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SEMILINEAR AND SEMIALGEBRAIC LOCI OF O-MINIMAL SETS

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ABSTRACT. We consider some semilinear (= semiaffine) and semialgebraic loci of o-minimal sets in euclidean spaces. Semilinear loci have good properties. Some of these properties hold for semialgebraic loci when we restrict to a smaller class of analytically o-minimal sets.

1. O-minimal sets and geometric loci

A *structure* τ on the set of real numbers $\mathbf{R}$ is a family τ_n ($n \in \mathbf{N}$) such that (cf. [4] or [5]):

(S1) τ_n is a boolean algebra of subsets of $\mathbf{R}^n$,
(S2) if $A \in \tau_n$, then $\mathbf{R} \times A$, $A \times \mathbf{R}$ belong to τ_{n+1},
(S3) if $A \in \tau_{n+1}$, then $\pi(A) \in \tau_n$, where $\pi\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ is the natural projection obtained by dropping the last component,
(S4) the diagonals $\{x \in \mathbf{R}^n : x_i = x_j\}$ for $1 \le i < j \le n$ belong to τ_n.

If also

(S5) singletons $\{r\}$ for $r \in \mathbf{R}$ belong to τ_1,
(S6) the linear order $\{(x,y) \in \mathbf{R}^2 : x \le y\}$ belongs to τ_2,
(S7) every set from τ_1 is a finite union of intervals (of any type),

then this structure is *o-minimal*. We will say that $A \subset \mathbf{R}^n$ (not necessary a proper subset of $\mathbf{R}^n$) belongs to τ if $A \in \tau_n$.

The following two examples of o-minimal structures are widely known (see, for example, [5]):

- The system of *semilinear sets*.
- The system of *semialgebraic sets*.

For a finite collection $\mathcal{F}$ of subsets of $\mathbf{R}, \mathbf{R}^2, \mathbf{R}^3, \ldots$, we define $\mathbf{Tarski}(\mathcal{F})$ to be the smallest structure on $\mathbf{R}$ containing $\mathcal{F}$ and the semialgebraic sets. We call $\mathcal{F}$ (or a single set A) *o-minimal* if $\mathbf{Tarski}(\mathcal{F})$ (or $\mathbf{Tarski}(A)$) is o-minimal. In Section 3 we prove:

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THEOREM 1.1. *If A is o-minimal, $B \supset A$ is semilinear, and all germs of A at points of B are semilinear, then A is semilinear.*

We assume that all stratifications considered here have connected strata. A structure τ admits *analytic stratification* if, for every finite collection of elements of any τ_n, there exists a finite analytic stratification of the ambient space $\mathbf{R}^n$ into elements of τ_n that is compatible with the given finite collection of sets. (This notion is equivalent to admitting analytic cell decomposition.) A set $A \subset \mathbf{R}^n$ is *analytically o-minimal* if $\mathbf{Tarski}(A)$ is contained in a structure admitting analytic stratification. Recently, J.-Ph. Rolin, P. Speissegger, and A. Wilkie have constructed new o-minimal structures which give examples of o-minimal sets that are not analytically o-minimal, and o-minimal sets A_1, A_2 such that $\{A_1, A_2\}$ is not o-minimal (see [3]). In Section 4 we prove:

THEOREM 1.2. *If A is analytically o-minimal, $B \supset A$ is semialgebraic, and all germs of A at points of B are semialgebraic, then A is semialgebraic.*

Let $A \subset \mathbf{R}^n$. Its tangent semicone[1] at $a \in \mathbf{R}^n$ is the set

$$C_a^+(A) = \{0\} \cup \{x \in \mathbf{R}^n \mid \exists b_k \in A \setminus \{a\}\, \exists \lambda_k > 0 : b_k \to a, \lambda_k(b_k - a) \to x\}.$$

We introduce the *semilinear* and the *semialgebraic tangent semicone locus* of A:

$$SLC^+(A) = \{a \in \mathbf{R}^n \mid \text{the tangent semicone } C_a^+(A) \text{ is semilinear}\},$$
$$SC^+(A) = \{a \in \mathbf{R}^n \mid \text{the tangent semicone } C_a^+(A) \text{ is semialgebraic}\}.$$

