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DIFFERENTIAL TRANSCENDENCE OF A CLASS OF GENERALIZED DIRICHLET SERIES

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ABSTRACT. We investigate differential transcendence properties for a generalized Dirichlet series of the form $\sum_{n=0}^{\infty} a_n\lambda_n^{-s}$. Our treatment of this series is purely algebraic and does not rely on any analytic properties of generalized Dirichlet series. We establish differential transcendence theorems for a certain class of generalized Dirichlet series. These results imply that the Hurwits zeta-function $\zeta(s,a)$ does not satisfy an algebraic differential equation with complex coefficients.

1. Introduction and statement of results

Let λ_n ($n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$) be a strictly increasing sequence of positive numbers which tends to infinity as n tends to infinity, and let a_n ($n \in \mathbb{N}_0$) be an arbitrary sequence of complex numbers. We consider a series of the form

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda_n^s}, \tag{1.1}$$

which is called a (formal) Dirichlet series. Here s is an abstract symbol (or a complex variable if the series converges for some complex number $s = s_0$). Let a be a positive real parameter. The series

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \tag{1.2}$$

which is a particular case of (1.1), is called the Hurwitz zeta-function. Note that $\zeta(s,1) = \zeta(s)$ is the Riemann zeta-function. The aim of the present paper is to study the differential transcendence of a certain class of generalized Dirichlet series, which includes the Hurwitz zeta-functions. Our main results are stated at the end of this section. We emphasize that our arguments are

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purely algebraic and do not appeal to any analytic properties of generalized Dirichlet series.

We denote by $\mathcal{G}$ the set of all generalized Dirichlet series. The complex field $\mathbb{C}$ can be regarded as a subset of $\mathcal{G}$ by defining $1^s = 1$. Then the set $\mathcal{G}$ forms an algebra over $\mathbb{C}$ under the following operations of addition and multiplication (or convolution). For any elements $f(s) = \sum_{l=0}^{\infty} a_l \lambda_l^{-s}$ and $g(s) = \sum_{m=0}^{\infty} b_m \mu_m^{-s}$ in $\mathcal{G}$, let κ_n $(n \in \mathbb{N}_0)$ be the strictly increasing sequence consisting of all elements λ_l and μ_m $(l, m \in \mathbb{N}_0)$, and let ν_n $(n \in \mathbb{N}_0)$ be the strictly increasing sequence consisting of all elements of the form $\lambda_l \mu_m$ $(l, m \in \mathbb{N}_0)$. Then the addition and the multiplication of f and g are defined by

$$f(s) + g(s) = \sum_{n=0}^{\infty} c_n \kappa_n^{-s}, \qquad c_n = a'_n + b'_n$$

and

$$f(s)g(s) = \sum_{n=0}^{\infty} d_n \nu_n^{-s}, \qquad d_n = \sum_{\substack{l,m \in \mathbb{N}_0 \\ \lambda_l \mu_m = \nu_n}} a_l b_m,$$

respectively, where $a'_n = a_k$ if $\lambda_k = \kappa_n$ for some k and $a'_n = 0$ otherwise, and b'_n is defined analogously. Further, the derivative with respect to s of the series $f(s)$ given by (1.1) is defined by

$$f'(s) = \sum_{n=0}^{\infty} \frac{a_n(-\log \lambda_n)}{\lambda_n^s},$$

and for any $j \in \mathbb{N}$ the j-th derivative $f^{(j)}(s)$ is defined inductively as the j-th iterate of the derivative.

Let r be a positive integer and let $\mathcal{H}$ be any subalgebra of $\mathcal{G}$. A collection of generalized Dirichlet series $f_1, \ldots, f_r$ is called differentially algebraically independent over $\mathcal{H}$ if there are no nontrivial algebraic relations with coefficients in $\mathcal{H}$ among the series $f_l^{(j)}$ $(j = 0, 1, 2, \ldots; l = 1, 2, \ldots, r)$. In particular, if this condition holds in the case $r = 1$, the series f_1 is said to be differentially transcendental over $\mathcal{H}$.

