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INTEGRATION IN VECTOR SPACES

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ABSTRACT. We define an integral of a vector-valued function $f : \Omega \longrightarrow X$ with respect to a bounded countably additive vector-valued measure $\nu : \Sigma \longrightarrow Y$ and investigate its properties. The integral is an element of $X \check{\otimes} Y$, and when f is ν-measurable we show that f is integrable if and only if $\|f\| \in L_1(\nu)$. In this case, the indefinite integral of f is of bounded variation if and only if $\|f\| \in L_1(|\nu|)$. We also define the integral of a weakly ν-measurable function and show that such a function f satisfies $x^* f \in L_1(\nu)$ for all $x^* \in X^*$ and is $|y^*\nu|$-Pettis integrable for all $y^* \in Y^*$.

1. Introduction and notation

R.G. Bartle [1] introduced an integral in which both the function to be integrated and the measure take values in normed linear spaces; the integral of an X-valued function with respect to a Y-valued (finitely) additive measure appears as an element of a Banach space Z via a bilinear mapping $X \times Y \longrightarrow Z$. The integral possesses some of the properties usually associated with the Lebesgue theory of integration; in particular, the Vitali and Bounded Convergence theorems remain valid in this setting, while the Lebesgue Dominated Convergence theorem fails.

In this paper we define the integral of an X-valued function with respect to a Y-valued measure as an element of the injective tensor product of $X \otimes Y$.

We begin by defining the integral of a strongly measurable function, an analogue of the Bochner integral [2], and closely connected to the integral of D. R. Lewis [3]. Next, we extend the integral to less measurable functions, namely, functions which are not essentially separably valued. This extension requires a different approach to the integral, and our setup follows that of [2, Section I.3].

Given a Banach space X, its closed unit ball is denoted by B_X, and its dual by X^*. If X and Y are Banach spaces, the space of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$, and $\mathcal{K}(X, Y)$ denotes the closed

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subspace of all compact linear operators. $\mathcal{B}(X,Y)$ represents the space of all bounded bilinear functionals on $X \times Y$, and the completion of the tensor product $X \otimes Y$ with respect to the least reasonable cross norm is $X \check{\otimes} Y$.

If (Ω, Σ) is a measurable space and $\nu : \Sigma \longrightarrow Y$ a countably additive measure, its semivariation on a set $E \in \Sigma$ is given by $\|\nu\|(E) = \sup\{|y^*\nu|(A) : y^* \in B_{Y^*}\}$, where $|y^*\nu|$ is the variation of the scalar measure $y^*\nu$. The measure ν is called bounded if $\|\nu\|(\Omega) < \infty$. The variation of ν, denoted by $|\nu|$, is given by $|\nu|(E) = \sup_\pi \sum_{A\in\pi} \|\nu(A)\|$, where the supremum is taken over all finite partitions π of E into pairwise disjoint members of Σ.

By a theorem of Rybakov [2, Section IX.2, Theorem 2], there exists $y^* \in B_{Y^*}$ such that $|y^*\nu| \leq \|\nu\| \ll |y^*\nu|$. As defined in [3], a function $f : \Omega \longrightarrow \mathbb{R}$ is said to have a *generalized integral with respect to* ν if f is $y^*\nu$-integrable for all $y^* \in Y^*$. The *generalized* ν*-integral of* f *over* $E \in \Sigma$ is an element $y_E^{**} \in Y^{**}$ such that

$$y_E^{**}(y^*) = \int_E f \, dy^*\nu$$

for all $y^* \in Y^*$. The function is called ν*-integrable* if y_E^{**} belongs to the image of Y in Y^{**}. In [6] it was shown that the space of all (equivalence classes) of functions having a generalized integral with respect to ν is a Banach space when equipped with the norm

$$\|f\|_\nu = \sup\left\{ \int_\Omega |f| \, d|y^*\nu| \, : \, y^* \in B_{Y^*} \right\}.$$

We denote this space by w-$L_1(\nu)$. The space $L_1(\nu)$ of all ν-integrable functions is a closed subspace of w-$L_1(\nu)$.

If $\mu : \Sigma \longrightarrow \mathbb{R}$ is countably additive and finite, $L_1(\mu, X)$ denotes the Banach space of all (equivalence classes) of μ-Bochner integrable functions $f : \Omega \longrightarrow X$ with norm

$$\|f\| = \int_\Omega \|f\| \, d\mu.$$

If $x^*f \in L_1(\mu)$ for all $x^* \in X^*$ then f is said to be μ-Dunford integrable. In this case, the mapping

$$T_f : X^* \longrightarrow L_1(\mu),$$

defined by $T_f(x^*) = x^*f$, is bounded. The μ-Dunford integral of f over a set $E \in \Sigma$ is an element $x_E^{**} \in X^{**}$ such that for all $x^* \in X^*$,

$$x_E^{**}(x^*) = \int_E x^* f \, d\mu.$$

If $x^{**} \in X \subset X^{**}$ then f is said to be μ-Pettis integrable. $P(\mu, X)$ denotes the completion of the vector space of all (equivalence classes of) μ-Pettis integrable functions $f : \Omega \longrightarrow X$ with norm

$$\|f\|_P = \sup\left\{ \int_\Omega |x^*f| \, d\mu \, : \, \|x^*\| \leq 1 \right\}.$$

