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EVERY LOCALLY BOUNDED SPACE WITH TRIVIAL DUAL IS THE QUOTIENT OF A RIGID SPACE

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ABSTRACT. Letting T_p denote the class of separable p-Banach spaces (for $0 < p < 1$) with trivial dual, we show that T_p does not have any projective spaces, i.e., there is no space X in T_p such that every space in T_p is a quotient of X. In lieu of a projective space we construct the $L_p(w)$ spaces, which are structurally similar to the space L_p. We then define a particularly well behaved type of $L_p(w)$ space, namely the uniform $L_p(w)$ spaces, and we show that every space in T_p is a quotient of some uniform $L_p(w)$ space. We then define a badly behaved type of $L_p(w)$ space, namely the unbalanced biuniform $\mathrm{L_p(w)}$ spaces. If $L_p(w)$ is unbalanced biuniform and C denotes the one dimensional subspace of constant functions, then $L_p(w)/C$ is a rigid space. We then show that each space in T_p is a quotient of one of these rigid spaces. This last result is used in an essential way to prove the nonexistence of a projective space in T_p.

1. Introduction

Every separable Banach space is a quotient of l_1. This can be generalized to p-Banach spaces with $0 \leq p < 1$, i.e., every separable p-Banach space is a quotient of l_p. In other words, in the class of separable p-Banach spaces, $0 < p \leq 1$, l_p is projective (see for instance [3]). With $0 < p < 1$ fixed, we shall let T_p denote the class of separable p-Banach spaces X such that X has trivial dual (i.e., the dual of X consists of only the zero functional). It is natural to ask whether there is a space X that is projective in T_p, i.e., so that every space in T_p is a quotient of X. If (P) is a property such that every quotient of a space with property (P) also has property (P), then either every space in T_p has property (P) or X is an example of a space failing to have property (P). Questions involving "quotient friendly" properties have already been posed and resolved. Let (P_1) be the property that X is not the domain of a nonzero compact operator, and let (P_2) be the property that X is a needlepoint space. Both of these are quotient friendly properties. N. J. Kalton

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and J. H. Shapiro [5] showed that there is a space in T_p failing property (P_1), and Kalton [1] showed that there is a space in T_p failing property (P_2). Of course, any projective space in T_p would fail to have both properties. It turns out that there is no projective space in T_p and there is a natural reason for suspecting this. It frequently happens that if $X \in T_p$ there is a $Y \in T_p$ so that there is only the trivial continuous linear operator from X to Y, i.e., $L(X,Y) = \{0\}$. In [4] Kalton and the author produced a cardinality c collection $\{X_\alpha : \alpha \in [0,1]\}$ of subspaces of L_p such that if $\alpha \neq \beta, L(X_\alpha, X_\beta) = \{0\}$. In Theorem 4.4 below we show that, given X in T_p, there is a space Y in T_p such that $L(X,Y) = \{0\}$. Thus there is no projective space in T_p. In lieu of a projective space we shall produce a *projective class* of spaces in T_p. (We call a subclass S of T_p a *projective class* if, whenever Y is in T_p, there is an X in S such that Y is a quotient of X.) To carry this out, we shall use a class of spaces $L_p(w)$ which are generalizations of the space L_p. The idea is to give some intervals a weight (given by the weight function w) different from their usual length. This involves an infimum norm construction. The details of this, including the formal definition of the $L_p(w)$ spaces, are carried out in Section 2. In Section 3 we define the uniform $L_p(w)$ spaces. We show that this particularly nice class of spaces is projective in T_p. We also show that for every X in T_p there is a compact operator from some uniform $L_p(w)$ space into X. In Section 4 we introduce the unbalanced biuniform $L_p(w)$ spaces. We show that the quotient of one of these spaces by the one dimensional space of constant functions is rigid. We also show that this class of rigid spaces is projective in T_p. In particular, every space in T_p is the quotient of a rigid space. We then use this fact to show that T_p has no projective spaces. Our notation will be fairly standard. Throughout the paper all scalars will be real, although all the results hold for complex scalars. If X is a vector space and $0 < p \leq 1$, a function $\|\cdot\| \colon X \to [0,\infty)$ is called a *p-seminorm* if

(1) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,
(2) $\|\alpha x\| = |\alpha|^p \|x\|$ for all $\alpha \in \mathbb{R}$, $x \in X$.