We define the (formal) Dirichlet series ring $\mathcal{D}$, a subalgebra of $\mathcal{G}$, as the set of all series $f \in \mathcal{G}$ of the form $f = \sum_{n=1}^{\infty} a_n n^{-s}$. Note that the Riemann zeta-function $\zeta(s)$ belongs to this subalgebra. The study of differential algebraic independence of elements of $\mathcal{D}$ has a long history, which dates back to a problem of Hilbert posed at the International Congress of Mathematicians held on 1902 (see Ostrowski [4]). Let $\mathcal{A}$ be the set of all arithmetic functions, which forms a ring under the usual addition, subtraction and convolution. It is readily seen that $\mathcal{A}$ is isomorphic to $\mathcal{D}$ through the homomorphism

$$\mathcal{A} \ni a(n) \longmapsto \sum_{n=1}^{\infty} a(n) n^{-s} \in \mathcal{D}.$$

Popken [5] studied independence problems on the algebra $\mathcal{A}$. More systematic and thorough investigations of the structure of $\mathcal{A}$ (and hence $\mathcal{D}$) have been carried out by Shapiro [6], and Shapiro and Sparer [7]. Laohakosol [2] gave a refinement of an independence criterion for arithmetic functions, due to Popken [5], and proved certain differential independence results in quantitative form for the elements of $\mathcal{D}$.

A Dirichlet series $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$ is called a Dirichlet polynomial if $a_n = 0$ for all sufficiently large n, and the subring of $\mathcal{D}$ consisting of all Dirichlet polynomials is denoted by $\mathcal{D}_0$. One of the simplest results in this direction is the following theorem (see, for e.g., [2, Theorem 4]).

THEOREM A. *Let $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$ be such that the set*

$$\{p \;\; \textit{prime}: \;\; p \textit{ divides some } n \textit{ with } a_n \neq 0\}$$

is an infinite set. Then f is differentially transcendental over $\mathbb{C}$.

Let $U = \{u_n\}_{n \in \mathbb{N}}$ be a sequence of complex numbers. The U-operation ∂_U on $\mathcal{D}$ is defined for $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$ by

$$(\partial_U f)(s) = \sum_{n=1}^{\infty} \frac{a_n u_n}{n^s},$$

and for any $j \in \mathbb{N}$, $\partial_U^j f$ denotes the j-th iterate of the U-operation. We note that ∂_U becomes the derivative (with respect to s) when $u_n = -\log n$. Following an argument in [7], one can easily reformulate the statement of Theorem A by using the U-operation and $\mathcal{D}_0$ instead of the derivative (with respect to s) and $\mathbb{C}$, respectively.

THEOREM B. *Let $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{D}$ be such that the set*

$$\{p \textit{ prime}: \;\; p \textit{ divides some } n \textit{ with } a_n \neq 0\}$$

is an infinite set. Let $U = \{u_n\}_{n \in \mathbb{N}}$ be a sequence of complex numbers that are pairwise distinct. Then, for any $r \in \mathbb{N}_0$, the series $f, \partial_U f, \partial_U^2 f, \ldots, \partial_U^r f$ are algebraically independent over $\mathcal{D}_0$.

In the present paper we generalize Theorem A to a certain class of (formal) generalized Dirichlet series, which includes a family of Hurwitz zeta functions $\zeta(s, a)$.

Our first theorem implies the differential transcendence of $\zeta(s, a)$ for transcendental a.