Similarly, we define the *semilinear* and the *semialgebraic germ locus* of A:

$$SL(A) = \{x \in \mathbf{R}^n \mid \text{the germ } A_x \text{ is semilinear}\},$$
$$S(A) = \{x \in \mathbf{R}^n \mid \text{the germ } A_x \text{ is semialgebraic}\}.$$

The *semilinear* and the *semialgebraic grassmannian* of A *at* $a \in \mathbf{R}^n$ are:

$$SL_a G_n^k(A) = \{l \in \mathbf{G}_n^k \mid (a + l) \cap A \text{ is semilinear}\},$$
$$S_a G_n^k(A) = \{l \in \mathbf{G}_n^k \mid (a + l) \cap A \text{ is semialgebraic}\}.$$

For $A \subset \mathbf{R}^{m+n}$ and given $m, n \in \mathbf{N}$, we introduce the *semilinear* and the *semialgebraic fiber locus* of A as follows:

$$SL_m^n(A) = \{x \in \mathbf{R}^m \mid \text{the fiber } A(x) = \{y \in \mathbf{R}^n \mid (x, y) \in A\} \text{ is semilinear}\},$$
$$S_m^n(A) = \{x \in \mathbf{R}^m \mid \text{the fiber } A(x) = \{y \in \mathbf{R}^n \mid (x, y) \in A\} \text{ is semialgebraic}\}.$$

In Section 3 we prove:

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[1]This notion is a little more delicate than the tangent cone where the λ_k's are any real numbers. Several possibilities of defining the tangent cone are presented in [7].

THEOREM 1.3. *For an o-minimal set $A \subset \mathbf{R}^{m+n}$, the following sets belong to* **Tarski**(A)*, and hence are o-minimal:*

$$SL^n_m(A), \quad SL_0G^n_{m+n}(A), \quad SLC^+(A).$$

PROPOSITION 1.4. *If $A \subset \mathbf{R}^n$ is o-minimal, then so is $SL(A)$. Moreover, we have $SL(SL(A)) = SL(A)$.*

We will call a germ A_a of a set $A \subset \mathbf{R}^n$ *trivial* if it is equal to the germ at a of the empty set or the whole space $\mathbf{R}^n$.

If A_a is semialgebraic ($a \in S(A)$) or semilinear ($a \in SL(A)$) then we can define the semialgebraic (semilinear) complexity of this germ as follows: Given a description of the type

$$A_a = \bigcup_{i=1}^{k} \bigcap_{j=1}^{l_i} D_{i,j},$$

where $D_{i,j} = \{x \in \mathbf{R}^n : \operatorname{sgn} f_{i,j}(x) = T_{i,j}\}_a$, $T_{i,j} \in T = \{+, 0, -\}$ and $f_{i,j}$ are non-zero polynomials (affine functions), the *complexity of the description* is defined as $\sum_{i,j} \deg(f_{i,j})$.

The *semialgebraic (semilinear) complexity* of the germ A_a is the least complexity of all possible descriptions of A_a.

REMARKS. A semilinear (semialgebraic) germ is trivial iff its complexity is zero. If a germ is semialgebraic (semilinear) and nontrivial, then we can avoid polynomials of degree zero in its description. The semilinear complexity is not the semialgebraic complexity restricted to semilinear germs. The germ of a singleton in $\mathbf{R}^3$ has semialgebraic complexity 2 and semilinear complexity 3.

In Section 3 we prove:

PROPOSITION 1.5. *If A is o-minimal, then the semilinear complexity of its semilinear germs is bounded.*

For an o-minimal A, the family of germs A_x, $x \in SL(A)$, is uniformly semilinear in the sense of Theorem 3.1 below.

We recall the following result:

PROPOSITION 1.6 ([2, Theorem 2]). *If A is analytically o-minimal, then the semialgebraic complexity of its semialgebraic germs is bounded.*

For an analytically o-minimal A, the family of germs A_x, $x \in S(A)$, is uniformly semialgebraic in the sense of Theorem 4.4 below.

