Illinois Journal of Mathematics
Volume 45, Number 3, Fall 2001, Pages 907–923
S 0019-2082

COMPACTNESS ARGUMENTS FOR SPACES OF p-INTEGRABLE FUNCTIONS WITH RESPECT TO A VECTOR MEASURE AND FACTORIZATION OF OPERATORS THROUGH LEBESGUE-BOCHNER SPACES

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Abstract. If λ is a vector measure with values in a Banach space and $p > 1$, we consider the space of real functions $L_p(\lambda)$ that are p-integrable with respect to λ. We define two different vector valued dual topologies and we prove several compactness results for the unit ball of $L_p(\lambda)$. We apply these results to obtain new Grothendieck-Pietsch type factorization theorems.

1. Introduction

The Grothendieck-Pietsch factorization theorem for p-summing operators is a fundamental tool in the theory of Banach spaces. From a technical point of view, the proof of this classical result is closely related to the weak* compactness of the unit ball of the dual of a Banach space. The aim of this paper is to apply similar arguments in order to obtain a factorization theorem for operators defined on Köthe (Banach) function spaces through spaces of Bochner integrable functions. We also use compactness properties (but with respect to different topologies) of the unit ball of a particular class of Köthe function spaces (spaces of p-integrable functions with respect to a vector measure). In a recent paper, A. Defant [5] proposed a general and unified point of view for understanding the relation between vector-valued norm inequalities and factorization properties of (homogeneous) operators on Köthe function spaces in the context of the Maurey-Rosenthal theorem. In particular, Defant obtained several results on the factorization properties for operators on spaces of Bochner integrable functions (see 4.4 in [5]). Although the subject we consider here is closely related, our point of view is different, since we restrict our attention to the particular case of factorizations through spaces of p-integrable functions with respect to a vector measure. In this case the spaces of Bochner integrable functions appear in a natural way.

Received July 20, 2000; received in final form May 2, 2001.
2000 *Mathematics Subject Classification.* Primary 46G10, 46B42.

2. Preliminaries and notation

Let (Ω, Σ) be a measurable space, X a (real) Banach space and $\lambda\colon \Sigma \to X$ a countably additive vector measure. Following the definition of Bartle, Dunford and Schwartz [1] and Lewis [12], we consider the space $L_1(\lambda)$ of (classes of) real functions that are integrable with respect to λ. This space has been studied by Kluvánek and Knowles [10], Okada [15], and Curbera ([4], [3] and [2]). In this paper we use the same construction in order to define for real numbers $p > 1$ the spaces $L_p(\lambda)$ of real functions that are p-integrable with respect to λ. In Section 2 we investigate several elementary lattice properties of the spaces $L_p(\lambda)$ in order to get an easy description of their dual spaces. Section 3 is devoted to proving topological properties of these spaces by means of a new "vector-dual" space that can be defined using integration with respect to vector measures. In Section 4 we apply these results to obtain our factorization theorem.

Throughout this paper we will use several well-known results about general Vector Measure Theory. The reader can find these results in the book of Diestel and Uhl [8], and the results about Measure Theory in the book of Halmos [9].

The notation for Banach spaces and vector measures is standard. A good reference for general questions on this subject is the book of Wojtaszczyk [19]. Aspects related to locally convex topologies can be found in [11]. For p-absolutely summing operators we refer the reader to [18], [7] and [6].

If $A \in \Sigma$, we shall write χ_A for the characteristic function of A. Throughout this paper every vector measure will be countably additive. If (Ω, Σ) is a measurable space and $\lambda\colon \Sigma \to X$ is a countably additive vector measure, the semivariation of λ is the set function $\|\lambda\|(A) = \sup\{|\langle \lambda, x'\rangle|(A) : x' \in B_{X'}\}$, where $|\langle \lambda, x'\rangle|$ is the variation of the scalar measure $\langle \lambda, x'\rangle$ and $B_{X'}$ is the closed unit ball of the dual space X' of X.

A measurable real function f defined on Ω is *integrable with respect to* λ (λ-*integrable* for short) [12] if it is $\langle \lambda, x'\rangle$-integrable for each $x' \in X'$ and for every $A \in \Sigma$ there is an element $\int_A f\, d\lambda$ of X such that

$$\left\langle \int_A f\, d\lambda, x' \right\rangle = \int_A f\, d\left\langle \lambda, x' \right\rangle, \qquad x' \in X'.$$

The Banach lattice $L_1(\lambda)$ coincides with the completion of the normed space of equivalence classes of simple functions that are equal if the set where they differ has zero semivariation with respect to the norm

$$\|f\|_\lambda = \sup\left\{ \int_\Omega |f|\, d\,|\langle \lambda, x'\rangle| : x' \in B_{X'} \right\}$$

(see [12] and [13]). The order in this lattice is the $\|\lambda\|$-almost everywhere order. The following norm is equivalent to the one defined above:

$$|||f|||_\lambda = \sup_{A \in \Sigma} \left\| \int_A f\, d\lambda \right\|, \qquad f \in L_1(\lambda).$$

In particular, $|||f|||_\lambda \le \|f\|_\lambda \le 2|||f|||_\lambda$.

If λ is a vector measure and $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ is a Σ-simple function (where $\{A_i\} \subset \Sigma$ are pairwise disjoint sets), we define an integral operator by

$$\int_\Omega f\, d\lambda = \sum_{i=1}^n \alpha_i \lambda(A_i).$$

This definition can be extended to all elements f of the space $L_1(\lambda)$. Various properties of the integral operator $f \to \int_\Omega f\, d\lambda$ have been studied by Okada and Ricker in [16] and [17].

We extend the definition of $L_1(\lambda)$ to $L_p(\lambda)$ as follows.

Definition 1. Let $1 < p < \infty$ and let λ be a countably additive vector measure. We say that a measurable real function f defined on Ω is *p-integrable with respect to* λ if $|f|^p$ is λ-integrable.

A norm can be defined for the vector space of simple functions (more precisely, equivalence classes of functions that are equal $\|\lambda\|$-a.e.) by

$$\|f\|_{p,\lambda} = \sup\left\{ \left(\int_\Omega |f|^p\, d\, |\langle \lambda, x' \rangle| \right)^{1/p} : x' \in B_{X'} \right\}.$$

This norm is equivalent to the norm defined by

$$|||f|||_{p,\lambda} = \sup_{A \in \Sigma} \left\| \int_A |f|^p\, d\lambda \right\|^{1/p}.$$

Definition 2. $L_p(\lambda)$ denotes the set of (equivalence classes of) p-integrable functions with respect to λ, endowed with the topology given by the norm $\|\cdot\|_{p,\lambda}$.