THEOREM 1. *Let $f = \sum_{n=0}^{\infty} a_n \lambda_n^{-s}$ be a generalized Dirichlet series such that $a_n \neq 0$ for infinitely many $n \in \mathbb{N}_0$. Suppose that λ_n $(n \in \mathbb{N}_0)$ and p_n $(n \in \mathbb{N})$ are multiplicatively independent, where p_n is the n-th prime number. Then f is differentially transcendental over $\mathcal{D}_0$.*

Let K be an algebraic number field of finite degree, and let $\mathcal{O}_K$ be the ring of integers in K. For any $\alpha \in K$ we write $N(\alpha) = \mathrm{Norm}_{K/\mathbb{Q}}(\alpha)$. Our second theorem implies the differential transcendence of $\zeta(s, a)$ for algebraic a.

THEOREM 2. *Let $f = \sum_{n=0}^{\infty} a_n \lambda_n^{-s}$ be a generalized Dirichlet series which satisfies the following four conditions:*

(i) *$\lambda_n \in K$ for all $n \in \mathbb{N}_0$;*
(ii) *there exists a positive integer D such that $D\lambda_n \in \mathcal{O}_K$ for all $n \in \mathbb{N}_0$;*
(iii) *the sequence $\{|N(\lambda_n)|\}_{n\in\mathbb{N}_0}$ is strictly increasing except for finitely many initial terms;*
(iv) *if D is a positive integer satisfying* (ii)*, the set*

$$\{p \ \ \text{prime}: \ \ p \text{ divides } N(D\lambda_n) \text{ for some } n \text{ with } a_n \neq 0\}$$

is an infinite set.

Then f is differentially transcendental over $\mathcal{D}_0$.

We give two corollaries of the theorems. To state the first corollary, which is a consequence of Theorem 2, we extend the notion of the norm to any polynomial $P(x) = \sum_{j=0}^{m} \alpha_j x^j \in K[X]$ by setting

$$(\mathrm{Norm}_{K/\mathbb{Q}}\, P)(x) := \prod_{i=1}^{d} \sum_{j=0}^{m} \sigma_i(\alpha_j) x^j,$$

where $\sigma_1, \ldots, \sigma_d$ are the automorphisms from K into $\mathbb{C}$.

COROLLARY 1. *Let K be a real algebraic number field of finite degree over $\mathbb{Q}$, and let $P(x)$ be a non-constant polynomial with coefficients in K such that $P(n) > 0$ for all $n \in \mathbb{N}_0$. Let a_n $(n \in \mathbb{N}_0)$ be a sequence of complex numbers satisfying $a_n \neq 0$ for all sufficiently large $n \in \mathbb{N}_0$. Then the generalized Dirichlet series*

$$f(s) = \sum_{n=0}^{\infty} \frac{a_n}{P(n)^s}$$

is differentially transcendental over $\mathcal{D}_0$. If we suppose in addition that the norm for $P(x)$ contains a factor $Q(x) \in \mathbb{Z}[x]$ with at least two simple roots, then the same conclusion holds under the weaker condition that $a_n \neq 0$ for infinitely many n.

REMARK. The formal series $f(s)$ in Corollary 1 is, in fact, a generalized Dirichlet series since the set $\{n' \in \mathbb{N}_0 : \ P(n) = P(n')\}$ is a finite set for any $n \in \mathbb{N}_0$.

If $\lambda_n = n + a$ $(n \in \mathbb{N}_0)$ with a positive real number a, we can apply Theorem 1 or Theorem 2 according as a is transcendental or algebraic, and then obtain the following corollary.

COROLLARY 2. *Let a be a positive real number, and let a_n $(n \in \mathbb{N}_0)$ be a sequence of complex numbers satisfying $a_n \neq 0$ for infinitely many n. Then the generalized Dirichlet series*

$$g(s) = \sum_{n=0}^{\infty} \frac{a_n}{(n+a)^s} \tag{1.3}$$

is differentially transcendental over $\mathcal{D}_0$.