2. Frontiers of any rank

In this section we present a tool needed to deal with the loci of semialgebraic and semilinear germs.

Let Z be a set in a topological space. Then $\partial Z = \overline{Z} \setminus Z$ is called the frontier of Z. Analogously, $\partial' Z = Z \setminus \mathrm{int}(Z)$ will be called the *cofrontier* of Z.

For any *analytically o-minimal* set $A \subset \mathbf{R}^n$, we define its collection of semialgebraic frontiers as follows: For any $x \in \mathbf{R}^n$, we take the smallest algebraic subset of $\mathbf{R}^n$ containing $A \cap U$ for some neighborhood U of x in $\mathbf{R}^n$, and denote it by $\mathrm{Zar}_x(A)$. By the Identity Principle, $\mathrm{Zar}_x(A)$ is constant along a stratum of an analytic stratification compatible with A, so the collection $\{\mathrm{Zar}_x(A) : x \in \mathbf{R}^n\}$ is finite (cf. the proof of Theorem 1 in [2]). We set

$$\partial^0_x A = \text{the frontier of } A \cap \mathrm{Zar}_x(A) \text{ in } \mathrm{Zar}_x(A),$$
$$\partial^1_x A = \text{the cofrontier of } A \cap \mathrm{Zar}_x(A) \text{ in } \mathrm{Zar}_x(A).$$

These sets form a finite collection of *semialgebraic frontiers of rank* 1 *for* A.

The *semialgebraic frontiers of rank* $k+1$ *for* A are the semialgebraic frontiers of rank 1 for all semialgebraic frontiers of rank k for A. We restate a known fact:

THEOREM 2.1 ([2, Theorem 1,3 and Lemma 3]). *If $A \subset \mathbf{R}^n$ is analytically o-minimal, then so is $S(A)$. Moreover, $S(S(A)) = S(A)$.*

Let $\mathcal{S}$ be a finite analytic stratification of $\mathbf{R}^n$ compatible with A and all semialgebraic frontiers up to rank n for A. A point $x \in \mathbf{R}^n$ belongs to $S(A)$ if and only if, for each stratum $S \in \mathcal{S}$, with $x \in \overline{S}$ and S maximal (in the sense defined below) for A or any of these frontiers, S is a Nash manifold.

Given a C^1 stratification $\mathcal{S}$ of the space $\mathbf{R}^n$ compatible with a set F, a stratum $S \in \mathcal{S}$ will be called *maximal* for F if $S \subset F$ and, for any stratum $T \in \mathcal{S}$, $S \subset \partial T$ implies $T \subset \mathbf{R}^n \setminus F$. A stratum S is *flat at* $x \in \overline{S}$ if there is a neighborhood U of x such that $y \mapsto T_y S$ is constant on $U \cap S$. The *flat part* of S consists of the points of this stratum at which S is flat.

Now, assume that a finite *o-minimal* collection $\mathcal{F}$ of subsets of some $\mathbf{R}^n$ is given. Let us take a C^1 stratification $\mathcal{S}$ compatible with $\mathcal{F}$. Take a C^1 stratification $\mathcal{S}'$ compatible with $\mathcal{S}$ and with the flat parts of the strata of $\mathcal{S}$ that are maximal for elements of $\mathcal{F}$. We assume that both $\mathcal{S}$ and $\mathcal{S}'$ are taken from $\mathbf{Tarski}(\mathcal{F})$; this is possible, as proved in [6]. The strata of $\mathcal{S}'$ which are maximal for elements of $\mathcal{F}$ are open subsets of strata of $\mathcal{S}$ maximal for elements of $\mathcal{F}$, and hence are flat or everywhere nonflat.