2. Definition of the integral and basic properties

Throughout this section, let X and Y be real Banach spaces, (Ω, Σ) a measurable space and $\nu : \Sigma \longrightarrow Y$ a bounded and countably additive measure. We assume that the measurable space is complete with respect to $|y^*\nu|$, where $y^* \in B_{Y^*}$ is chosen such that $\|\nu\| \ll |y^*\nu|$. A function $f : \Omega \longrightarrow X$ is said to be *ν-measurable* if there exists a sequence (ϕ_n) of simple functions such that $\lim_n \|f - \phi_n\| = 0$ $\|\nu\|$-almost everywhere. Similarly, a function $f : \Omega \longrightarrow X$ is *weakly ν-measurable* if for each $x^* \in X^*$ the scalar function x^*f is $\|\nu\|$-measurable. Clearly, f is ν-measurable if and only if it is $|y^*\nu|$-measurable. Thus we have the following Pettis type measurability theorems (see [2, Section II.1]).

Theorem A. *A function $f : \Omega \longrightarrow X$ is ν-measurable if and only if*

(1) *f is $\|\nu\|$-essentially separably valued, and*

(2) *f is weakly ν-measurable.*

Corollary B. *A function $f : \Omega \longrightarrow X$ is ν-measurable if and only if f is the ν-almost everywhere uniform limit of a sequence of countably valued ν-measurable functions.*

Let $\phi = \sum x_i \chi_{A_i}$ be an X-valued simple function and let $E \in \Sigma$. We define $\int_E \phi \, d\nu$ by the equation

$$\int_E \phi \, d\nu = \sum x_i \otimes \nu(E \cap A_i).$$

Since ν is additive, $\int_E \phi \, d\nu$ does not depend on the representation of ϕ. Furthermore, for any element $x^* \otimes y^* \in B_{X^*} \times B_{Y^*}$, we get

$$\begin{aligned} \left| x^* \otimes y^* \left(\int_E \phi \, d\nu \right) \right| &= \left| \sum x^*(x_i) \cdot y^*\nu(E \cap A_i) \right| \\ &\leq \sum |x^*(x_i)| \cdot |y^*\nu| (E \cap A_i) \\ &\leq \sum \|x_i\| \cdot |y^*\nu| (E \cap A_i) \\ &= \int_E \|\phi\| \, d |y^*\nu| . \end{aligned}$$

Therefore, if we view $\int_E \phi \, d\nu$ as an element of $X \check{\otimes} Y$, then

$$\left\| \int_E \phi \, d\nu \right\| \leq \sup \left\{ \int_E \|\phi\| \, d |y^*\nu| \, : \, y^* \in B_{Y^*} \right\}.$$

Definition 1. A ν-measurable function $f : \Omega \longrightarrow X$ is called *$\check{\otimes}$-integrable*, if there exists a sequence (ϕ_n) of simple functions such that

$$\lim_n \sup \left\{ \int_\Omega \|f - \phi_n\| \, d |y^*\nu| \, : \, y^* \in B_{Y^*} \right\} = 0. \tag{1}$$

In this case, the sequence $\left(\int_E \phi_n \, d\nu\right)$ is a Cauchy sequence in $X\check{\otimes}Y$ for each $E \in \Sigma$. The limit,

$$\int_E f \, d\nu = \lim_n \int_E \phi_n \, d\nu, \tag{2}$$

is called the $\check{\otimes}$-*integral of* f *over* E *with respect to* ν. Since the integral of a simple function does not depend on the representation of this function, the above limit is well defined and independent of the defining sequence $\left(\int_E \phi_n \, d\nu\right)$. To simplify the notation, we set

$$\mathbf{N}(f) = \sup \left\{ \int_\Omega \|f\| \, d\,|y^*\nu| \; : \; y^* \in B_{Y^*} \right\},$$

whenever $f : \Omega \longrightarrow X$ is ν-measurable.

THEOREM 1. *A ν-measurable function f is $\check{\otimes}$-integrable if and only if $\|f\|$ is ν-integrable.*

Proof. To prove necessity, let f be a $\check{\otimes}$-integrable function and (ϕ_n) a sequence of simple functions such that $\lim_n \mathbf{N}(f - \phi_n) = 0$. If we denote the essential supremum of $\|\phi_n(\cdot)\|$ by M_n, then $\mathbf{N}(\phi_n) \le M_n \|\nu\|(\Omega)$ and consequently, $\mathbf{N}(f) < \infty$. It follows that $\|f\|$ has a generalized integral with respect to ν; that is, $\|f\| \in w - L_1(\nu)$. But$|\, \|f\| - \|\phi_n\| \,| \le \|f - \phi_n\|$, and therefore $\| \, \|f\| - \|\phi_n\| \, \|_\nu \le \mathbf{N}(f - \phi_n)$. Thus, $(\|\phi_n\|)$ converges to $\|f\|$ in w-$L_1(\nu)$. Since each $\|\phi_n\| \in L_1(\nu)$, and $L_1(\nu)$ is a closed subspace of w-$L_1(\nu)$, $\|f\|$ is in fact an element of $L_1(\nu)$.