REMARK. Let x be a real parameter. The Lerch zeta-function $L(x,a,s)$ is a particular case by (1.3), with the choice $a_n = e^{2\pi i n x}$. Using the universality properties of Lerch zeta-functions, Garunkštis and Laurinčikas [1] recently established the functional independence of $L(x,a,s)$ when a is rational or transcendental (see also the very recent work of Laurinčikas and Matsumoto [3]). For instance, when a is transcendental, their result asserts that if F_m, $m = 0, 1, \ldots, n$, are continuous functions and for all s

$$\sum_{m=0}^{n} s^m F_m(L(x,a,s), L'(x,a,s), \ldots, L^{(N-1)}(x,a,s)) = 0,$$

then $F_m \equiv 0$ for $m = 0, 1, \ldots, n$. Here N is an arbitrary positive integer, and $L^{(j)}(x,a,s)$, $j = 0, 1, \ldots, N-1$, denotes the j-th derivative with respect to the variable s.

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We prove Theorem 1 in the next section. Theorem 2 is proved in Section 3, and the final section is devoted to the proofs of the corollaries.

2. Proof of Theorem 1

In order to prove Theorem 1, we need a slight generalization of the Jacobian criterion for algebraic independence of elements in $\mathcal{D}$, due to Shapiro and Sparer (see Theorem 3.1 of [7, Section 3]).

Let $\mathcal{H}$ be an arbitrary subalgebra of $\mathcal{G}$. A derivation d over $\mathcal{H}$ is defined to be a mapping from $\mathcal{H}$ into itself satisfying

$$d(f_1 f_2) = d(f_1) f_2 + f_1 d(f_2), \qquad d(c_1 f_1 + c_2 f_2) = c_1 d(f_1) + c_2 d(f_2)$$

for all f_1 and f_2 in $\mathcal{H}$, and any complex constants c_1 and c_2. For any given elements $f_1, f_2, \ldots, f_r$ in $\mathcal{H}$, the Jacobian of the f_i relative to the d_i is defined as the $r \times r$ determinant

$$J(f_1, \ldots, f_r | d_1, \ldots, d_r) = \det(d_i(f_j))_{1 \le i,j \le r}.$$

Lemma 1. *Let $f_1,\dots,f_r$ be given elements in $\mathcal{H}$, and let $d_1,\dots,d_r$ be derivations over $\mathcal{H}$ which annihilate all elements of a subring $\mathcal{K}$ of $\mathcal{H}$. If $J(f_1,\dots,f_r|d_1,\dots,d_r)\neq 0$, then the elements $f_1,\dots f_r$ are algebraically independent over $\mathcal{K}$.*

Remark. The proof of [7, Theorem 3.1] remains valid if $\mathcal{A}$ and $\mathcal{E}$ in the proof are replaced, as above, by $\mathcal{H}$ and $\mathcal{K}$, respectively.

We proceed to the proof of Theorem 1. Let $f=\sum_{n=0}^{\infty} a_n\lambda_n^{-s}$ be a generalized Dirichlet series satisfying the assumptions of Theorem 1. By removing the subsequence of those numbers λ_n for which $a_n = 0$, we may suppose without loss of generality that $a_n\neq 0$ for all $n\in\mathbb{N}_0$.

Let r be a nonnegative integer, and let $P(X_0,X_1,\dots,X_r)$ be a nonzero polynomial with coefficients in $\mathcal{D}_0$. We will show that