Remark 3. Note that, if $p > 1$, each function $f \in L_p(\lambda)$ also belongs to $L_1(\lambda)$. To verify this, let us define for a function $f \in L_p(\lambda)$ the set $E(f) = \{\omega \in \Omega : |f(\omega)| \le 1\}$. It is clear that $\chi_{E(f)} \in L_1(\lambda)$, and hence $|f|^p + \chi_{E(f)} \in L_1(\lambda)$. Since $|f| \le |f|^p + \chi_{E(f)}$, it follows from the lattice property of $L_1(\lambda)$ that $f \in L_1(\lambda)$.

We now establish several basic results on the lattice structure of the spaces of p-integrable functions with respect to a vector measure. First we prove that this set is indeed a Banach space.

Proposition 4. *Let $p \ge 1$ and λ be a vector measure. Then $L_p(\lambda)$ is a Banach space and the vector space consisting of (equivalence classes of) simple functions is dense in it.*

Proof. First we show that the simple functions are dense in $L_p(\lambda)$. Of course, every simple function is p-integrable with respect to λ. We will use the fact that this set is dense in $L_1(\lambda)$ (see [12]). If $\epsilon > 0$ and $f \in L_p(\lambda)$, there is a simple function f_0 such that

$$\sup_{x' \in B_{X'}} \left(\int_\Omega ||f|^p - |f_0|^p| \; d\,|\langle \lambda, x' \rangle| \right)^{1/p} < \epsilon.$$

A standard argument using the properties of the integrable functions with respect to a scalar measure, the decomposition of f into its positive and negative parts and the inequality $|a-b|^p \leq |a^p - b^p|$ for every $a, b \in [0, \infty]$ shows that there is a simple function f_1 such that $|f_1|^p = |f_0|^p$ and

$$\sup_{x' \in B_{X'}} \left(\int_\Omega |f - f_1|^p \; d\,|\langle \lambda, x' \rangle| \right)^{1/p} < 2\epsilon.$$

This means that the set of p-integrable functions is in the closure of the normed space of simple functions with respect to the norm $\| \cdot \|_{p,\lambda}$. Now we show that the limit of each Cauchy sequence of p-integrable functions is also p-integrable. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $L_p(\lambda)$ and $\epsilon > 0$. We can suppose that $\|f_n\|_{p,\lambda} \leq 1$ for every n. For each n, consider the canonical decomposition of the measurable function f_n into its positive and negative parts, $f_n = f_n^+ - f_n^-$. It is clear that $\|f_n^+\|_{p,\lambda} \leq 1$ and $\|f_n^-\|_{p,\lambda} \leq 1$ for every n. Moreover, the definition of the norm $\| \cdot \|_{p,\lambda}$ implies that the sequences $(f_n^+)_{n=1}^\infty$ and $(f_n^-)_{n=1}^\infty$ are also Cauchy sequences, since $|f_n - f_m| \geq |f_n^+ - f_m^+|$ and $|f_n - f_m| \geq |f_n^- - f_m^-|$ pointwise. Choose a natural number n_0 such that for all $n, m \geq n_0$

$$\sup_{x' \in B_{X'}} \left(\int_\Omega \left| f_n^+ - f_m^+ \right|^p \; d\,|\langle \lambda, x' \rangle| \right)^{1/p} < \epsilon.$$

Let $x'_0 \in B_{X'}$ and $A \in \Sigma$. Consider the scalar measure μ defined as $\mu(B) = \langle \lambda(B), x'_0 \rangle$ for each $B \in \Sigma$. Then there are positive measures μ_1 and μ_2 such that $\mu(B) = \mu_1(B) - \mu_2(B)$ for each $B \in \Sigma$, and we get for $i = 1, 2$,

$$\begin{aligned} \epsilon > \left(\int_A \left| f_n^+ - f_m^+ \right|^p \; d\,|\langle \lambda, x'_0 \rangle| \right)^{1/p} &\geq \left(\int_A \left| f_n^+ - f_m^+ \right|^p \; d\mu_i \right)^{1/p} \\ &\geq \left| \left(\int_A |f_n^+|^p \, d\mu_i \right)^{1/p} - \left(\int_A |f_m^+|^p \, d\mu_i \right)^{1/p} \right| \\ &\geq \frac{1}{p} \left| \left(\int_A |f_n^+|^p \, d\mu_i \right) - \left(\int_A |f_m^+|^p \, d\mu_i \right) \right|, \end{aligned}$$

where the last inequality holds since $|a^p - b^p| \le p|a-b|$ for all $a, b \in [0,1]$. Then,

$$\begin{aligned} 2p\epsilon &> \left| \left(\int_A |f_n^+|^p \, d\mu_1 \right) - \left(\int_A |f_m^+|^p \, d\mu_1 \right) \right| \\ &\quad + \left| \left(\int_A |f_m^+|^p \, d\mu_2 \right) - \left(\int_A |f_n^+|^p \, d\mu_2 \right) \right| \\ &\ge \left| \left(\int_A |f_m^+|^p \, d\langle \lambda, x_0' \rangle \right) - \left(\int_A |f_n^+|^p \, d\langle \lambda, x_0' \rangle \right) \right| \\ &= \left| \left\langle \left(\int_A |f_m^+|^p \, d\lambda \right) - \left(\int_A |f_n^+|^p \, d\lambda \right), x_0' \right\rangle \right|. \end{aligned}$$

Thus, $\sup_{A \in \Sigma} \left\| \int_A |f_m^+|^p \, d\lambda - \int_A |f_n^+|^p \, d\lambda \right\| < 2p\epsilon$, and $\left(|f_n^+|^p\right)_{i=1}^\infty$ is a Cauchy sequence in $L_1(\lambda)$. Since $L_1(\lambda)$ is a Banach space, there is a function h^+ that is the limit of $\left(|f_n^+|^p\right)_{i=1}^\infty$ in $L_1(\lambda)$. Moreover, we can suppose that h^+ is a positive function. Let us define $f^+ := (h^+)^{1/p}$. We can use the same argument to find a limit h^- of the sequence $\left(|f_n^-|^p\right)_{i=1}^\infty$ in $L_1(\lambda)$ and a function $f^- := (h^-)^{1/p}$. Consider the measurable function $f := f^+ - f^-$. It is clear that $f \in L_p(\lambda)$. Moreover, if we denote by $\lambda_{x'}$ the measures $|\langle \lambda, x' \rangle|$ for each $x' \in X'$, we obtain for every n,

$$\begin{aligned} &\sup_{x' \in B_{X'}} \left(\int_\Omega \left| |f_n^+|^p - h^+ \right| \, d\lambda_{x'} \right)^{1/p} + \sup_{x' \in B_{X'}} \left(\int_\Omega \left| |f_n^-|^p - h^- \right| \, d\lambda_{x'} \right)^{1/p} \\ &\quad \ge \sup_{x' \in B_{X'}} \left(\int_\Omega |f_n^+ - f^+|^p \, d\lambda_{x'} \right)^{1/p} + \sup_{x' \in B_{X'}} \left(\int_\Omega |f_n^- - f^-|^p \, d\lambda_{x'} \right)^{1/p} \\ &\quad \ge \sup_{x' \in B_{X'}} \left(\int_\Omega \left| f_n^+ - f_n^- - (f^+ - f^-) \right|^p \, d\lambda_{x'} \right)^{1/p}, \end{aligned}$$

where we have used again the inequality $|a-b|^p \le |a^p - b^p|$ for every $a, b \in [0, \infty]$. This shows that $f \in L_p(\lambda)$ is the limit of the sequence $(f_n)_{n=1}^\infty$. □

