} w \leq K \operatorname{ess\,inf} w$. This means that there exist constants c_1 and c_2 such that $c_1 \leq w \leq c_2$ a.s. Therefore (21) can be written as

$$c' \, \|f_\infty\|_\Phi \leq \|S(f)\|_\Phi \leq C' \, \|f_\infty\|_\Phi \qquad (f \in \mathcal{M})$$

with some constants c' and C'. According to Theorem 1, the upper Boyd index β_Φ must be less than one, or equivalently Φ must satisfy the ∇_2-condition. This completes the proof. □

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