$$P(f,f',f'',\dots,f^{(r)})\neq 0. \tag{2.1}$$

Set $S_n=p_n^{-s}$ for $n=1,2,3,\dots$, where p_n denotes the n-th prime number. Then there exists a positive integer l such that the coefficients of P belong to $\mathbb{C}[S_1,S_2,\dots,S_l]$. Let $T_n=\lambda_n^{-s}$ $(n\in\mathbb{N}_0)$ be the infinite sequence of variables. Since λ_n $(n\in\mathbb{N}_0)$ and p_n $(n\in\mathbb{N})$ are multiplicatively independent, the numbers T_n $(n\in\mathbb{N}_0)$ are algebraically independent over $\mathcal{K}=\mathbb{C}[S_1,S_2,\dots,S_l]$. The ring $\mathcal{H}=\mathcal{K}[[T_n:\ n\in\mathbb{N}_0]]$ $(\subset\mathcal{G})$ can therefore be regarded as a formal power series ring over $\mathcal{K}$ with T_n $(n\in\mathbb{N}_0)$ as the variables. This allows us to define, for any $n\in\mathbb{N}_0$, a derivation $\partial_n=\partial/\partial T_n$ over $\mathcal{H}$, which is the usual differentiation with respect to T_n. We consider now the Jacobian $J_m=J(f,f',\dots,f^{(r)}|\partial_m,\partial_{m+1},\dots,\partial_{m+r})$ for a positive integer m satisfying $\lambda_n>1$ for all $n\geq m$. On writing $b_n=-\log\lambda_n$ we have

$$\begin{aligned} J_m &= \det(\partial_{m+i}(f^{(j)})_{0\leq i,j\leq r} = \det(a_{m+i}b_{m+i}^{j})_{0\leq i,j\leq r}\\ &= a_m\cdots a_{m+r}\det(b_{m+i}^{j})_{0\leq i,j\leq r}\neq 0. \end{aligned}$$

By Lemma 1, this implies the algebraic independence of $f,f',\dots,f^{(r)}$ over $\mathcal{K}$, and hence (2.1). The proof of Theorem 1 is therefore complete. □

3. Proof of Theorem 2

We first assume that $\lambda_0>1$ and $\lambda_n\in\mathcal{O}_K$ with $|N(\lambda_n)|>1$ for all $n\in\mathbb{N}_0$, and set $\Lambda=\{\lambda_n\}_{n\in\mathbb{N}_0}$. Before beginning the proof, we introduce some notation.

Let $\mathcal{I}$ be the set of all finite non-decreasing sequences in $\mathbb{N}_0$ with the convention that $\mathcal{I}$ includes the empty sequence $\emptyset$. For any $I=(i_1,i_2,\dots,i_m)\in\mathcal{I}$ we define

$$\tau(I)=\tau(I;\Lambda)=\prod_{l=1}^{m}\lambda_{i_l}$$

with the convention $\tau(\emptyset) = 1$. Then the following lemma follows immediately form conditions (i), (ii) and (iii) of Theorem 2.

LEMMA 2. *For any $I \in \mathcal{I}$ and any $n \in \mathbb{N}$ we have:*
(i) $\#\{I' \in \mathcal{I} :\ \tau(I') = \tau(I)\} < \infty$;
(ii) $\#\{(n', I') \in \mathbb{N} \times \mathcal{I} :\ n'\tau(I') = n\tau(I)\} < \infty$;
(iii) $\#\{(n', I') \in \mathbb{N} \times \mathcal{I} :\ |N(n'\tau(I'))| = |N(n\tau(I))|\} < \infty$.

We introduce an equivalence relation on the set $\mathcal{I}$ with respect to Λ as follows. Let I and I' be elements of $\mathcal{I}$. We say that I is equivalent to I' if and only if $\tau(I) = \tau(I')$ holds. The equivalence class of I is denoted by $[I]$, and the set of all equivalence classes by $\mathcal{I}/\Lambda$. By (i) of Lemma 2 any series of the form

$$\sum_{[I]\in\mathcal{I}/\Lambda} \frac{a([I])}{\tau(I)^s} \qquad (a([I]) \in \mathbb{C})$$

makes sense as an element of $\mathcal{G}$, and the set of all such elements is denoted by $\mathcal{H}_\Lambda$.