PROPOSITION 5. *Let $p \ge 1$. Then $L_p(\lambda)$ is a Köthe function space.*

Proof. Let μ be a control measure for λ (for instance, a Rybakov measure; see IX.2 of [8]). We will show that $L_p(\lambda)$ is a Köthe function space over μ (see Def. 1.b.17 in [14]). Suppose that f is a μ-measurable function and $g \in L_p(\lambda)$ such that $|f| \le |g|$ μ-a.e.. Then $|g|^p \in L_1(\lambda)$. Since $|f|^p \le |g|^p$, we get $|f|^p \in L_1(\lambda)$ by applying Theorem 1 in [2]. Moreover,

$$\int_\Omega |f|^p \, d|\langle \lambda, x' \rangle| \le \int_\Omega |g|^p \, d|\langle \lambda, x' \rangle|$$

for every $x' \in X'$. This means that $\|f\|_{p,\lambda} \leq \|g\|_{p,\lambda}$. For each $A \in \Sigma$, we have $\chi_A \in L_p(\lambda)$, and $\|\chi_A\|_{p,\lambda}$ is equivalent to $\|\lambda\|^p(A)$, as a consequence of the equivalent expression for the norm given in Definition 1. $\square$

PROPOSITION 6. *Let $p \geq 1$. Then $L_p(\lambda)$ is an order continuous Banach lattice with weak order unit.*

Proof. The following proof is similar to the proof of Theorem 1 in [2] but we include it for the sake of completeness. $L_p(\lambda)$ is a Banach lattice with the natural order, i.e., the pointwise λ-a.e. order. We use the characterization of order continuity given in 1.a.8 of [14]: a Banach lattice X is order continuous if every increasing order bounded sequence is convergent in the norm topology of X. Take such a sequence $(f_n)_{n=1}^\infty$ in $L_p(\lambda)$. We can suppose that $0 \leq f_n \leq f_{n+1} \leq g$, where $g \in L_p(\lambda)$. Let us define $f(\omega) = \sup_n f_n(\omega)$, for $\omega \in \Omega$. On the one hand, for each $x' \in X'$, $(f_n)_{n=1}^\infty$ is order bounded (as $g \in L_p(|\langle \lambda, x' \rangle|)$, and thus $f \in L_p(|\langle \lambda, x' \rangle|)$, since $L_p(|\langle \lambda, x' \rangle|)$ is order continuous. On the other hand, the measure $\Phi_{|g|^p}(A) = \int_A |g|^p \, d\lambda$ is absolutely continuous with respect to the semivariation $\|\lambda\|$ (using the fact that a function f which is scalarly integrable is integrable with respect to λ if and only if the measure $\Phi_f(A) = \int_A f \, d\lambda$ is absolutely continuous with respect to $\|\lambda\|$; see [13]). Therefore we get

$$\left\| \int_A |f|^p \, d\lambda \right\| \leq \sup \left\{ \int_A |f|^p \, d\,|\langle x', \lambda \rangle| : x' \in B_{X'} \right\} \leq \left\| \Phi_{|g|^p} \right\| (A).$$

This means that $|f|^p \in L_1(\lambda)$, and thus $f \in L_p(\lambda)$. It remains to check that $\|f_n - f\|_{p,\lambda} \to 0$. Let $\epsilon > 0$. Since the measure $\Phi_{|f-f_1|^p}$ is absolutely continuous with respect to μ (as shown in the proof of Proposition 5), there is a $\delta_\epsilon > 0$ such that if $\mu(A) < \delta_\epsilon$ then $\Phi_{|f-f_1|^p}(A) < \epsilon$. Egoroff's Theorem gives a subset $A_0 \in \Sigma$ such that $\mu(A_0) < \delta_\epsilon$ and the convergence of $(f_n)_{n=1}^\infty$ is uniform in $\Omega - A_0$. Thus, if, for each $x' \in X$, we denote by $\lambda_{x'}$ the measures $|\langle \lambda, x' \rangle|$, we can write

$$\begin{aligned} \||f_n - f|^p\|_\lambda &= \sup_{x' \in B_{X'}} \int_\Omega |f_n - f|^p \, d\lambda_{x'} \\ &\leq \sup_{x' \in B_{X'}} \int_{\Omega - A_0} |f_n - f|^p \, d\lambda_{x'} + \sup_{x' \in B_{X'}} \int_{A_0} |f_1 - f|^p \, d\lambda_{x'} \\ &\leq \epsilon \|\lambda\| \, (\Omega - A_0) + \left\| \Phi_{|f_1 - f|^p} \right\| (A_0) \end{aligned}$$

if $n \geq n_0$, where n_0 only depends on ϵ. This gives the result.

Finally, let us show that χ_Ω is a weak order unit. An element $e \geq 0$ of a Banach lattice L is said to be a weak order unit of L if $e \wedge x = 0$ for $x \in L$ implies $x = 0$, where $y \wedge z$ denotes the greatest lower bound for $y, z \in L$ (see [14]). Since $L_p(\lambda)$ is an order continuous Banach lattice, it is σ-complete (see Proposition 1.a.8 in [14]). The elements of $L_p(\lambda)$ are functions, and the order is the pointwise order. Thus, the projection P_{χ_Ω} associated to χ_Ω is defined as the multiplication operator $P_{\chi_\Omega}(f) := \chi_\Omega f$. Then χ_Ω is a weak order unit since obviously $\chi_\Omega f = f$ for every $f \in L_p(\lambda)$ (see p. 9 of [14]). $\square$

The results of this section lead to an easy representation of the dual space of $L_p(\lambda)$. Let μ be a Rybakov measure for λ. As in the case of $L_1(\lambda)$ (see Theorem 1.b.14 in [14], [13], [15] or Proposition 1.1 in [16]), we can obtain the dual space $(L_p(\lambda))'$ as the Köthe function space of μ-measurable functions h that satisfy

$$\sup_{\|g\|_{p,\lambda} \leq 1} \left| \int_\Omega gh \, d\mu \right| < \infty.$$

This expression defines a norm for the dual space, and the duality is given by $\langle g, h \rangle = \int_\Omega gh \, d\mu$ (see Lemma 1 in [5]).