To prove sufficiency, assume $\|f\|$ is ν-integrable. By [3, Theorem 2.2], the indefinite integral of $\|f\|$ with respect to ν is a countably additive Y-valued measure and $\lim_{\|\nu\|(E)\to 0} \mathbf{N}(f \cdot \chi_E) = 0$.

Using Corollary B, we obtain a sequence (f_n) of countably valued functions such that $\|f - f_n\| \le 1/n$ $\|\nu\|$-almost everywhere. Then $\|f_n\| \le \|f\| + 1/n$ and so, by [6, Proposition 5], $\|f_n\|$ is ν-integrable for all n. In particular,

$$\lim_{\|\nu\|(E)\to 0} \mathbf{N}(f_n \cdot \chi_E) = 0. \tag{3}$$

Write

$$f_n = \sum_{k=1}^{\infty} x_{n,k} \chi_{E_{n,k}},$$

where $E_{n,i} \cap E_{n,j} = \emptyset$ if $i \ne j$, $E_{n,k} \in \Sigma$ and $x_{n,k} \in X$. For each n, equation (3) above allows us to choose p_n large enough so that

$$\sup_{\|y^*\| \le 1} \int_{\cup_{k>p_n} E_{n,k}} \|f_n\| \, d\,|y^*\nu| \; < \; \frac{\|\nu\|(\Omega)}{n}.$$

If we let $\phi_n = \sum_{k\le p_n} x_{n,k}\chi_{E_{n,k}}$, then ϕ_n is a simple function and

$$\mathbf{N}(f-\phi_n) \le \mathbf{N}(f-f_n) + \mathbf{N}(f_n-\phi_n) \le \frac{2\|\nu\|(\Omega)}{n}. \qquad \square$$

Corollary 1. *If f is ν-measurable and bounded, then f is $\check{\otimes}$-integrable.*

Corollary 2. *Let f and g be two ν-measurable functions. If g is $\check{\otimes}$-integrable and $\|f\| \le \|g\|$ $\|\nu\|$-almost everywhere, then f is $\check{\otimes}$-integrable.*

The following result gives some fundamental properties of the $\check{\otimes}$-integral.

Theorem 2. *If f is a $\check{\otimes}$-integrable function, then the set function μ_f defined on Σ by*

$$\mu_f(E) = \int_E f\,d\nu$$

is a countably additive measure. Furthermore, we have:

(1) $\|\mu_f\|(E) = \sup\left\{\int_E |x^*f|\ d\,|y^*\nu|\ :\ x^* \in B_{X^*}, y^* \in B_{Y^*}\right\}$;
(2) $\lim_{\|\nu(E)\|\to 0} \|\mu_f\|(E) = 0$;
(3) *μ_f is of bounded variation if and only if $\|f\| \in L_1(|\nu|)$, in which case*

$$|\mu_f|(E) = \int_E \|f\|\,d|\nu|.$$

Proof. To prove that μ_f is countably additive it suffices to show that μ_f is weakly countably additive, in view of the Orlicz-Pettis theorem. To this end, let (E_n) be a sequence of pairwise disjoint sets in Σ, let $E = \cup_n E_n$, and fix an element $x^*\otimes y^* \in X^*\otimes Y^*$. Equation (2) above shows that for any $F\in\Sigma$

$$(x^*\otimes y^*)\int_F f\,d\nu = \int_F x^*f\,dy^*\nu,$$

and therefore

$$\left|(x^*\otimes y^*)\mu_f(E) - \sum_{n=1}^{k}(x^*\otimes y^*)\mu_f(E_k)\right| = |(x^*\otimes y^*)\mu_f(\cup_{n>k}E_n)|$$
$$\le \int_{\cup_{n>k}E_n} |x^*f|\,d|y^*\nu|.$$

Clearly $\lim_n \int_{\cup n>kE_n} |x^*f|\,d|y^*\nu| = 0$, and therefore $(x^*\otimes y^*)\mu_f$ is countably additive. Since $x^*\otimes y^*$ was arbitrary, a theorem of Lewis [4, Lemma 1.1] allows us to conclude that μ_f is weakly countably additive.

To prove (1), we use the fact that any element u^* of $(X\check{\otimes}Y)^*$ is of integral type; that is, for any $u\in X\check{\otimes}Y$,

$$u^*(u) = \int_{B_{X^*}\times B_{Y^*}} x^*\otimes y^*(u)\,d\mu(x^*,y^*) \quad \text{and} \quad \|u^*\| = |\mu|(B_{X^*}\times B_{Y^*}),$$

where μ is a regular Borel measure on the compact space $(B_{X^*},\text{weak}^*) \times (B_{Y^*},\text{weak}^*)$. Let π be a partition of a set E in Σ and u^* an element of the unit ball of $(X\check{\otimes}Y)^*$. Then

$$\begin{aligned}
\sum_{A\in\pi} |u^*\mu_f(A)| &= \sum_{A\in\pi} \left| \int_{B_{X^*}\times B_{Y^*}} x^*\otimes y^*\,(\mu_f(A))\; d\mu(x^*,y^*) \right| \\
&\leq \int_{B_{X^*}\times B_{Y^*}} \sum_{A\in\pi} \left| \int_A x^* f\, dy^*\nu \right| d|\mu|(x^*,y^*) \\
&\leq \int_{B_{X^*}\times B_{Y^*}} \sum_{A\in\pi} \int_A |x^*f|\; d\,|y^*\nu|\; d|\mu|(x^*,y^*) \\
&= \int_{B_{X^*}\times B_{Y^*}} \left(\int_E |x^*f|\; d\,|y^*\nu| \right) d|\mu|(x^*,y^*) \\
&\leq \sup\left\{ \int_E |x^*f|\; d\,|y^*\nu| \,:\, x^*\in B_{X^*}, y^*\in B_{Y^*} \right\} \cdot |\mu|(B_{X^*}\times B_{Y^*}) \\
&\leq \sup\left\{ \int_E |x^*f|\; d\,|y^*\nu| \,:\, x^*\in B_{X^*}, y^*\in B_{Y^*} \right\}.
\end{aligned}$$