We now prove Theorem 2 under the assumption stated above. Let r be a nonnegative integer, and let

$$P(X_0, X_1, \ldots, X_r) = \sum_{J=(j_0,\ldots,j_r)\in\mathcal{J}_0} P_J X_0^{j_0} X_1^{j_1} \cdots X_r^{j_r}$$

be a nonzero element of $\mathcal{D}_0[X_0, X_1, \ldots, X_r]$, where

$$P_J = \sum_{m=1}^{m(J)} \frac{p_J(m)}{m^s} \in \mathcal{D}_0 \qquad (m(J) \in \mathbb{N};\ p_J(m) \in \mathbb{C}),$$

and $\mathcal{J}_0$ is a finite subset of $\mathcal{J}$ defined by

$$\mathcal{J} = \{J = (j_0, j_1, \ldots, j_r) :\ j_l \in \mathbb{N}_0\ (0 \le l \le r)\}.$$

Then we show that

$$F := P(f, f', f'', \ldots, f^{(r)}) \neq 0.$$

For any $J = (j_0, j_1, \ldots, j_r) \in \mathcal{J}$ one can represent $f^{j_0}(f')^{j_1} \cdots (f^{(r)})^{j_r}$ in the form

$$f^{j_0}(f')^{j_1} \cdots (f^{(r)})^{j_r} = \sum_{[I]\in\mathcal{I}/\Lambda} \frac{a_J([I])}{\tau(I)^s} \qquad (a_J([I]) \in \mathbb{C}),$$

and hence

$$F = \sum_{J\in\mathcal{J}_0} \sum_{m=1}^{m(J)} \frac{p_J(m)}{m^s} \sum_{[I]\in\mathcal{I}/\Lambda} \frac{a_J([I])}{\tau(I)^s}.$$

Let M be a positive integer such that $M \ge \max_{J\in\mathcal{J}_0} m(J)$. Then by (ii) of Lemma 2 we can consider the algebra

$$\mathcal{H} := \mathcal{H}_\Lambda[m^{-s} :\ m = 2, 3, \ldots, M](\subset \mathcal{G}),$$

whose elements are given by

$$\sum_{m,[I]} \frac{c(m,[I])}{(m\tau(I))^s},$$

where $c(m,[I]) \in \mathbb{C}$ and m is bounded. We define a map from $\mathcal{H}$ to $\mathcal{D}$ by

$$\sigma:\ \mathcal{H} \ni \sum_{m,[I]} \frac{c(m,[I])}{(m\tau(I))^s} \longmapsto \sum_{m,[I]} \frac{c(m,[I])}{|N(m\tau(I))|^s} \in \mathcal{D}.$$

From (iii) of Lemma 2 and the multiplicativity of the norm it is seen that the map σ is a well-defined homomorphism. The image $\sigma(F)$ is written as

$$\sum_{J\in\mathcal{J}_0} \sum_{m=1}^{m(J)} \frac{p_J(m)}{|N(m)|^s} \prod_{l=0}^{r} \Big(\sum_{n=0}^{\infty} \frac{a_n(-\log\lambda_n)^l}{|N(\lambda_n)|^s} \Big)^{j_l}.$$

The desired assertion therefore follows (under the assumption made at the beginning of the section), since, by Theorem B, the series

$$\sum_{n=0}^{\infty} \frac{a_n(-\log\lambda_n)^l}{|N(\lambda_n)|^s} \qquad (l=0,1,\dots,r)$$

are algebraically independent over $\mathcal{D}_0$.

To show the assertion in the general case, suppose on the contrary that

$$P(f,f',\dots,f^{(r)})=0$$

for some nonzero $P(X_0,X_1,\dots,X_r) \in \mathcal{D}_0[X_0,X_1,\dots,X_r]$. Let D be a positive integer such that $D\lambda_0 > 1$ and $D\lambda_n \in \mathcal{O}_K$ with $|N(D\lambda_n)| > 1$ for all $n \in \mathbb{N}_0$, and define

$$g(s) = \sum_{n=0}^{\infty} \frac{a_n}{(D\lambda_n)^s}.$$

It follows from the preceding argument that $g(s)$ is differentially transcendental over $\mathcal{D}_0$. Since $f(s) = (1/D)^{-s} g(s)$, we have

$$\begin{aligned} f^{(j)}(s) &= \sum_{k=0}^{j} \binom{j}{k} \frac{(\log D)^k}{(1/D)^s} g^{(j-k)}(s) \\ &= (1/D)^{-s} \sum_{k=0}^{j} \binom{j}{k} (\log D)^k g^{(j-k)}(s) \qquad (j=0,1,2,\dots). \end{aligned}$$