Take $x \in \mathbf{R}^n$. We say that x is *linearly Nash* for $F \in \mathcal{F}$ ($x \in \mathrm{LNash}(F)$) if the germ F_x is contained in some germ G_x where G is the union of a finite number of affine subspaces and $\dim(G_x)$=$\dim(F_x)$. This happens if and only if each stratum $S \in \mathcal{S}'$ that is maximal for F and has x in its closure is flat. Each set $\mathrm{LNash}(F)$ is open, belongs to $\mathbf{Tarski}(F)$, and its complement is a finite union of closures of everywhere nonflat strata of $\mathcal{S}'$. If $x \in \mathrm{LNash}(F)$, then the smallest finite union of affine subspaces containing F_x, denoted by $\mathrm{LZar}_x(F)$, has the same dimension as the germ F_x, and depends only on the stratum on which x lies. Thus the collection of frontiers and cofrontiers of $F \cap \mathrm{LZar}_x(F)$ in $\mathrm{LZar}_x(F)$ for all $F \in \mathcal{F}$ and $x \in \bigcap_{F\in\mathcal{F}} \mathrm{LNash}(F)$ is a finite collection of o-minimal sets. This collection will be denoted $\mathrm{LFron}^1(\mathcal{F})$ and called the collection of *semilinear frontiers of rank* 1 for $\mathcal{F}$. Their dimensions are strictly smaller than the dimensions of respective elements of $\mathcal{F}$. Now, $\mathrm{LFron}^{k+1}(\mathcal{F}) = \mathrm{LFron}^1(\mathrm{LFron}^k(\mathcal{F}))$ and $\mathrm{LFron}^0(\mathcal{F}) = \mathcal{F}$. The following proposition is obvious:

Proposition 2.2. *Let $\mathcal{S}$ be a finite C^1 stratification of $\mathbf{R}^n$ compatible with A and all semilinear frontiers up to rank n for $\{A\}$. A point $x \in \mathbf{R}^n$ belongs to $SL(A)$ if and only if each stratum $S \in \mathcal{S}$ that has x in the closure and is maximal for A or any of these frontiers is flat at x.*

3. Semilinear loci

This section contains the proofs in the case of semilinear loci.

Proof of Theorem 1.1.

Case 1: B is an open set. We use induction on $l = \dim A$. The theorem is obvious for $l \leq 0$.

Assume $l > 0$. The set A is a finite union of C^1 submanifolds S, and is also contained in a countable union of linear subspaces of dimension at most l. Each of the l-dimensional manifolds S is contained in a single subspace L_i of dimension l. The intersection $A \cap L_i$ is semilinear iff both its frontier and cofrontier in L_i are semilinear. But these, as well as $A \setminus \bigcup_i L_i$, have dimensions less than l.

Case 2: General case. We use the cell decomposition theorem for the structure of semilinear sets. □

Proof of Theorem 1.3. The set $SL^n_m(A)$ belongs to $\mathbf{Tarski}(A)$: We can assume that $\dim A(x)$ is constant. Let us apply induction on $l = \dim A(x)$, where $x \in \mathbf{R}^m$. For $l \leq 0$ the statement is obvious. If $l > 0$, then both $R = \{(x,y) \in \mathbf{R}^{m+n} \mid y \in \mathrm{Reg}^1(A(x))\}$ and $S = \{(x,y) \in \mathbf{R}^{m+n} \mid y \in \mathrm{Sing}^1(A(x))\}$ belong to $\mathbf{Tarski}(A)$. (Here Reg^1 denotes the set of C^1-regular points of the highest dimension.) The fiber $A(x)$ is semilinear iff

(i) $\mathrm{Sing}^1(A(x))$ is semilinear,
(ii) the collection of tangent spaces (of dimension l) at points of $\mathrm{Reg}^1(A(x))$ is finite,
(iii) for each affine subspace L of dimension l in $\mathbf{R}^n$, the frontier and cofrontier of $L \cap \mathrm{Reg}^1(A(x))$ in L are semilinear.

By the induction assumption and simple facts on o-minimal structures, the set of all $x \in \mathbf{R}^m$ such that $A(x)$ is semilinear belongs to $\mathbf{Tarski}(A)_m$.