Hence

$$\|\mu_f\|(E) \leq \sup\left\{ \int_E |x^*f|\; d\,|y^*\nu| \,:\, x^*\in B_{X^*}, y^*\in B_{Y^*} \right\}.$$

To establish the reverse inequality, note that

$$\begin{aligned}
\|\mu_f\|(E) &= \sup\{|u^*\mu_f|(E) \,:\, \|u^*\|\leq 1\} \\
&\geq \sup\{|(x^*\otimes y^*)\mu_f|(E) \,:\, \|x^*\|, \|y^*\|\leq 1\} \\
&= \sup\left\{ \int_E |x^*f|\, d|y^*\nu| \,:\, x^*\in B_{X^*}, y^*\in B_{Y^*} \right\}.
\end{aligned}$$

To prove (2), observe that if $y^*\in B_{Y^*}$ is chosen so that $\|\nu\| \ll |y^*\nu|$, the countably additive measure μ_f vanishes on sets of $|y^*\nu|$-measure zero. Thus, by [2, Theorem I.2.1], μ_f is $|y^*\nu|$-continuous, and hence ν-continuous.

To prove (3), let us first assume that $\|f\|\in L_1(|\nu|)$ and fix $E\in\Sigma$. If π is a finite partition of E, then

$$\sum_{A\in\pi} \|\mu_f(A)\| \leq \sum_{A\in\pi} \int_A \|f\|\, d|\nu| = \int_E \|f\|\, d|\nu|.$$

Thus, μ_f is of bounded variation and $|\mu_f|(E) \leq \int_E \|f\|\, d|\nu|$.

For the converse, suppose μ_f is of bounded variation. If we view $\mu_f(E)$ as an element of $\mathcal{L}(Y^*, X)$, then for any fixed $y^*\in Y^*$, $\mu_f(\cdot)(y^*)$ is a countably additive X-valued measure. If fact, for any $E\in\Sigma$,

$$\mu_f(E)(y^*) = \int_E f\, dy^*\nu,$$

which is the Bochner integral of f with respect to $y^*\nu$. If π is a finite partition of E, then

$$\sum_{A\in\pi} \|\mu_f(A)(y^*)\| \le \sum_{A\in\pi} \|\mu_f(A)\| \cdot \|y^*\|,$$

and hence

$$\int_E \|f\|\, d|y^*\nu| \le \|y^*\| \cdot |\mu_f|(E).$$

Fix $E \in \Sigma$ and let $A \in \Sigma$ be a subset of E. Find $y^* \in Y^*$ with $\|y^*\| = 1$ such that $\|\nu(A)\| = |y^*\nu(A)|$. If $|a| \cdot \chi_A \le \|f\|$, then

$$\|\nu(A)\| = |y^*\nu(A)| \le \int_A d|y^*\nu| \le |a|^{-1} \int_A \|f\|\, d|y^*\nu| \le |a|^{-1}|\mu_f|(A).$$

Consequently, $|a| \cdot |\nu|(E) \le |\mu_f|(E)$. It follows that for any real-valued, non-negative simple function ϕ satisfying $\phi \le \|f\|$ we have

$$\int_E \phi\, d|\nu| \le |\mu_f|(E).$$

Therefore, $\|f\| \in L_1(|\nu|)$ and $\int_E \|f\|\, d|\nu| \le |\mu_f|(E)$. □

Theorem 3 (Dominated Convergence Theorem). *Let (f_n) be a sequence of $\check{\otimes}$-integrable functions which converges $\|\nu\|$-a.e to a function f. If there exists a $\check{\otimes}$-integrable function g such that $\|f_n\| \le \|g\|$ $\|\nu\|$-a.e., then f is $\check{\otimes}$-integrable and*

$$\lim_n \int_E f_n\, d\nu = \int_E f\, d\nu, \qquad E \in \Sigma.$$

In fact, the limit is uniform with respect to $E \in \Sigma$.