A p-seminorm is a *p-norm* if, in addition,

(3) for $x \in X$, $\|x\| = 0$ implies $x = 0$.

If X has a p-norm $\|\cdot\|$ such that the metric $d(x,y) = \|x-y\|$ is a complete metric on X, then $(X, \|\cdot\|)$ is called a *p-Banach space*. Henceforth, we shall assume that p is in the range $0 < p < 1$. For instance, the space of all sequences $x = \langle x_n \rangle$ with

$$\|x\|_p = \sum_{n=1}^{\infty} |x_n|^p$$

is called l_p and $(l_p, \|\cdot\|_p)$ is a p-Banach space. The space of measurable functions $x = x(t)$ on $[0,1]$ such that

$$\|x\|_p = \int_0^1 |x(t)|^p < \infty$$

is called L_p and $(L_p, \|\cdot\|_p)$ is a p-Banach space.

If X and Y are p-Banach spaces we let $L(X,Y)$ denote the space of continuous linear operators from X to Y. If $X = Y$ we denote $L(X,Y)$ by $L(X)$, and if $Y = \mathbb{R}$, we set $L(X,\mathbb{R}) = X^*$. We say that X has *trivial dual* if $X^* = \{0\}$. Note that L_p has trivial dual but l_p does not (see [3]). Finally if I is an interval we let $|I|$ denote the length of I. If E is a finite set we let $|E|$ denote the cardinality of E. This should not cause any confusion. Before proceeding we note that the examples E_q constructed by Kalton in [2] are all $L_p(w)$ spaces in disguised form.

2. $L_p(w)$ spaces

We begin with the notion of infimum norm. This is defined in far more generality in [3].

Definition 2.1. A sequence $\langle (S_n, \|\cdot\|_n)\rangle$ is said to be a *stacked sequence* if it satisfies the following conditions:

(1) Each S_n is a finite dimensional space equipped with a p-norm $\|\cdot\|_n$.
(2) For every n, $S_n \subset S_{n+1}$.
(3) If $x \in S_n$, then $\|x\|_n \leq \|x\|_{n+1}$.

Definition 2.2. If $\langle (S_n, \|\cdot\|_n)\rangle$ is a stacked sequence we define the *infimum norm* $\|\cdot\|$ on $S = \cup S_n$ by

$$\|x\| = \inf\left\{\sum_{k=0}^{n} \|x_k\|_k : x = \sum_{k=0}^{n} x_k \text{ with } x_k \in S_k, 0 \leq k \leq n\right\},$$

and we say that $\|\cdot\| = \inf \|\cdot\|_n$.

Proposition 2.1. *Suppose $\langle (S_n, \|\cdot\|_n)\rangle$ is a stacked sequence and suppose that $\|\cdot\| = \inf \|\cdot\|_n$. Then we have:*

(1) *$\|\cdot\|$ is a p-norm on $S = \cup S_n$.*
(2) *If $|\cdot|$ is a p-seminorm on S such that for all n and for every $x \in S_n$, $|x| \leq \|x\|_n$, then for every $x \in S$ $|x| \leq \|x\|$.*
(3) *If for each $x \in S_n$ we let*

$$N_n(x) = \inf\left\{\sum_{k=0}^{n} \|x_k\|_k : \sum_{k=0}^{n} x_k = x \text{ with } x_k \in S_k, 0 \leq k \leq n\right\},$$

then $N_n(x) = \|x\|$.

(4) *For every $x \in S_n$, there exist $x_k \in S_k, 0 \leq k \leq n$, such that*

$$\|x\| = \sum_{k=0}^{n} \|x_k\|_k.$$

Furthermore, $\|x_k\|_k = \|x_k\|$ *for* $0 \leq k \leq n$.

Proof. (1) It is easily verified from the definition that $\|\cdot\|$ is a p-seminorm. If $x \in S$ and $x \neq 0$, $\|x\| > 0$ will follow once we have established (4).

(2) If $x \in S$ and $x = \sum_{k=0}^{n} x_k$ with $x_k \in S_k$, $0 \leq k \leq n$, then

$$|x| \leq \sum_{k=0}^{n} |x_k| \leq \sum_{k=0}^{n} \|x_k\|.$$

By taking the infimum, we obtain $|x| \leq \|x\|$.