Also, the set $SL_0G^n_{m+n}(A)$ belongs to $\mathbf{Tarski}(A)$, since it is a finite union of sets of the form $SL^n_m(\Psi^{-1}(A))$, where $\Psi\colon U \times \mathbf{R}^n \to \mathbf{R}^{m+n}$ is an appropriate semialgebraic mapping which, for a given subspace $l \in U \subset \mathbf{G}^n_{m+n}$ and the coordinates of a point in l, produces the coordinates of this point in $\mathbf{R}^{m+n}$.

The set $SLC^+(A)$ belongs to $\mathbf{Tarski}(A)$ as a semilinear fiber locus of the set $\{(x,y) \in \mathbf{R}^{2(m+n)} : y \in C^+_x(A)\}$. $\square$

Let i be the natural inclusion of the grassmannian space $\mathbf{G}^k_n$ into the projective space $\mathbf{P}(\bigwedge^k \mathbf{R}^n)$, and let $\mathrm{pr}\colon \mathrm{Cone}(\bigwedge^k \mathbf{R}^n) \to \mathbf{P}(\bigwedge^k \mathbf{R}^n)$ be the projectivization mapping for cones in $\bigwedge^k \mathbf{R}^n$. A set $B \subset \mathbf{G}^k_n$ will be called *semilinear* if $(\mathrm{pr}^{-1} \circ i)(B)$ is semilinear in $\bigwedge^k \mathbf{R}^n$.

Remark. Considering mappings of type $\widetilde{\Psi} = \mathrm{id}_{\mathbf{R}^{m+n}} \times \Psi$, we can show that the set of points where the semilinear grassmannian of a given definable set is semilinear also belongs to $\mathbf{Tarski}(A)$.

Proof of Proposition 1.4. By Proposition 2.2 we have

$$SL(A) = \bigcap_{i=0}^{n} \bigcap_{F \in \mathrm{LFron}^i(\{A\})} \mathrm{LNash}(F),$$

so $SL(A)$ belongs to $\mathbf{Tarski}(A)$.

Another proof is as follows. The set $SL(A)$ is a projection of the locus of semilinear fibers of the set $\tilde{A} = \{(x,y,\epsilon) : y \in A, |x-y| < \epsilon\}$ (an observation by L. van den Dries). Also $SL(SL(A)) = SL(A)$ holds because the complement of $SL(A)$ is a union of closures of manifolds that are everywhere nonflat. $\square$

Proof of Proposition 1.5. If the semilinear complexity of some semilinear $B, C \subset \mathbf{R}^n$ is bounded (from above) by k, then their boolean combinations and unions of their connected components have semilinear complexity bounded by some $M(k)$. The same applies to germs. The theorem follows by induction on the dimension of A, using the notion of semilinear frontiers of rank 1. $\square$

Now take an o-minimal set $A \subset \mathbf{R}^n$. By the existence of definable Skolem functions in o-minimal structures (in our sense), there exists a function $\varepsilon\colon \mathbf{R}^n \to \mathbf{R}$, with graph belonging to $\mathbf{Tarski}(A)$, such that $\varepsilon(x) = 0$ if $x \notin SL(A)$, $\varepsilon(x) > 0$ for $x \in SL(A)$, and the complexity of each semilinear germ A_x is realized by $A \cap C(x, \varepsilon(x))$ in $C(x, \varepsilon(x))$. (Here $C(x,r)$ denotes the open cube $\mathop{\times}\limits_{i=1}^{n} (x_i - r, x_i + r)$.) Set $\hat{A} = \{(x,y) \in \mathbf{R}^{2n} : y \in A \cap C(x, \varepsilon(x))\}$, so that $\hat{A} \in \mathbf{Tarski}(A)_{2n}$ and each fiber $\hat{A}(x)$ is semilinear. Notice that $\hat{A}(x)_x = A_x$ if $x \in SL(A)$, and $\hat{A}(x)_x = \emptyset_x$ otherwise. Using again the existence of definable Skolem functions and Proposition 1.5, we obtain the following theorem:

THEOREM 3.1. *There are* $p \in \mathbf{N}$ *and functions*

$$f_{i,j}(x,y) = \sum_{k=1}^{n} a_{i,j,k}(x) y_k + b_{i,j}(x),$$

where $a_{i,j,k}$ *and* $b_{i,j}$ $(i,j = 1,\dots,p)$ *belong to* **Tarski**(A)*, such that for each* $x \in \mathbf{R}^n$ *we have*

$$\hat{A}(x) = \bigcup_{i=1}^{p} \bigcap_{j=1}^{p} \{y \in \mathbf{R}^n : \operatorname{sgn} f_{i,j}(x,y) = T_{i,j}\}$$

with $T_{i,j} \in \{+,0,-\}$.

4. Semialgebraic loci

This section gives some examples and contains the proofs for the case of semialgebraic loci.

EXAMPLE 1. Let $f(x,y) = x^y = \exp(y \log x)$ for $x, y \in (1,2)$. Then $f(\cdot,y)$ is semialgebraic iff $y \in \mathbf{Q}$. This function belongs to the o-minimal structure **Tarski**(e), where $e = \exp|_{[-1,1]}$ (we identify functions with their graphs).

EXAMPLE 2. Let $\{q_i\}_{i\in\mathbf{N}^*}$ be a sequence of algebraically independent real numbers such that $|q_i| < \frac{1}{2}$ for all i. Let

$$f(x,y) = \sum_{m=1}^{\infty} x^m (y-q_1)\cdot\dots\cdot(y-q_m).$$

This function is analytic and globally subanalytic on $(-\frac{1}{2},\frac{1}{2})^2$. For $y = q_i$, $f(\cdot,y)$ is a polynomial; if $y \neq q_i$ for all i, then $f(\cdot,y)$ is nonsemialgebraic.

PROPOSITION 4.1. *Let* $n \geq 2$ *and* $m \geq 1$*. There exist globally subanalytic sets* $\Gamma_1, \Gamma_2 \subset \mathbf{R}^{m+n}$ *such that* $S^n_m(\Gamma_1)$ *and* $S_0 G^n_{m+n}(\Gamma_2)$ *are not o-minimal.*

If $m+n \geq 4$*, then there exists a globally subanalytic set* Γ_3 *such that* $SC^+(\Gamma_3)$ *is not o-minimal.*

Proof. Let f be the function from Example 1 or Example 2. Let Γ_1 be the set

$$\{(x_1,\dots,x_{m+n}) \in \mathbf{R}^{m+n} : x_{m+n} = f(x_{m+n-1}, x_1)\},$$

and define Γ_2 as

$$\{(x_{m+n-1}\sin(\pi x_1), x_2, \dots, x_{m+n-2}, x_{m+n-1}\cos(\pi x_1), x_{m+n}) : \\ (x_1,\dots,x_{m+n}) \in \Gamma_1, x_1 > 0, x_{m+n-1} > 0\}.$$

Finally, take Γ_3 to be the following "semicone with parameter"

$$\{(x_1, 0, \dots, 0, \lambda, \lambda x_{m+n-1}, \lambda f(x_{m+n-1}, x_1)) : \\ (x_{m+n-1}, x_1) \in \operatorname{dom} f, \lambda \in [0,+\infty)\}. \quad \square$$

We state a result of similar type:

PROPOSITION 4.2. *Let τ be an o-minimal structure containing all semialgebraic sets whose field of exponents $k(\tau)$ (defined in* [1]*) is a proper subfield of* $\mathbf{R}$. *Then, for $m \geq 1$ and $n \geq 2$, there exists $A \subset \mathbf{R}^{m+n}$ belonging to* $\mathbf{Tarski}(\exp)$, *such that the locus of τ-definable fibers, $\tau^n_m(A) = \{x \in \mathbf{R}^m \mid A(x) \in \tau_n\}$, is not o-minimal.*

However, for low dimensional ambient spaces, we have o-minimality of the semialgebraic tangent semicone locus. This follows from a more general fact:

PROPOSITION 4.3. *If $A \subset \mathbf{R}^3$ is o-minimal, then $SLC^+(A)$ is cofinite.*