Proof. Note that $\|f\| \le \|g\|$ $\|\nu\|$-a.e. Hence, by Corollary 2, f is $\check{\otimes}$-integrable. Fix $\epsilon > 0$, and for each n let

$$E_n = \{\, \omega \in \Omega : \|f(\omega) - f_n(\omega)\| \ge \epsilon\}.$$

For any $E \in \Sigma$ and $(x^*, y^*) \in B_{X^*} \times B_{Y^*}$ we have

$$\begin{aligned}\left|\int_E x^*(f - f_n)\, dy^*\nu\right| &\le \left|\int_{E\setminus E_n} x^*(f - f_n)\, dy^*\nu\right| + \left|\int_{E\cap E_n} x^*(f - f_n)\, dy^*\nu\right| \\ &\le \epsilon \cdot \|\nu\|(E \setminus E_n) + 2\|\mu_g\|(E \cap E_n) \\ &\le \epsilon \cdot \|\nu\|(\Omega) + 2\|\mu_g\|(E_n).\end{aligned}$$

Hence

$$\left\|\int_E f\, d\nu - \int_E f_n\, d\nu\right\| \le \epsilon \cdot \|\nu\|(\Omega) + 2\|\mu_g\|(E_n).$$

Since $\lim_n \|\mu_g\|(E_n) = 0$, and ϵ can be chosen arbitrarily small, the result follows. □

Let $L_1(\nu, X, Y)$ denote the vector space of all ($\|\nu\|$-equivalence classes of) $\check{\otimes}$-integrable functions equipped with the norm $\mathbf{N}(\cdot)$.

THEOREM 4. *$L_1(\nu, X, Y)$ is a Banach space.*

Proof. If (f_n) is a Cauchy sequence in $L_1(\nu, X, Y)$ then (f_n) is uniformly Cauchy in $L_1(|y^*\nu|, X)$ for all $y^* \in B_{Y^*}$. Let f_{y^*} be the limit of (f_n) in $L_1(|y^*\nu|, X)$.

Find $z^* \in B_{Y^*}$ such that $\|\nu\| \ll |z^*\nu|$. There exists a set $E_{z^*} \in \Sigma$ of $|z^*\nu|$-measure zero and a subsequence (f_{n_k}) of (f_n) such that

$$\lim_k f_{n_k}(\omega) = f_{z^*}(\omega)$$

off E_{z^*}. Similarly, for any $y^* \in B_{Y^*}$ there exists a set $E_{y^*} \in \Sigma$ of $|y^*\nu|$-measure zero and a subsequence $(f_{n_{k,j}})$ of (f_{n_k}) such that

$$\lim_j f_{n_{k,j}}(\omega) = f_{y^*}(\omega)$$

off E_{y^*}. Then $f_{y^*}(\omega) = f_{z^*}(\omega)$ off $E_{y^*} \cup E_{z^*}$. Since $|y^*\nu|(E_{y^*} \cup E_{z^*}) = 0$, it follows that $f_{z^*} \in L_1(|y^*\nu|, X)$ and $f_{z^*} = f_{y^*}$ $|y^*\nu|$- a.e. Therefore, $f_{z^*} \in L_1(|y^*\nu|, X)$, for all $y^* \in B_{Y^*}$, and $\lim_n \mathbf{N}(f_{z^*} - f_n) = 0$. Set $f = f_{z^*}$.

It remains to show that f is $\check{\otimes}$-integrable. But each f_n is $\check{\otimes}$ integrable, so we can find a sequence (ϕ_n) of simple functions so that $\mathbf{N}(f_n - \phi_n) < 1/n$. Then

$$\begin{aligned}\mathbf{N}(f - \phi_n) &\le \mathbf{N}(f - f_n) + \mathbf{N}(f_n - \phi_n)\\ &< \mathbf{N}(f - f_n) + 1/n.\end{aligned}$$

Thus, f is $\check{\otimes}$-integrable. □

EXAMPLE 1. Take X to be any infinite-dimensional Banach space, and take $Y = \mathbb{R}$. Let $\Omega = [0, 1]$ and let ν be the Lebesgue measure.

There exists an unconditionally convergent series $\sum_n x_n$ in X that is not absolutely convergent. The function

$$f = \sum_n \frac{x_n}{\nu(E_n)} \chi_{E_n},$$

where (E_n) is any partition of $[0, 1]$ into sets of positive measures, is ν-Pettis integrable but not ν-Bochner integrable. If we let f_n be the n'th partial sum, then

(1) $\lim_n f_n = f$ everywhere, and
(2) $\lim_{\nu(E)\to 0} \sup \|\mu_{f_n}\|(E) = 0$.

Since f is not Bochner integrable, the usual formulation of the Vitali convergence theorem does not hold.

Let us consider the same example under more general assumptions.

EXAMPLE 2. Assume we have a sequence (f_n) of $\check{\otimes}$-integrable functions and a ν-measurable function f such that the following two conditions hold:

(1) $\lim_n f_n = f$ ν-almost everywhere, and
(2) $\lim_{\nu(E)\to 0} \sup \|\mu_{f_n}\|(E) = 0$.

What can we say about the function f?

CLAIM 1. For any $(x^*, y^*) \in X^* \times Y^*$, we have $f \in P(y^*\nu, X)$ and $x^*f \in L_1(\nu)$.

Indeed, the first assertion follows from [5, Theorem 2.10] and the second follows from [3, Lemma 2.3], once we realize that $(\int_E x^* f_n \, d\nu)$ is a Cauchy sequence in Y for all $E \in \Sigma$.

CLAIM 2. For any $E \in \Sigma$, the sequence $(\mu_{f_n}(E))$ is Cauchy in $X\check{\otimes}Y$.

To see this, fix $E \in \Sigma$. Since $f_n \longrightarrow f$ $\|\nu\|$- a.e, we have $\|f_n\| \longrightarrow \|f\|$ $\|\nu\|$-a.e, and hence $\|f_n\| \longrightarrow \|f\|$ $\|\nu\|$-almost uniformly.