(3) It suffices to show that for $x \in S_n$, $N_n(x) = N_{n+1}(x)$. Obviously $N_{n+1}(x) \leq N_n(x)$. Let $x \in S_n$. If $x_k \in S_k$, $0 \leq k \leq n+1$, such that

$$x = \sum_{k=0}^{n+1} x_k$$

then

$$x_{n+1} = x - \sum_{k=0}^{n} x_k \in S_n.$$

Thus

$$\begin{aligned}
\sum_{k=0}^{n+1} \|x_k\| &= \sum_{k=0}^{n-1} \|x_k\|_k + \|x_n\|_n + \|x_{n+1}\|_{n+1} \\
&\geq \sum_{k=0}^{n-1} \|x_k\|_k + \|x_n\|_n + \|x_{n+1}\|_n \\
&\geq \sum_{k=0}^{n-1} \|x_k\|_k + \|x_n + x_{n+1}\|_n \geq N_n(x).
\end{aligned}$$

Hence

$$N_n(x) \geq N_{n+1}(x).$$

(4) If $x \in S_n$, let K denote the set of all $(x_0, x_1, \ldots, x_n) \in \prod_{k=0}^{n} S_k$ such that $\|x_k\|_k \leq 2\|x\| + 1$, $0 \leq k \leq n$, and $\sum_{k=0}^{n} x_k = x$. Since each S_k is finite dimensional, K is compact. Define Φ on K by

$$\Phi((x_0, x_1, \ldots, x_n)) = \sum_{k=0}^{n} \|x_k\|_k.$$

Since Φ is continuous, it assumes its minimum at some $(x_0, x_1, \ldots, x_n)$, i.e., we have

$$\|x\| = N_n(x) = \sum_{k=0}^{n} \|x_k\|_k.$$

Since

$$\|x\| \le \sum_{k=0}^{n} \|x_k\| \le \sum_{k=0}^{n} \|x_k\|_k = \|x\|,$$

we obtain

$$\|x_k\|_k = \|x_k\|, \qquad 0 \le k \le n. \qquad \square$$

If $\langle (S_n, \|\cdot\|_n) \rangle$ is a stacked sequence with $\|\cdot\| = \inf \|\cdot\|_n$, we let $|||\cdot|||_n = \inf_{k \ge n} \|\cdot\|_k$, i.e., for $x \in S = \cup S_n$,

$$|||x|||_n = \inf \left\{ \sum_{k=n}^{N} \|x_k\|_k : \sum_{k=n}^{N} x_k = x, x_k \in S_k, n \le k \le N \right\}.$$

Proposition 2.2. *Let $\langle (S_n, \|\cdot\|_n) \rangle$ be a stacked sequence and let $\bar{S}$ denote the completion of $S = \cup S_n$ with respect to $\|\cdot\| = \inf \|\cdot\|_n$. Then we have:*

(1) *$|||\cdot|||_n$ is equivalent to $\|\cdot\|$ on S and therefore on $\bar{S}$.*
(2) *If $x_k \in S_k$ for $k \ge n$ such that*

$$\sum_{k=n}^{\infty} \|x_k\|_k < \infty,$$

then $\sum_{k=n}^{\infty} x_k$ is absolutely convergent in $\bar{S}$ and

$$\left|\left|\left| \sum_{k=n}^{\infty} x_k \right|\right|\right|_n \le \sum_{k=n}^{\infty} \|x_k\|_k.$$

(3) *If $x \in \bar{S}$ and $\varepsilon > 0$, then there exists a sequence $\langle x_n \rangle$, with $x_n \in S_n$ for each n, such that*

$$x = \sum_{k=0}^{\infty} x_k$$

and

$$\sum_{k=0}^{\infty} \|x_k\|_k < \|x\| + \varepsilon.$$

(4) *If $x \in \bar{S}$, then for each integer $n \ge 0$ there exist $y \in S_n$ and $z \in \bar{S}$ such that $x = y + z$ and $|||x|||_n = \|y\|_n + |||z|||_{n+1}$.*
(5) *If $x \in \bar{S}$, then for each integer $n \ge 0$ there exist $y \in S_n$ and $z \in \bar{S}$ such that $x = y + z$ and $\|x\| = \|y\| + |||z|||_{n+1}$.*