Hence the equality

$$(D^{-s})^N P(f,f',\dots,f^{(r)}) = 0$$

with a sufficiently large $N \in \mathbb{N}$ yields a differentially algebraic relation for $g(s)$ over $\mathcal{D}_0$, which is a contradiction. □

4. Proof of the corollaries

Proof of Corollary 1. The sequence $\lambda_n = P(n)$ ($n \in \mathbb{N}_0$) obviously satisfies conditions (i)–(iii) of Theorem 2. Hence it remains to verify condition (iv). Let D be a positive integer such that $\widetilde{P}(x) := D(\mathrm{Norm}_{K/\mathbb{Q}} P)(x) \in \mathbb{Z}[x]$. Since $N(\lambda_n) = (\mathrm{Norm}_{K/\mathbb{Q}} P)(n)$, to prove the first assertion of Corollary 1 we need to show

$$\limsup_{n\to\infty} \mathcal{P}[\widetilde{P}(n)] = +\infty,$$

where, for any integer $N \geq 2$, $\mathcal{P}[N]$ denotes the greatest prime factor of N. This is clear if $\widetilde{P}(x)$ does not have a constant term since $n \mid \widetilde{P}(n)$. Thus suppose $c := \widetilde{P}(0) \neq 0$. Given an arbitrarily large $k \in \mathbb{N}$, let $p_1, \ldots, p_k$ be the first k prime numbers. It is readily seen that $\widetilde{P}(c(p_1 \cdots p_k)^l)$ with a sufficiently large $l \in \mathbb{N}$ has a prime factor different from $p_1, \ldots, p_k$, and this shows the desired assertion.

Next we suppose, in addition, that $\mathrm{Norm}_{K/\mathbb{Q}} P$ contains a factor $Q(x) \in \mathbb{Z}[x]$ with at least two simple roots. Then a result of Siegel [8] gives

$$\lim_{n\to\infty} \mathcal{P}[Q(n)] = +\infty,$$

which ensures that condition (iv) is fulfilled in this case. The second assertion therefore follows, and the corollary is proved. □

Proof of Corollary 2. If a is transcendental or rational, then the assertion directly follows from Theorem 1 or Theorem 2, respectively. When a is algebraic but not rational, we can apply the second part of Corollary 1 with $P(x) = x + a$. This completes the proof. □

References

[1] R. Garunkštis and A. Laurinčikas, *On the Hilbert's problem for the Lerch zeta-function*, Publ. Inst. Math. (Beograd) (N.S.) **65(79)** (1999), 63–68.

[2] V. Laohakosol, *Dependence of arithmetic functions and Dirichlet series*, Proc. Amer. Math. Soc. **115** (1992), 637–645.

[3] A. Laurinčikas and K. Matsumoto, *The joint universality and the functional independence for Lerch zeta-functions*, Nagoya Math. J. **157** (2000), 211–227.

[4] A. Ostrowski, *Über Dirichletische Reihen und algebraische Differentialgleichungen*, Math. Z. **8** (1920), 241–298.

[5] J. Popken, *Algebraic dependence of arithmetic functions*, Proc. Kon. Ned. Acad. Wetensch. **65** (1962), 155–168.

[6] H.N. Shapiro, *On the convolution ring of arithmetic functions*, Comm. Pure. Appl. Math. **25** (1972), 287–336.

[7] H.N. Shapiro and G.H. Sparer, *On algebraic independence of Dirichlet series*, Comm. Pure. Appl. Math. **39** (1986), 695–745.

[8] C.L. Siegel, *Approximation algebraischer Zahlen*, Math. Z. **10** (1921), 173–213.

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