Let $\epsilon > 0$ and choose $\delta > 0$ such that $\sup_n \|\mu_{f_n}\|(F) < \epsilon$ whenever $\|\nu\|(F) < \delta$. Next, choose a set $F \in \Sigma$ with $\|\nu\|(F) < \delta$ such that $\|f_n\| \longrightarrow \|f\|$ uniformly off F. Then, for any $(x^*, y^*) \in B_{X^*} \times B_{Y^*}$,

$$\begin{aligned} |(x^* \otimes y^*)(\mu_{f_n}(E) - \mu_{f_m}(E))| &\leq \int_{E\cap F} |x^*(f_n - f_m)| \, d|y^*\nu| \\ &\quad + \int_{E\setminus F} |x^*(f_n - f_m)| \, d|y^*\nu| \\ &\leq 2\epsilon + \epsilon \cdot \|\nu\|(\Omega), \end{aligned}$$

for all sufficiently large n and m. Therefore,

$$\|\mu_{f_n}(E) - \mu_{f_m}(E)\| \leq 2\epsilon + \epsilon \cdot \|\nu\|(\Omega)$$

for all sufficiently large n and m.

Let u_E denote the limit of the sequence $(\mu_{f_n}(E))$. Then

$$(x^* \otimes y^*)(u_E) = \lim_n (x^* \otimes y^*)(\mu_{f_n}(E)) = \lim_n \int_E x^* f_n \, dy^*\nu = \int_E x^* f \, dy^*\nu.$$

Thus we have shown that, under conditions (1) and (2), the function f, even though it need not be not $\check{\otimes}$-integrable (as in Example 1), does have a weaker integral, namely $u_{(\cdot)} \in X\check{\otimes}Y$, such that for any x^* and y^*,

$$(x^* \otimes y^*)(u_E) = \int_E x^* f \, dy^*\nu.$$

We now turn our attention to this weaker integral.

3. The integral of weakly measurable functions

Let $f : \Omega \longrightarrow X$ be a weakly ν-measurable function and choose an element $y_0^* \in B_{Y^*}$ such that $\|\nu\| \ll |y_0^*\nu|$.

If $x^*f \in L_1(y^*\nu)$ for all $(x^*, y^*) \in X^* \times Y^*$ and we fix $x^* \in X^*$, then x^*f has a generalized integral with respect to ν. By [6, Proposition 2], the mapping

$$Y^* \ni y^* \longrightarrow x^* f \frac{dy*\nu}{d|y_0^*\nu|} \in L_1(|y_0^*\nu|)$$

is bounded. Similarly, the mapping

$$X^* \ni x^* \longrightarrow x^* f \frac{dy*\nu}{d|y_0^*\nu|} \in L_1(|y_0^*\nu|)$$

is bounded, because for any fixed $y^* \in Y^*$, the function f is $y^*\nu$-Dunford integrable. This means that if we define an operator $T_{f,\nu} : X^* \times Y^* \longrightarrow L_1(|y_0^*\nu|)$ by the equation

$$T_{f,\nu}(x^*, y^*) = x^* f \frac{dy^*\nu}{d|y_0^*\nu|},$$

then $T_{f,\nu}$ is separately continuous, and thus continuous. Hence, for every $g \in L_\infty(|y_0^*\nu|)$, the map ψ_g defined by

$$\psi_g(x^*, y^*) = \int_\Omega g \cdot x^* f \, dy^*\nu,$$

is an element of $\mathcal{B}(X^*, Y^*)$.

Definition 2. A weakly $\|\nu\|$-measurable function $f : \Omega \longrightarrow X$ is said to have a *generalized weak* $\otimes$*-integral* (*with respect to* ν) if $x^*f \in L_1(y^*\nu)$ for all $(x^*, y^*) \in X^* \times Y^*$. If f is such a function, and $E \in \Sigma$, the *generalized weak* $\otimes$*-integral of* f *over* E is defined by the element ψ_{χ_E}.

If $\psi_{\chi_E} \in X \check{\otimes} Y$ for all $E \in \Sigma$, then f is said to be *weakly* $\check{\otimes}$*-integrable* and ψ_{χ_E} is called the *weak* $\check{\otimes}$*-integral of* f *over* E and denoted by $\int_E f \, d\nu$.

The measure $\mu_f : \Sigma \longrightarrow \mathcal{B}(X^* \times Y^*)$, defined by $\mu_f(E) = \psi_{\chi_E}$, is not necessarily countably additive. A standard argument proves that μ_f is countably additive if and only if the operator $T_{f,\nu}$ is weakly compact.

The following theorem is an analogue of Theorem 2, whose proof applies with a few minor changes.

Theorem 5. *If f is weakly $\check{\otimes}$-integrable, then we have:*

(1) $\lim_{\|\nu\|(E) \to 0} \int_E f \, d\nu = 0$.

(2) *If (E_n) is a sequence of pairwise disjoint sets in Σ and $E = \bigcup_{n=1}^\infty E_n$, then*

$$\int_E f \, d\nu = \sum_{n=1}^\infty \int_{E_n} f \, d\nu,$$

where the sum on the right is unconditionally convergent.