Proof. Since S_n is finite dimensional there is a constant $C \geq 1$ so that $|||x|||_n \leq C\|x\|_k$ for each $x \in S_k$, $0 \leq k \leq n$. Thus $|||\cdot|||_n \leq C\|\cdot\|_k$ on S_k, for every $k \geq 0$. By (2) of Proposition 2.1 we have $|||\cdot|||_n \leq C\|\cdot\|$. Since $\|\cdot\| \leq |||\cdot|||_n$ on S, assertion (1) follows. (2) is obvious. To obtain (3) suppose $\langle \varepsilon_n \rangle$ is a positive sequence such that

$$2\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon.$$

Since $\cup S_n$ is dense in $\bar{S}$ we may select a sequence $N_1 \leq N_2 \leq \cdots$ with $y_n \in S_{N_n}$ so that

$$\left\| x - \sum_{n=1}^{m} y_n \right\| \leq \varepsilon_m$$

for each m. Note that

$$\begin{aligned} \|y_m\| &= \left\| \left(y - \sum_{n=1}^{m-1} y_n \right) - \left(y - \sum_{n=1}^{m} y_n \right) \right\| \\ &\leq \left\| y - \sum_{n=1}^{m-1} y_n \right\| + \left\| y - \sum_{n=1}^{m} y_n \right\| < \varepsilon_{m-1} + \varepsilon_m \end{aligned}$$

if $m \geq 2$. Since

$$\|y_1\| \leq \|x\| + \|x - y_1\| < \|x\| + \varepsilon_1,$$

we obtain

$$\sum_{n=1}^{\infty} \|y_n\| < \|x\| + 2\sum_{n=1}^{\infty} \varepsilon_n < \|x\| + \varepsilon.$$

If $y_n = \sum_{k=1}^{N_n} x_{kn}$ with $x_{kn} \in S_k$, $1 \leq k \leq N_n$, then

$$\|y_n\| = \sum_{k=0}^{N_n} \|x_{kn}\|_k.$$

Thus

$$\sum_{k,n} \|x_{kn}\|_k = \sum_{n=1}^{\infty} \|y_n\| < \|x\| + \varepsilon.$$

Let $x_k = \sum x_{kn}$. Then $x_k \in S_k$,

$$\sum_{k=0}^{\infty} \|x_k\|_k \leq \|x\| + \varepsilon$$

and $\sum_{k=0}^{\infty} x_k = x$. (4) is a special case of (5) once we observe that $|||\cdot|||_n = \inf_{k \geq n} \|\cdot\|_k$. To prove (5) we note that by (3) we have, for any $\varepsilon > 0$, $x = \sum_{k=0}^{\infty} x_k$ such that

$$\sum_{k=0}^{\infty} \|x_k\|_k < \|x\| + \varepsilon.$$

If we let

$$y = \sum_{k=0}^{n} x_k \in S_n \quad \text{and} \quad z = \sum_{k=n+1}^{\infty} x_k,$$

then $x = y + z$ and

$$\|y\| + |||\, z \,|||_n \leq \sum_{k=0}^{n} \|x_k\|_k + \sum_{k=n+1}^{\infty} \|x_k\|_k < \|x\| + \varepsilon.$$

Thus, for each n, we may select $y_k \in S_n$ and $z_k \in \bar{S}$ such that $x = y_k + z_k$ and

$$\|y_k\| + |||\, z_k \,|||_n \leq \|x\| + \frac{1}{k}.$$

By passing to a subsequence, if necessary, we may assume that $\lim y_k = y$ since S_n is finite dimensional. But then

$$\lim z_k = \lim(x - y_k) = x - y,$$

i.e., with $z = x - y$ we have

$$\|x\| = \|y\| + |||\, z \,|||_n\,,$$

by the continuity of the norms. □

Note that assertion (3) of Proposition 2.2 provides an infinite version of our definition of infimum norm; i.e., if $x \in \bar{S}$, then

$$\|x\| = \inf \sum_{k=0}^{\infty} \|x_k\|_k,$$

where $x_k \in S_k$ and $\sum_{k=0}^{\infty} x_k = x$.

It is often convenient to have a sequence $\langle x_k \rangle$, where $\|x\| = \sum_{k=0}^{\infty} \|x_k\|_k$.

Definition 2.3. Suppose $\langle (S_n, \|\cdot\|_n) \rangle$ is a stacked sequence and $x \in \bar{S}$.

(1) x is *robust* if there exists a sequence $\langle x_k \rangle$ such that $x_k \in S_k$ for each k, $\sum_{k=0}^{\infty} x_k = x$ and $\|x\| = \sum_{k=0}^{\infty} \|x_k\|_k$.