3. Vector measure duality and compactness arguments

Let $p > 1$. We denote by p' the real number that satisfies $1/p + 1/p' = 1$. If λ is a vector measure, let us fix a function $f \in L_{p'}(\lambda)$. This function defines a linear map $\widetilde{f} \colon L_p(\lambda) \to X$ via the expression $\widetilde{f} \colon g \to \int_\Omega fg \, d\lambda$, for $g \in L_p(\lambda)$. Indeed, the following inequalities and the density of the simple functions in $L_p(\lambda)$-spaces (Proposition 4) show that this map is well-defined and continuous. In particular, it is easy to see that the product fg gives an integrable function with respect to λ. Moreover, the multiplication map may be defined (and is also continuous) with images in $L_1(\lambda)$. For a simple function g, the required inequality (via Hölder's inequality for scalar measures) is

$$\begin{aligned} \left\| \int_A fg \, d\lambda \right\| &\leq \sup_{x' \in B_{X'}} \left| \int_\Omega |fg| \, d \, |\langle \lambda, x' \rangle| \right| \\ &\leq \left(\sup_{x' \in B_{X'}} \left| \int_\Omega |g|^p d \, |\langle \lambda, x' \rangle| \right|^{1/p} \right) \cdot \left(\sup_{x' \in B_{X'}} \left| \int_\Omega |f|^{p'} d \, |\langle \lambda, x' \rangle| \right|^{1/p'} \right), \end{aligned}$$

for each $A \in \Sigma$. These operators may be used in order to define a "vector valued duality" between the spaces $L_p(\lambda)$ and $L_{p'}(\lambda)$.

Definition 7. Let μ be a (finite) control measure for a vector measure λ and let L be a Köthe function space on (Ω, Σ, μ). Consider the (linear) space $L_0(\mu)$ of (equivalence classes of μ-a.e.) simple functions f that satisfy:

(1) The function fg is integrable with respect to λ for each $g \in L$.
(2) The norm $\|f\|_{L^\lambda} = \sup_{\|g\|_L \leq 1} \|fg\|_\lambda$ is finite.

We define the Banach space L^λ of μ-measurable functions as the completion of the space $L_0(\mu)$ with respect to the norm given in (2). The same expression can be used for every $f \in L^\lambda$.

Using the equivalent formula $|||\cdot|||_\lambda$ for the norm of $L_1(\lambda)$ (see Section 2) we see that the following norm is equivalent to the norm of L^λ defined above:

$$|||f|||_{L^\lambda} := \sup_{A\in\Sigma, \|g\|_L \le 1} \left\| \int_A f g \, d\lambda \right\|_X .$$

PROPOSITION 8. *Let $p>1$. Then $(L_p(\lambda))^\lambda = L_{p'}(\lambda)$.*

Proof. The inequalities at the begining of this section give $\|f\|_{(L_p(\lambda))^\lambda} \le \|f\|_{p',\lambda}$ and the rest of the conditions needed to assure that $L_{p'}(\lambda) \subset (L_p(\lambda))^\lambda$. Now suppose that $f \in L_{p'}(\lambda)$. Let us define the function

$$g = \frac{f^{p'-1}}{(\|f\|_{p',\lambda})^{p'/p}}.$$

On the one hand, since $p'-1 = p'/p$,

$$\|g\|_{p,\lambda} = \frac{\sup_{x'\in B_{X'}} \left(\int_\Omega |f|^{(p'-1)p} \, d\,|\langle \lambda, x'\rangle|\right)^{1/p}}{\sup_{x'\in B_{X'}} \left(\int |f|^{p'} \, d\,|\langle \lambda, x'\rangle|\right)^{1/p}} = 1$$

On the other hand, we have

$$\begin{aligned}\|f\|_{(L_p(\lambda))^\lambda} &\ge \|fg\|_\lambda \\ &= \sup_{x'\in B_{X'}} \int_\Omega |f| \frac{|f|^{p'-1}}{\sup_{x'\in B_{X'}} \left(\int |f|^{p'} \, d\,|\langle \lambda, x'\rangle|\right)^{1/p}} \, d\,|\langle \lambda, x'\rangle| \\ &= \sup_{x'\in B_{X'}} \left(\int_\Omega |f|^{p'} \, d\,|\langle \lambda, x'\rangle|\right)^{1/p'} = \|f\|_{p',\lambda}.\end{aligned}$$

These inequalities and the density of the set of the simple functions in both spaces give the result. $\square$

In particular, if we call a Banach function space L λ-reflexive when $(L^\lambda)^\lambda = L$, then the spaces $L_p(\lambda)$ are λ-reflexive for $p>1$. We will use the description of $L_p(\lambda)$ as $(L_p(\lambda)^\lambda)^\lambda$ in the following section. However, the dual space $(L_p(\lambda))'$ does not coincide with $L_{p'}(\lambda)$, even in the case when the range of λ is relatively compact. Of course, for each $x' \in X'$ there is an operator $I_{x'}$ from $L_{p'}(\lambda)$ to $(L_p(\lambda))'$ given by the formula $\langle I_{x'}(f), g\rangle := \left\langle \int_\Omega fg \, d\lambda, x' \right\rangle$. Moreover, if $h \in L_{p'}(|\langle \lambda, x'\rangle|)$ then the expression $\xi_{h,x'} \colon g \to \int_\Omega hg \, d\,|\langle \lambda, x'\rangle|$ defines an element of $(L_p(\lambda))'$, since

$$\begin{aligned}\left| \int_\Omega hg \, d\,|\langle \lambda, x'\rangle| \right| &\le \left(\int_\Omega |g|^p \, d\,|\langle \lambda, x'\rangle|\right)^{1/p} \left(\int_\Omega |h|^{p'} \, d\,|\langle \lambda, x'\rangle|\right)^{1/p'} \\ &\le K\|g\|_{p,\lambda}.\end{aligned}$$

The case when x_0' defines a Rybakov measure for λ leads in this way to a continuous inclusion of $L_{p'}(\lambda)$ into $(L_p(\lambda))'$ via $L_p(\lambda) \ni h \to \xi_{h,x_0'}$, since the dual space of a Köthe function space can be represented by means of the duality relation $\int fgd\mu$ for a Rybakov measure μ.