(3) *If* $\mu_f(E) = \int_E f\, d\nu$, *then* μ_f *is of bounded semivariation and*

$$\|\mu_f\|(E) = \sup\left\{ \int_E |x^* f|\, d|y^*\nu| \;:\; \|x^*\|, \|y^*\| \leq 1 \right\}.$$

It should be clear that if $X = \mathbb{R}$, then f is $\check{\otimes}$-integrable if and only if $f \in L_1(\nu)$. Similarly, if $Y = \mathbb{R}$, then f is $\check{\otimes}$-integrable if and only if f is ν-Pettis integrable.

In Claim 1 of Example 2 we showed that if a ν-measurable function f is $\check{\otimes}$-integrable, then $x^* f \in L_1(\nu)$ and $f \in P(|y^*\nu|, X)$ for all $(x^*, y^*) \in X^* \times Y^*$. This is a property shared by the weakly $\check{\otimes}$-integrable functions.

PROPOSITION 1. *Assume* $f : \Omega \longrightarrow X$ *is weakly* $\check{\otimes}$*-integrable. Then:*

(1) *For every* $y^* \in Y^*$, f *is* $|y^*\nu|$*-Pettis integrable.*
(2) *For every* $x^* \in X^*$, $x^* f \in L_1(\nu)$.

Proof. Let f be a weakly $\check{\otimes}$-integrable function. To prove (1), fix $y^* \in Y^*$. We want to show that, for every $E \in \Sigma$, the functional

$$x^* \longmapsto \int_E x^* f\, d|y^*\nu|$$

is an element of X. To do so, we show that this functional is weak*-to-weak continuous. Let (x^*_α) be a net in B_{X^*} converging weak* to $x^* \in B_{X^*}$. Then $x^*_\alpha \otimes y^*(u)$ converges to $x^* \otimes y^*(u)$ for all $u \in X \check{\otimes} Y$. In particular,

$$\begin{aligned}
\lim_\alpha x^*_\alpha \left(\int_E f\, dy^*\nu \right) &= \lim_\alpha \int_E x^*_\alpha f\, dy^*\nu \\
&= \lim_\alpha x^*_\alpha \otimes y^* \left(\int_E f\, d\nu \right) \\
&= x^* \otimes y^* \left(\int_E f\, d\nu \right) \\
&= \int_E x^* f\, dy^*\nu \\
&= x^* \left(\int_E f\, dy^*\nu \right).
\end{aligned}$$

To prove (2), fix an element x^* in X^*. The remarks preceding Definition 2 show that $x^* f$ is an element of w-$L_1(\nu)$. To prove that $f \in L_1(\nu)$, it suffices to verify that the indefinite integral

$$\mu_{x^* f}(E) = \int_E x^* f\, d\nu$$

is countably additive. Thus, let (E_n) be a sequence of pairwise disjoint sets in Σ and let $E = \cup E_n$. Then

$$\begin{aligned}\|\mu_{x^*f}(E) - \sum_{n=1}^{k} \mu_{x^*f}(E_n)\| &= \|\mu_{x^*f}(\cup_{n>k}E_n)\| \\ &= \sup_{y^*\in B_{Y^*}} \left| y^* \int_{\cup_{n>k}E_n} x^* f \, d\nu \right| \\ &\le \sup_{y^*\in B_{Y^*}} \int_{\cup_{n>k}E_n} |x^* f| \, d|y^*\nu| \\ &\le \|\mu_f\|(\cup_{n>k}E_n).\end{aligned}$$

Therefore

$$\lim_k \|\mu_{x^*f}(E) - \sum_{n=1}^{k} \mu_{x^*f}(E_n)\| \le \lim_k \|\mu_f\|(\cup_{n>k}E_n) = 0.$$

It follows that μ_{x^*f} is countably additive and, consequently, $x^*f \in L_1(\nu)$. $\square$

PROPOSITION 2. *If a ν-measurable function $f : \Omega \longrightarrow X$ has a generalized weak $\otimes$- integral (with respect to ν), then f is weakly $\check{\otimes}$-integrable if and only if $T_{f,\nu}$ is weakly compact. In this case $T_{f,\nu}$ is compact.*

Proof. First, we note that if a ν-measurable function f is $\check{\otimes}$-integrable then it is weakly $\check{\otimes}$-integrable and, for any $E \in \Sigma$, the two integrals over E are equal.

Let f be a ν-measurable function and assume f is weakly $\check{\otimes}$-integrable. By Theorem 5, the indefinite integral of f is countably additive, and hence $T_{f,\nu}$ is weakly compact.

Now, assume f has a generalized integral and $T_{f,\nu}$ is weakly compact. Then $\mu_f : \Sigma \longrightarrow \mathcal{B}(X^* \times Y^*)$ is countably additive. We want to show that μ_f takes its values in $X\check{\otimes}Y$. To this end, write f as a sum

$$f = \sum_{n=1}^{\infty} f \cdot \chi_{E_n},$$

where (E_n) is a sequence of pairwise disjoint sets in Σ such that $f\chi_{E_n}$ is bounded, and $\cup_{n=1}^{\infty} E_n = \Omega$. Since μ_f is countably additive, we have

$$\mu_f(E) = \sum_{n=1}^{\infty} \mu_f(E \cap E_n)$$

for all $E \in \Sigma$. But $\mu_f(E \cap E_n) = \int_E f\chi_{E_n}\, d\nu$ is an element of $X\check{\otimes}Y$ for all n, because $f \cdot \chi_{E_n}$ is $\check{\otimes}$-integrable. Consequently $\mu_f(E) \in X\check{\otimes}Y$ for all $E \in \Sigma$, and hence f is $\check{\otimes}$-integrable.