(2) x is *languid* if $\|x\| = |||\, x \,|||_n$ for every n.

We say that $\bar{S}$ is *robust* if every point in $\bar{S}$ is robust.

Note that a point can be both robust and languid; e.g., the origin is both robust and languid.

Proposition 2.3. *If $\langle (S_k, \|\cdot\|_k) \rangle$ is a stacked sequence and $x \in \bar{S}$, then $x = y + z$, where y is robust and z is languid.*

Proof. We inductively select a sequence $\langle x_k \rangle$ with $x_k \in S_k$ such that

$$x = \sum_{k=0}^{n} x_k + z_{n+1}$$

and

$$\|x\| = \sum_{k=0}^{n} \|x_k\|_k + |||\, z_{n+1} \,|||_{n+1} \,.$$

Suppose

$$x = \sum_{k=0}^{n-1} x_k + z_n$$

(or, when $n = 0$, $x = z_0$) such that

$$\|x\| = \sum_{k=0}^{n-1} \|x_k\|_k + |||\, z_n \,|||_n \,.$$

By assertion (4) of Proposition 2.2, $z_n = x_n + z_{n+1}$ with $x_n \in S_n$ such that $|||\, z_n \,|||_n = \|x_n\| + |||\, z_{n+1} \,|||_{n+1}$. Thus

$$\|x\| = \sum_{k=0}^{n} \|x_k\|_k + |||\, z_{n+1} \,|||_{n+1} \,.$$

Note that since

$$\begin{aligned} \|x\| &\leq \sum_{k=0}^{n} \|x_k\|_k + \|z_{n+1}\| \\ &\leq \sum_{k=0}^{n} \|x_k\| + |||\, z_{n+1} \,|||_{n+1} = \|x\|, \end{aligned}$$

we have $\|z_{n+1}\| = |||\, z_{n+1} \,|||_{n+1}$. If $k \leq n+1$, then $\|\cdot\| \leq |||\cdot|||_k \leq |||\cdot|||_{n+1}$, so that $\|z_{n+1}\| = |||\, z_{n+1} \,|||_k$ for all k, $0 \leq k \leq n+1$. Now let $y = \sum_{k=0}^{\infty} x_k$. Clearly $\|y\| = \sum_{k=0}^{\infty} \|x_k\|_k$ so that y is robust. We let $z = \lim z_n$. Since $|||\, z_n \,|||_k = \|z_n\|$ once $n \geq k$, we have $|||\, z \,|||_k = \|z\|$ for all k, so z is languid. $\square$

Proposition 2.4. *Suppose $\langle (S_k, \|\cdot\|_k) \rangle$ is a stacked sequence.*

(1) *If there is a sequence $\langle c_n \rangle$ such that for every $x \in S_n$, $\|x\|_{n+1} = c_n \|x\|_n$ and $\prod_{n=0}^{\infty} c_n = \infty$, then $\bar{S}$ is robust.*
(2) *If there is a sequence $\langle c_n \rangle$ such that for every $x \in S_n$, $\|x\|_{n+1} \geq c_n \|x\|_n$ and $\overline{\lim}\, c_n = \lambda > 1$, then $\bar{S}$ is robust.*

Proof. (1) Note that by assertion (3) of Proposition 2.1 applied to $|||\cdot|||_n = \inf_{k \geq n} \|\cdot\|_k$, we have $|||\, y \,|||_n = \|y\|_n$ if $y \in S_n$. Now suppose $x \neq 0$, $x \in \bar{S}$, and $x = \sum_{k=0}^{\infty} x_k$, so that $\sum_{k=0}^{\infty} \|x_k\|_k < \infty$. Let $\varepsilon > 0$ so that $3\varepsilon < \|x\|$. Choose N so that $\sum_{k=N+1}^{\infty} \|x_k\|_k < \varepsilon$. Note that if $n \geq N+1$, then

$$\left|\left|\left| \sum_{k=n}^{\infty} x_k \right|\right|\right|_n \leq \sum_{k=n}^{\infty} \|x_k\|_k < \varepsilon.$$