The following example shows that $L_{p'}(\lambda)$ and $(L_p(\lambda))'$ are different spaces. Let μ be the Lebesgue measure for $\Omega = [0,\infty]$ and $p > 1$. Consider the vector measure $\lambda\colon \Sigma \to l_2$ given by $\lambda(A) := \sum_{i=1}^{\infty} \frac{\mu(A\cap[i-1,i])}{2^{i/2}} e_i$, where $\{e_i\}_{i=1}^{\infty}$ defines the canonical basis of l_2. Take the element $x_0' = \sum_{i=1}^{\infty} \frac{1}{2^{i/2}} e_i \in l_2$ and the function $f := \sum_{i=1}^{\infty} \frac{2^{i/p'}}{i} \chi_{[i-1,i]}$. Then f belongs to $L_{p'}(|\langle \lambda, x_0'\rangle|) \subset (L_p(\lambda))'$, since

$$\int_{[0,\infty]} |f|^{p'}\, d\,|\langle \lambda, x_0'\rangle| = \sum_{i=1}^{\infty} \frac{2^i}{i^{p'}} \cdot \frac{1}{2^i} = \sum_{i=1}^{\infty} \frac{1}{i^{p'}} < \infty.$$

However, f is not an element of $L_{p'}(\lambda)$. Let us show that $|f|^{p'}$ is not λ-integrable. Since l_2 does not contain an isomorphic copy of c_0, the λ-integrability of $|f|^{p'}$ is equivalent to its $|\langle \lambda, x'\rangle|$-integrability, for each $x' \in X'$ (see Theorem 1 on p. 31 of [10]). Thus it is enough to find an element $x_1' \in X'$ such that $|f|^{p'}$ is not $|\langle \lambda, x_1'\rangle|$-integrable. Take the sequence $x_1' = \sum_{i=1}^{\infty} \frac{i^{p'/p}}{2^{i/2}} e_i$, in which case x_1' is an element of l_2. However, a direct calculation of the integral of $|f|^{p'}$ with respect to $|\langle \lambda, x_1'\rangle|$ gives the series

$$\sum_{i=1}^{\infty} \left(\frac{2^i}{i^{p'}}\right) \cdot \frac{1}{2^{i/2}} . \frac{i^{p'/p}}{2^{i/2}} = \sum_{i=1}^{\infty} i^{p'/p - p'} = \sum_{i=1}^{\infty} \frac{1}{i},$$

which does not converge. This means that

$$\int_{[0,\infty]} |f|^{p'}\, d\,|\langle \lambda, x_1'\rangle| = \infty.$$

Thus, f is not an element of $L_{p'}(\lambda)$.

The aim of the rest of this section is to obtain compactness results for the unit ball of the spaces $L_p(\lambda)$ endowed with a topology that is coarser than the norm topology. In order to do this we define two locally convex topologies for the spaces $L_p(\lambda)$. Note that, for $p > 1$, $L_p(\lambda)$ can also be represented (isometrically) as $(L_{p'}(\lambda))^{\lambda}$, as a consequence of Proposition 8. Then we can use the equivalent norm $|||\cdot|||_{(L_{p'}(\lambda))^{\lambda}}$ given after Definition 7 for the space $L_p(\lambda)$. From now on, we will use the norm $|||\cdot|||_{(L_{p'}(\lambda))^{\lambda}}$ for the space $L_p(\lambda)$, and we will denote it briefly by $\|\cdot\|_{L_p(\lambda)}$. For simplicity of notation, the definition of the unit ball $B_{L_p(\lambda)}$ of $L_p(\lambda)$ will also be with respect to this norm. We will write $B_{p,\lambda}$ for the unit ball defined by the usual norm $\|\cdot\|_{p,\lambda}$.

Lemma 9. *Let $p > 1$. Then*

$$\|g\|_{L_p(\lambda)} := |||g|||_{(L_{p'}(\lambda))^{\lambda}} = \sup_{\|f\|_{p',\lambda}\le 1} \left\| \int_{\Omega} fg\, d\lambda \right\|,$$

for every $g \in L_p(\lambda)$. Thus the unit ball $B_{L_p(\lambda)}$ of the space $L_p(\lambda)$ (with respect to this norm) can then be represented as

$$B_{L_p(\lambda)} = \left\{ g \in L_p(\lambda) : \sup_{\|f\|_{p',\lambda} \leq 1} \left\| \int_\Omega fg \, d\lambda \right\| \leq 1 \right\}.$$

Proof. The result is a direct consequence of Proposition 8 and the definition of the equivalent norm for the space $L_{p'}(\lambda)^\lambda$ given after Definition 7. The lattice property of the space $L_{p'}(\lambda)$ implies that $\|f\chi_A\|_{p',\lambda} \leq \|f\|_{p',\lambda}$ for each function $f \in L_{p'}(\lambda)$ and $A \in \Sigma$. This means that

$$\sup_{A \in \Sigma, \|f\|_{p',\lambda} \leq 1} \left\| \int_A fg \, d\lambda \right\|_X = \sup_{\|f\|_{p',\lambda} \leq 1} \left\| \int_A fg \, d\lambda \right\|_X . \qquad \square$$

Definition 10. Let $p > 1$ and consider the space $L_p(\lambda)$. Given $g_0 \in L_p(\lambda)$, $\epsilon > 0$, $n \in \mathbf{N}$, and $f_1, f_2, \dots, f_n \in L_{p'}(\lambda)$, we define the set

$$\xi_{\epsilon, f_1, \dots, f_n}(g_0) := \left\{ g \in L_p(\lambda) : \left\| \int_\Omega f_i (g - g_0) \, d\lambda \right\|_X < \epsilon, \forall i = 1, \dots, n \right\}.$$

It is easy to see that a (Hausdorff) locally convex topology on $L_p(\lambda)$ can be defined if we consider the class of all the sets $\xi_{\epsilon, f_1, \dots, f_n}(g_0)$ for every $g_0 \in L_p(\lambda)$ as a basis of neighbourhoods. We call this topology the λ-topology for the space $L_p(\lambda)$.

It is obvious that all the multiplication operators $T_f \colon L_p(\lambda) \to X$ defined by $T_f(g) = \int_\Omega fg \, d\lambda$, where $f \in L_{p'}(\lambda)$, are continuous with respect to the λ-topology.

Definition 11. Let $p > 1$. Given a function $g_0 \in L_p(\lambda)$, $\epsilon > 0$, $n \in \mathbf{N}$, $x_1', \dots, n_n' \in X'$, and $f_1, f_2, \dots, f_n \in L_{p'}(\lambda)$, we define the set

$$\xi_{\epsilon, f_1, \dots, f_n, x_1', \dots, x_n'}(g_0) := \left\{ g \in L_p(\lambda) : \left| \left\langle \int_\Omega f_i (g - g_0) \, d\lambda, x_i' \right\rangle \right| < \epsilon, \forall i = 1, \dots, n \right\}.$$

Then the λ-weak topology for the space $L_p(\lambda)$ is the Hausdorff locally convex topology which has as a basis of neighbourhoods the family of sets

$$\xi_{\epsilon, f_1, \dots, f_n, x_1', \dots, x_n'}(g_0).$$

It is easy to see that the λ-topology is coarser than the norm topology and finer than the λ-weak topology. The λ-weak topology is also coarser than the weak topology of $L_p(\lambda)$, as the following argument shows. Let $f \in L_{p'}(\lambda)$ and $x' \in X'$, and consider the continuous linear form $\phi(g) = \left\langle \int_\Omega fg \, d\lambda, x' \right\rangle$. If μ is a Rybakov measure for λ, then λ is absolutely continuous with respect to μ, and so there is a Radon-Nikodym derivative f_0 (depending on f and x')

for the scalar measure $\langle \lambda(A), x' \rangle = \int_A f_0 \, d\mu$. Thus $\phi(g) = \int_\Omega g f f_0 \, d\mu$ and $f f_0$ defines an element of $(L_p(\lambda))'$ since

$$\sup_{\|g\|_{p,\lambda} \leq 1} \left| \int_\Omega g f f_0 \, d\mu \right| = \sup_{\|g\|_{p,\lambda} \leq 1} \left| \left\langle \int_\Omega g f \, d\lambda, x' \right\rangle \right| \leq \|x'\| \cdot \|f\|_{L_{p'}(\lambda)}.$$