To prove that $T_{f,\nu}$ is, in fact, compact, choose a sequence f_n of countably valued functions such that $\|f - f_n\| \le 1/n$. Then each f_n is weakly

$\check{\otimes}$-integrable. Indeed, f is weakly $\check{\otimes}$-integrable, $f - f_n$ is $\check{\otimes}$-integrable and therefore $f_n = f - (f - f_n)$ is weakly $\check{\otimes}$-integrable. Then we know that the indefinite integral, μ_{f_n}, is countably additive for all n. Write f_n as a sum

$$f_n = \sum_{k=1}^{\infty} x_{n,k}\chi_{E_{n,k}},$$

where $E_{n,i} \cap E_{n,j} = \emptyset$ if $i \neq j$ and $\cup_k E_{n,k} = \Omega$, and note that

$$\|\mu_{f_n}\|(E) \leq \frac{\|\nu\|(E)}{n} + \|\mu_f\|(E).$$

Therefore $\lim_{\|\nu\|(E)\to 0} \sup_n \|\mu_{f_n}\|(E) = 0$. For each n find an integer p_n so that $\|\mu_{f_n}\|(\cup_{k>p_n} E_{n,k}) < 1/n$, and let $\phi_n = \sum_{k\leq p_n} x_{n,k}\chi_{E_{n,k}}$.

Now, for any $g \in L_\infty(|y_0^*\nu|)$,

$$\begin{aligned}
\|T^*_{f,\nu}(g) - T^*_{\phi_n,\nu}(g)\| &\leq \sup_{\|x^*\|,\|y^*\|\leq 1} \int_\Omega |g||x^*(f - f_n)|\, d|y^*\nu| \\
&\qquad + \sup_{\|x^*\|,\|y^*\|\leq 1} \int_\Omega |g||x^*(f_n - \phi_n)|\, d|y^*\nu| \\
&\leq \|g\| \cdot \frac{\|\nu\|(\Omega)}{n} + \|g\| \cdot \|\mu_{f_n}\|(\cup_{k>p_n} E_{n,k}) \\
&\leq \|g\| \cdot \left(\frac{|\nu\|(\Omega)}{n} + \frac{1}{n} \right).
\end{aligned}$$

Hence $T^*_{f,\nu}$ is the uniform operator limit of the sequence $(T^*_{\phi_n,\nu})$. Since each ϕ_n has a finite range, each $T^*_{\phi_n,\nu}$ is a finite rank operator, and thus compact. It follows that $T^*_{f,\nu}$ is compact. Consequently, $T_{f,\nu}$ is compact as well. □

A Banach space Y is said to be *accessible* if, given a compact set K in Y and $\epsilon > 0$, there is a finite rank bounded linear operator $u : Y \longrightarrow Y$ such that $\|u(y) - y\| < \epsilon$ for any $y \in K$. It is known that Y is accessible if and only if, for every Banach space X, we have $X^*\check{\otimes}Y = \mathcal{K}(X, Y)$.

Example 3. Suppose that a weakly $\|\nu\|$-measurable function f has a generalized weak $\otimes$-integral with respect to ν. Assume further that f satisfies conditions (1) and (2) of Proposition 1. Then, for every $E \in \Sigma$, we can consider the (generalized) integral ψ_E as an element u of $\mathcal{L}(X^*, Y)$ or as an element v of $\mathcal{L}(Y^*, X)$. Since $u^* = v$ and $v^* = u$, both u and v are weak*-to-weak continuous, and if either u or v is compact, both are compact.

Consider the case where u is compact and Y is accessible. For given $\epsilon > 0$ we can find a finite rank operator $w : Y \longrightarrow Y$ such that $\|y - w(y)\| < \epsilon$ for all $y \in \overline{B_{X^*}}$. Thus $\|u - wu\| < \epsilon$ and wu is a finite rank operator. But the adjoint, $(wu)^* = u^*w^*$, takes its values in X, and therefore $wu \in X \otimes Y$. Hence $u \in \overline{X \otimes Y} = X\check{\otimes}Y$. Similarly, if v is compact and X is accessible, $v \in X\check{\otimes}Y$. In either case, the integral ψ_E is an element of $X\check{\otimes}Y$.

THEOREM 6. *Suppose that X or Y is accessible and $f : \Omega \longrightarrow X$ is weakly ν-measurable. The following statements are equivalent:*

(1) *f is weakly $\check{\otimes}$-integrable.*
(2) *For every $y^* \in Y^*$, we have $f \in P(y^*\nu, X)$ and $\{\int_E x^* f \, d\nu : \|x^*\| \leq 1\}$ is a compact subset of Y.*
(3) *For every $x^* \in X^*$, we have $x^* f \in L_1(\nu)$ and $\{\int_E f \, dy^*\nu : \|y^*\| \leq 1\}$ is a compact subset of X.*

Proof. This is a direct consequence of Proposition 1 and Example 3. □

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