For $n \geq N+1$,

$$
\begin{aligned}
|||\, x \,|||_n &\geq \left|\left|\left| \sum_{k=0}^{N} x_k \right|\right|\right|_n - \left|\left|\left| \sum_{k=N+1}^{n-1} x_k \right|\right|\right|_n - \left|\left|\left| \sum_{k=n}^{\infty} x_k \right|\right|\right|_n \\
&\geq \left|\left|\left| \sum_{k=0}^{N} x_k \right|\right|\right|_n - \left|\left|\left| \sum_{k=N+1}^{n-1} x_k \right|\right|\right|_n - \varepsilon \\
&\geq \left|\left|\left| \sum_{k=0}^{N} x_k \right|\right|\right|_n - \sum_{k=N+1}^{n-1} |||\, x_k \,|||_n - \varepsilon \\
&= \left\| \sum_{k=0}^{N} x_k \right\|_n - \sum_{k=N+1}^{n-1} \|x_k\|_n - \varepsilon \\
&= c_N \dots c_{n-1} \left\| \sum_{k=0}^{N} x_k \right\|_n - \sum_{k=N+1}^{n-1} c_k \dots c_{n-1} \|x_k\|_k - \varepsilon \\
&\geq c_N \dots c_{n-1} \left(\left\| \sum_{k=0}^{N} x_k \right\|_n - \sum_{k=N+1}^{n-1} \|x_k\|_k \right) - \varepsilon \\
&\geq c_N \dots c_{n-1} \left(\left\| \sum_{k=0}^{N} x_k \right\| - \varepsilon \right) - \varepsilon \\
&\geq c_N \dots c_{n-1} \left(\left\| \sum_{k=0}^{N} x_k \right\| - 2\varepsilon \right) \\
&\geq c_N \dots c_{n-1} \left(\|x\| - 3\varepsilon \right).
\end{aligned}
$$

Since $\lim_{n\to\infty} c_N \dots c_{n-1} = \infty$, we obtain $\lim_{n\to\infty} |||\, x \,|||_n = \infty$. Thus x is not languid. Since 0 is the only languid element in $\bar{S}$, $\bar{S}$ is robust by Proposition 2.3.

To prove (2) we may assume $\|x\| = 1$. Chose $\varepsilon > 0$ so that $2\varepsilon < 1 - 1/\lambda_0$, i.e., so that $\lambda_0(1-2\varepsilon) > 1$ where $1 < \lambda_0 < \lambda$. Now select a sequence $\langle x_k \rangle$ with each $x_k \in S_k$ such that $\sum_{k=0}^{\infty} x_k = x$ and $\sum_{k=0}^{\infty} \|x_k\|_k < 1 + \varepsilon$. For n sufficiently large with $c_n \geq \lambda_0$ we have $\sum_{k=n+1}^{\infty} \|x_k\|_k < \varepsilon$. Let $y = \sum_{k=0}^{n} x_k$ and $z = \sum_{k=n+1}^{\infty} x_k$, so that $|||\, z \,|||_{n+1} < \varepsilon$. Then

$$
\begin{aligned}
|||\, x \,|||_{n+1} &\geq |||\, y \,|||_{n+1} - |||\, z \,|||_{n+1} \geq |||\, y \,|||_{n+1} - \varepsilon \\
&= \|y\|_{n+1} - \varepsilon \geq \lambda_0 \|y\|_n - \varepsilon \geq \lambda_0 \|y\| - \varepsilon \\
&\geq \lambda_0(\|x\| - \varepsilon) - \varepsilon > \lambda_0(1 - 2\varepsilon) > 1 = \|x\|,
\end{aligned}
$$

so x is not languid. By Proposition 2.3, $\bar{S}$ is robust. $\square$

We are now ready to define the spaces $L_p(w)$.

DEFINITION 2.4. For each nonnegative integer n we let Π_n denote a partition of $[0,1]$ into a finite number of intervals such that

(1) $\Pi_0 = \{[0,1]\}$;
(2) Π_{n+1} refines Π_n;
(3) if $I \in \Pi_n$, then the intervals in Π_{n+1} that subdivide I are of equal length.

We let $\Pi = \cup_{n=0}^{\infty} \Pi_n$. A function $w\colon \Pi \to (0,\infty)$ is said to be a *weight function* if

(4) $w([0,1]) = 1$;
(5) if $I \in \Pi_n$ and I is subdivided into intervals $I_1, I_2, \ldots, I_m$ from Π_{n+1} then $w(I) \le \sum_{k=1}^{m} w(I_k)$;
(6) $\lim_{n\to\infty} \max_{I\in\Pi_n} w(I)/|I|^p = 0$.