PROPOSITION 12. *The unit ball of the space $L_p(\lambda)$ is closed for the λ-weak topology, and thus for the λ-topology.*

Proof. Let $\hat{B}^\lambda_{L_p(\lambda)}$ be the closure of $B_{L_p(\lambda)}$ with respect to the λ-weak topology. Suppose that $B_{L_p(\lambda)} \neq \hat{B}^\lambda_{L_p(\lambda)}$. Then there is a function $g_0 \in \hat{B}^\lambda_{L_p(\lambda)} - B_{L_p(\lambda)}$, and hence a $\delta > 0$ such that $\|g_0\|_{L_p(\lambda)} > 1 + \delta$. Thus there exists a function $f_0 \in B_{p',\lambda}$ and a norm one element $x'_0 \in X'$ such that $\left| \left\langle \int_\Omega g_0 f_0 \, d\lambda, x'_0 \right\rangle \right| > 1 + \delta$. On the other hand, since $\hat{B}^\lambda_{L_p(\lambda)}$ is closed for the λ-weak topology, there exists a sequence $(g_n)_{n=1}^\infty \subset B_{L_p(\lambda)}$ such that $\left| \left\langle \int_\Omega f_0 (g_0 - g_n) \, d\lambda, x'_0 \right\rangle \right| < 1/n$ for every natural number n. But

$$\begin{aligned} \frac{1}{n} &> \left| \left\langle \int_\Omega f_0 g_0 \, d\lambda - \int_\Omega f_0 g_n \, d\lambda, x'_0 \right\rangle \right| \\ &\geq \left| \left| \left\langle \int_\Omega f_0 g_0 \, d\lambda, x'_0 \right\rangle \right| - \left| \left\langle \int_\Omega f_0 g_n \, d\lambda, x'_0 \right\rangle \right| \right| > 1 + \delta - \|g_n\|_{L_p(\lambda)} \geq \delta, \end{aligned}$$

and we get a contradiction. □

PROPOSITION 13. *The unit ball $B_{L_p(\lambda)}$ is compact for the λ-weak topology.*

Proof. The proof follows the lines of the classical proof of Alaoglu's theorem. For each $x' \in X'$, consider the product space $S_{x'} := \Pi_{f \in L_{p'}(\lambda)} B_{x',f}$, where

$$B_{x',f} = \left\{ \left\langle \int_\Omega f g \, d\lambda, x' \right\rangle : g \in B_{L_p(\lambda)} \right\} \subset R.$$

These sets are compact, since $\left\langle \int_\Omega f(\cdot) \, d\lambda, x' \right\rangle$ defines a linear form in $L_p(\lambda)$. Thus, the product space $S_{x'}$ is compact as a consequence of Tychonov's Theorem. Now consider the product space $\Pi_{x' \in X'} S_{x'}$. Another application of Tychonov's Theorem gives the compactness of this topological space. We can identify the elements $g \in B_{L_p(\lambda)}$ via their coordinates $\left\langle \int_\Omega f g \, d\lambda, x' \right\rangle$ as elements of the product space. The definition of the λ-weak topology makes it clear that it coincides with the product topology in the topological subspace of the product defined by $B_{L_p(\lambda)}$. Then the closure of $B_{L_p(\lambda)}$ in this topology is compact. Proposition 12 then yields the desired conclusion. □

We are interested in the case when the unit ball is also compact for the λ-topology. This is not true in general, as shown by the following example. Consider the measure space $(\Sigma, [0,1], \mu)$, where Σ is the σ-algebra of Borel subsets and μ is the Lebesgue measure, and the vector measure of bounded variation $\lambda\colon \Sigma \to L_1(\mu)$ given by $\lambda(A) = \chi_A$. The range $\mathrm{rg}(\lambda)$ of λ is closed but not compact (see p. 261 of [8]). If $p > 1$, a direct calculation shows that in this case $L_{p'}(\lambda) = L_{p'}(\mu)$, and the λ-topology is finer than the weak topology. However, $B_{L_p(\lambda)}$ is not a compact set for the λ-topology as this would imply that the range of the vector measure λ is compact. Indeed, if f is a function in $L_{p'}(\lambda)$, then the integral map $T_f\colon L_p(\lambda) \to X$ given by $T_f(g) := \int fg\, d\lambda$ is obviously continuous for the λ-topology. We also have that $\chi_{[0,1]} \in B_{L_{p'}(\lambda)}$ and, for each $A \in \Sigma$, $\|\chi_A\|_{p,\lambda} \le 1$. Then if $B_{L_p(\lambda)}$ is λ-compact the set $B_{\chi_{[0,1]}} = \left\{\int_\Omega g\, d\lambda \in L_1(\mu) : g \in B_{L_p(\lambda)}\right\}$ would also be compact. In fact, in this case $L_p(\lambda)$ coincides with $L_p(\mu)$.

The following theorem characterizes the situation when the unit ball is compact for the λ-topology in terms of the different topologies we have defined for $L_p(\lambda)$.

THEOREM 14. *Let $p > 1$ and λ be a vector measure. The following are equivalent.*

(1) *The unit ball $B_{L_p(\lambda)}$ is compact for the λ-topology.*
(2) *For each $f \in L_{p'}(\lambda)$, the operator $g \to T_f(g) = \int_\Omega fg\, d\lambda$, from $L_p(\lambda)$ into X, is compact.*
(3) *The λ-topology coincides with the λ-weak topology.*

Proof. First we show the equivalence of (1) and (2). We apply the same argument that we used to prove Proposition 13. Consider the product space $\Pi_{f \in L_{p'}(\lambda)} \hat{B}_f$, where, for $f \in L_{p'}(\lambda)$, $\hat{B}_f$ is the closure in X of the set

$$B_f := \left\{ \int fg\, d\lambda \in X : g \in B_{L_p(\lambda)} \right\}.$$

We can identify the element $g \in B_{L_p(\lambda)}$ with its coordinates $\left(\int_\Omega fg\, d\lambda\right)_{f \in L_{p'}(\lambda)}$ as elements of the product space. This space is compact, by Tychonov's Theorem. From the definition of the λ-topology it is clear that this topology is exactly the product topology when restricted to the topological subspace $B_{L_p(\lambda)}$. Since the latter space is closed, it is compact.