Proof. Let $\mathcal{S}$ be a finite C^1 stratification of $\mathbf{R}^n$, compatible with A, and whose strata are cells in $\mathbf{Tarski}(A)$ (cf. [6, 4.8(1)]). Let us consider $C^+_x(A)$ for some $x \in \mathbf{R}^n$. The interesting case holds when x belongs to some one-dimensional stratum Γ. As $C^+_x(A) = \overline{C^+_x(A)} = C^+_x(\overline{A})$, $C^+_x(A \cup B) = C^+_x(A) \cup C^+_x(B)$, and the boundary $\mathrm{bd}(C^+_x(A))$ of $C^+_x(A)$ is contained in $C^+_x(\mathrm{bd}(A))$, we can assume that $A_x = \Gamma_x \cup \Gamma'_x$, where Γ' is some two-dimensional stratum. Notice that for all but finitely many points $x \in \Gamma$ the tangent spaces $T_y\Gamma'$ with $\Gamma' \ni y \to x$ have a limit in $\mathbf{G}^2_3$ (since the set of limit points is finite and connected). If the limit plane exists, then the tangent cone $C^+_x(A)$ is contained in this plane, so it is semilinear. □

If $A \subset \mathbf{R}^2$ is an o-minimal set, then $SC^+(A) = SLC^+(A) = \mathbf{R}^2$.

Proof of Theorem 1.2. We can assume that B is the whole space $\mathbf{R}^n$. We use induction on n. The theorem is obvious for $n = 0, 1$.

Case 1. A is nowhere dense.

The set A is a finite union of connected analytic submanifolds which is contained in a countable union of nowhere dense algebraic sets. By Baire's theorem, it is contained in a finite union of algebraic sets. Taking a good direction, we can express A as a subset of a finite union of graphs of semialgebraic functions. Applying the induction assumption to the domains of these functions, we obtain that A is semialgebraic.

Case 2. A is open. The analytically o-minimal set $\mathrm{bd}(A)$ is nowhere dense. By Case 1, it is semialgebraic, and so is A.

Case 3. General case. The set A is a union of an open set and a nowhere dense set, both of them semialgebraic. □

Take an analytically o-minimal set $A \subset \mathbf{R}^n$. By the existence of definable Skolem functions in o-minimal structures, there exists a function $\epsilon \colon \mathbf{R}^n \to \mathbf{R}$, with analytically o-minimal graph, such that $\epsilon(x) = 0$ if $x \notin S(A)$, $\epsilon(x) > 0$ for $x \in S(A)$, and the complexity of each semialgebraic germ A_x is realized by $A \cap B(x, \epsilon(x))$ in $B(x, \epsilon(x))$. ($B(x, r)$ denotes the open ball centered at x with radius r). Set $\tilde{A} = \{(x,y) \in \mathbf{R}^{2n} : y \in A \cap B(x, \epsilon(x))\}$, so each fiber $\tilde{A}(x)$ is semialgebraic. Notice that $\tilde{A}(x)_x = A_x$ if $x \in S(A)$, and $\tilde{A}(x)_x = \emptyset_x$ otherwise. Again, by the existence of definable Skolem functions and Proposition 1.6, we conclude that the following theorem holds:

Theorem 4.4. *There are $p \in \mathbf{N}$ and functions*

$$f_{i,j}(x,y) = \sum_{|\beta| \leq p} a_{i,j,\beta}(x) y^{\beta},$$

where $\beta \in \mathbf{N}^n$ and all $a_{i,j,\beta}$ $(i,j = 1, \ldots, p, |\beta| \leq p)$ belong to some structure admitting analytic stratification, such that for each $x \in \mathbf{R}^n$ we have

$$\tilde{A}(x) = \bigcup_{i=1}^{p} \bigcap_{j=1}^{p} \{y \in \mathbf{R}^n : \operatorname{sgn} f_{i,j}(x,y) = T_{i,j}\}$$

with $T_{i,j} \in \{+, 0, -\}$.

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