The converse is obvious (since each X-valued, linear function $\int_\Omega f(\cdot)\, d\lambda$ is continuous for the λ-topology).

Now we prove the equivalence of (1) and (3). Clearly (3) implies (1) since $B_{L_p(\lambda)}$ is compact for the λ-weak topology. To see the converse, we will use the following property, which is easily proved: if B is a compact subset (for the norm topology) of a Banach space, then for each $\epsilon > 0$ there is a finite number of norm one linear functionals $x_i' \in X'$, $i = 1, \dots, n$ such that, for each $x \in B$, there is a number $i \in \{1, \dots, n\}$ satisfying that $\|x\| \le |\langle x, x_i' \rangle| + \epsilon$.

Let

$$\xi_{\epsilon,f_1,\dots,f_n}(0) := \left\{ g \in L_p(\lambda) : \left\| \int_\Omega f_i g \, d\lambda \right\|_X < \epsilon, \forall i = 1, \dots, n \right\}$$

(where $f_i \in L_{p'}(\lambda)$ for $i = 1, \dots, n$) be a basic neighbourhood of $0 \in B_{L_p(\lambda)}$ for the λ-topology. Then we just need to find a neighbourhood of 0 for the λ-weak topology V such that $V \subset \xi_{\epsilon,f_1,\dots,f_n}(0)$. Take $i_0 \in \{1, \dots, n\}$, and consider the set $A_{i_0} := \left\{ \int_\Omega f_{i_0} g \, d\lambda : g \in B_{L_p(\lambda)} \right\}$. This set is well-defined and bounded (see Lemma 9). Moreover, it is a compact set in X, and so we can find finitely many elements $x'_1, \dots, x'_m$ of the dual space X' satisfying the above property for $\epsilon/2$. Thus, for each $g \in B_{L_p(\lambda)}$, there is an index $j \in \{1, \dots, m\}$ such that $\left\| \int f_{i_0} g \, d\lambda \right\| \leq \epsilon/2 + \left| \left\langle \int_\Omega f_{i_0} g \, d\lambda, x'_j \right\rangle \right|$. Therefore, the λ-weak neighbourhood of 0 defined by

$$V_{i_0} := \bigcap_{j=1,\dots,m} \left\{ g \in B_{L_p(\lambda)} : \left| \left\langle \int f_{i_0} g \, d\mu, x'_j \right\rangle \right| < \frac{\epsilon}{2} \right\}$$

is contained in the λ-neighbourhood $\xi_{\epsilon,f_{i_0}}(0)$. Since

$$\xi_{\epsilon,f_1,\dots,f_n}(0) = \bigcap_{i=1,\dots,n} \xi_{\epsilon,f_i}(0),$$

we see that $V = \bigcap_{i_0=1,\dots,n} V_{i_0}$ is the desired λ-weak neighbourhood. $\square$

Definition 15. A vector measure λ satisfying any one of the conditions of Theorem 14 will be called p-compact.

If $L_p(\lambda)$ is a reflexive space, then λ is a p-compact vector measure if and only if the λ topology is coarser than the weak topology. This is a direct consequence of Theorem 14 and the fact that the weak topology is finer than the λ-weak topology.

4. Factorizations through spaces of Bochner integrable functions

In this section we apply the above results to obtain our factorization theorem for Köthe function spaces. First we show how we can use the results for operators defined in the spaces $L_p(\lambda)$.

Lemma 16. *Let $p > 1$ and $q \geq 1$, and let λ be an X-valued p'-compact vector measure. Let Y be a Banach space and T a continuous linear operator $T\colon L_p(\lambda) \to Y$. Then the following statements are equivalent:*

(1) *There is a constant K such that, for each finite sequence $(g_i)_{i=1}^n \subset L_p(\lambda)$, we have*

$$\left(\sum_{i=1}^n \| T(g_i) \|_Y^q \right)^{1/q} \leq K \sup_{\|f\|_{L_{p'}(\lambda)} \leq 1} \left(\sum_{i=1}^n \left\| \int_\Omega g_i f \, d\lambda \right\|_X^q \right)^{1/q}.$$

(2) *There is a constant K and a regular probability measure μ_0 defined on the Borel sets of $B_{L_{p'}(\lambda)}$ such that, for every $g \in L_p(\lambda)$,*

$$\|T(g)\|_Y \le K \left(\int_{B_{L_{p'}(\lambda)}} \left\| \int_\Omega fg \, d\lambda \right\|_X^q \, d\mu_0(f) \right)^{1/q}.$$

Proof. First we give an easy proof of the implication (2) $\Rightarrow$ (1). Let $(g_i)_{i=1}^n$ be a finite sequence of functions in $L_p(\lambda)$. Then

$$\begin{aligned} \sum_{i=1}^n \|T(g_i)\|_Y^q &\le K^q \int_{B_{L_{p'}(\lambda)}} \sum_{i=1}^n \left\| \int_\Omega f g_i \, d\lambda \right\|_X^q \, d\mu_0(f) \\ &\le K^q \sup_{f \in B_{L_{p'}(\lambda)}} \left(\sum_{i=1}^n \left\| \int_\Omega f g_i \, d\lambda \right\|_X^q \right). \end{aligned}$$

For the converse, we use a classical separation argument based on Ky Fan's Lemma (see [18] or [7]). Consider the topological space $B_{L_{p'}(\lambda)}$ endowed with the λ-topology, and the space $M(B_{L_{p'}(\lambda)})$ of regular Borel measures. Riesz' Theorem states that this space is the dual of the space of continuous functions $C(B_{L_{p'}(\lambda)})$. Consider the subset $P(B_{L_{p'}(\lambda)}) \subset M(B_{L_{p'}(\lambda)})$ of probability measures. This is a convex set which is compact if we endow $M(B_{L_{p'}(\lambda)})$ with its weak* topology. In this context, we need to define an appropriate set of functions N on $P(B_{L_{p'}(\lambda)})$ satisfying the properties that are required to apply Ky Fan's Lemma (see p. 190 of [7]). For each finite sequence $(g_i)_{i=1}^n$ from $L_p(\lambda)$ we can define a function $\mu \to \Phi_{g_1,\dots,g_n}(\mu)$, for $\mu \in M(B_{L_{p'}(\lambda)})$, by

$$\Phi_{g_1,\dots,g_n}(\mu) := \sum_{i=1}^n \|T(g_i)\|_Y^q - K^q \int_{B_{L_{p'}(\lambda)}} \sum_{i=1}^n \left\| \int_\Omega f g_i \, d\lambda \right\|_X^q \, d\mu(f).$$

We define N to be the set of all such functions. Then we have:

(a) Each function $\Phi_{g_1,\dots,g_n}(\cdot)$ is clearly convex, and continuous in the weak* topology of $M\big(B_{L_{p'}(\lambda)}\big)$, since for each $g \in L_p(\lambda)$ the function $\Psi_g(f) := \big\| \int_\Omega fg \, d\lambda \big\|^q$ belongs to $C\big(B_{L_{p'}(\lambda)}\big)$.

(b) A direct calculation shows that each convex combination of two functions from N gives another function in N (see, for example, p. 192 of [7]).

(c) Consider a function $\Phi_{g_1,\dots,g_n}(\cdot) \in N$. Assumption (1), the compactness of $B_{L_{p'}(\lambda)}$ and the continuity of the functions Ψ_g of (a) give a function $f_0 \in B_{L_{p'}(\lambda)}$ such that

$$\sum_{i=1}^n \|T(g_i)\|_Y^q - K^q \sum_{i=1}^n \left\| \int_\Omega f_0 g_i \, d\lambda \right\|_X^q \le 0.$$

Then the discrete measure δ_{f_0} satisfies $\Phi_{g_1,\dots,g_n}\big(\delta_{f_0}\big) \le 0$. An application of Ky Fan's Lemma gives the desired probability measure μ_0. $\square$

Let Q be a Hausdorff compact topological space and let μ be a regular Borel probability measure on Q. If X is Banach space, consider the space of X-valued continuous functions $C(Q,X)$ and the space $L_p(Q,\mu,X)$ of X-valued Bochner μ-integrable functions. We write I_p for the natural inclusion map $I_p\colon C(Q,X) \to L_p(Q,\mu,X)$. It is well known that this map is continuous, injective and $\|I_p\| \le 1$.

If we consider the space of continuous functions $C\big(B_{L_{p'}(\lambda)}, X\big)$ the results of Section 3 make it clear that the map $\mathrm{Id}\colon L_p(\lambda) \to C\big(B_{L_{p'}(\lambda)}, X\big)$ defined as $\mathrm{Id}(g) := \int g(\cdot)\, d\lambda$ is an isometry. Thus we can identify $L_p(\lambda)$ with the subspace $\mathrm{Id}(L_p(\lambda))$.

The following theorem is just the "factorization form" of the above lemma.

Theorem 17. *Let $p > 1$ and $q \ge 1$, and λ be an X-valued p'-compact vector measure. Let $T\colon L_p(\lambda) \to Y$ be a continuous linear operator satisfying* (1) *of Lemma 16. Then there is a probability measure $\mu_0 \in M\big(B_{L_{p'}(\lambda)}\big)$ such that T factorizes as follows:*

$$\begin{tikzcd}
L_p(\lambda) \arrow[r, "T"] \arrow[d, "\mathrm{Id}"'] & Y \\
G \subset C(B_{L_{p'}}, X) \arrow[r, "I_q"] & I_q(G) \subset L_q(B_{L_{p'}}, \mu_0, X) \arrow[u, "T_1"']
\end{tikzcd}$$

where G and $I_q(G)$ are the subspaces of $C\big(B_{L_{p'}(\lambda)}, X\big)$ and $L_q\big(B_{L_{p'}(\lambda)}, \mu_0, X\big)$, respectively, defined by the functions of $L_p(\lambda)$.

Proof. The map $\mathrm{Id}\colon L_p(\lambda) \to C\big(B_{L_{p'}(\lambda)}, X\big)$ is a continuous operator, and its image $G := \mathrm{Id}(L_p(\lambda)$ is closed. Consider the Bochner space $L_q\big(B_{L_{p'}(\lambda)}, \mu_0, X\big)$, where μ_0 is a probability measure given by Lemma 16. The restriction of the inclusion I_q to the subspace G is also continuous. Finally, the operator T_1 defined by $T_1(g) := T(g)$ for the functions $g \in I_q(G)$ is also well defined (as both Id and I_q are injective) and continuous, since $\|T(g)\|_Y^q \le K^q \int_{B_{L_{p'}(\lambda)}} \big\|\int_\Omega fg\, d\lambda\big\|_X^q \, d\mu_0(f)$. $\square$

The converse is also true; i.e., if we have such a factorization, then the operator T satisfies (1) of Lemma 16. In fact, as for the case of q-absolutely summing operators, the canonical map satisfying this condition is $I_q\colon C\big(B_{L_{p'}(\lambda)}, X\big) \to L_q\big(B_{L_{p'}(\lambda)}, \mu_0, X\big)$, and the Ideal Property is obviously true for the factorization scheme of the theorem.

We conclude this paper with a natural extension of the above results to operators defined on Köthe function spaces. The condition we need is a previous canonical factorization through a space $L_p(\lambda)$.

Definition 18. Let L be a Köthe function space over (Ω,Σ,μ), Y a Banach space, and $p>1$. Let $\lambda\colon\Sigma\to X$ be a μ-continuous vector measure. We say that an operator $T\colon L\to Y$ is (λ,p)-representable if

(1) the natural map $I_{(\lambda,p)}\colon L\to L_p(\lambda)$, given by $I_{(\lambda,p)}(g):=g$, is well-defined, continuous, and has dense range, and
(2) there is an operator $T_1\colon L_p(\lambda)\to Y$ such that $T_1 o I_{(\lambda,p)}=T$.

Corollary 19. *Let L be a Köthe function space over (Ω,Σ,μ), Y a Banach space, $q\geq 1$ and $p>1$. Let $\lambda\colon\Sigma\to X$ be a p'-compact vector measure. If $T\colon L\to Y$ is a (λ,p)-representable operator, then the following are equivalent.*

(1) *There is a constant K such that, for each finite sequence $(g_i)_{i=1}^n\subset L$,*

$$\left(\sum_{i=1}^n \|Tg_i\|_Y^q\right)^{1/q} \leq K \sup_{\|f\|_{L_{p'}(\lambda)}\leq 1} \left(\sum_{i=1}^n \left\|\int_\Omega g_i f\,d\lambda\right\|_X^q\right)^{1/q}.$$

(2) *If G is the subspace $I_{(\lambda,p)}(L)$, then there is a probability measure μ_0 such that the operator T factorizes as follows:*

$$\begin{tikzcd} L \arrow[r, "T"] \arrow[d, "I_{(\lambda,p)}"'] & Y \\ G\subset C(B_{L_{p'}},X) \arrow[r, "I_q"] & I_q(G)\subset L_q(B_{L_{p'}},\mu_0,X) \arrow[u, "A"'] \end{tikzcd}$$

An obvious example of this result is the Grothendieck-Pietsch factorization theorem for p-summing operators on L_p spaces. For a finite positive measure μ every operator defined on $L_p(\mu)$ is obviously (μ,p)-representable by considering the factorization through the same $L_p(\mu)$. We have just shown that this is also true for spaces $L_p(\lambda)$, where λ has range in a finite dimensional space. In this case λ is obviously p'-compact, and the (λ,p)-representability of the operator T can be easily checked.

Acknowledgement. The author thanks the referee for his thorough and careful reading of the manuscript and for many suggestions and comments.

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