If w is a weight function on Π, we let S_n denote the Π_n*-step functions*, i.e.,

$$S_n = \left\{ \sum_{k=1}^{m} \alpha_k 1_{I_k} : \alpha_1, \ldots, \alpha_n \text{ are scalars} \right\},$$

where $\Pi_n = \{I_1, \ldots, I_m\}$. We define $\|\cdot\|_n$ on S_n by

$$\left\| \sum_{k=1}^{m} \alpha_k 1_{I_k} \right\|_n = \sum_{k=1}^{m} |\alpha_k|^p w(I_k).$$

Condition (2) in our definition of the weight function ensures that $\langle (S_n, \|\cdot\|_n)\rangle$ is a stacked sequence. We call $S = \cup_{n=0}^{\infty} S_n$ the set of Π-*step functions*, and define $\|\cdot\|_w = \inf_{n\ge 0} \|\cdot\|_n$ and $L_p(w) = \bar{S}$.

Note that each space $(S_n, \|\cdot\|_n)$ is isometrically isomorphic to l_p^m where $m = |\Pi_n|$. Also notice that if we take $w(I) = |I|$ then $L_p(w)$ is isometrically isomorphic to L_p. Recall that a p-Banach space X is said to have *trivial dual* if its dual consists of only the zero functional, i.e., if $X^* = \{0\}$.

PROPOSITION 2.5. *The $L_p(w)$ spaces have trivial dual.*

Proof. Let $\lambda \in L_p(w)^*$ such that λ is nonexpansive, let $I \in \Pi$ and let $\varepsilon > 0$. By condition (6) there exists n suitably large so that if $J \in \Pi_n$, then $w(J) < \varepsilon |J|^p$. Let $I_1, \ldots, I_m$ denote the intervals from Π_n that are contained in I. Then for each k, $1 \le k \le m$, we have

$$\left\| \frac{1}{|I_k|} 1_{I_k} \right\|_w \le \frac{w(I_k)}{|I_k|^p} < \varepsilon.$$

Thus

$$\lambda\left(\frac{1}{|I|}1_I\right) = \lambda\left(\sum_{k=1}^m \frac{|I_k|}{|I|}\left(\frac{1}{|I_k|}1_{I_k}\right)\right)$$
$$= \sum_{k=1}^m \frac{|I_k|}{|I|}\lambda\left(\frac{1}{|I_k|}1_{I_k}\right) < \varepsilon \sum_{k=1}^m \frac{|I_k|}{|I|} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\lambda(1_I) = 0$. Thus $\lambda(S) = \{0\}$. Since $\bar{S} = L_p(w)$, $\lambda = 0$. □

The major virtue of the spaces $L_p(w)$ is that it is easy to construct maps from these spaces to other trivial dual spaces. The following simple but useful proposition justifies this point.

Proposition 2.6. *Suppose w is a weight function on Π, X is a p-Banach space and $\langle T_n\rangle$ is a sequence of linear maps $T_n\colon S_n \to X$ satisfying:*

(1) *For $I \in \Pi_n$, $\|T_n(1_I)\|_X \le w(I)$, or equivalently*

$$\left\|T_n\left(\frac{1}{|I|}1_I\right)\right\|_X \le \frac{w(I)}{|I|^p}.$$

(2) *If $I \in \Pi_n$ and $I_1, \dots, I_m$ are the intervals in Π_{n+1} which partition I, then $T_n(1_I) = \sum_{k=1}^m T_{n+1}(1_{I_k})$, or equivalently*

$$T_n\left(\frac{1}{|I|}1_I\right) = \sum_{k=1}^m \frac{|I_k|}{|I|}T_n\left(\frac{1}{|I_k|}1_{I_k}\right).$$

Then there is a unique nonexpansive $T \in L(L_p(w), X)$ such that for every $x \in S_n$, $T(x) = T_n(x)$.

Proof. First note that each $T_n \in L(S_n, X)$ is nonexpansive, i.e., if $\Pi_n = \{I_1, \dots I_m\}$ and $x = \sum_{k=1}^m \alpha_k 1_{I_k}$, then

$$\|T_n(x)\|_X = \left\|\sum_{k=1}^m \alpha_k T_n(1_{I_k})\right\|_X$$
$$\le \sum_{k=1}^m |\alpha_k|^p \|T_n(1_{I_k})\|_X$$
$$\le \sum_{k=1}^m |\alpha_k|^p w(I_k) = \|x\|_n.$$

Clearly, by condition (2), there is a linear map $T\colon S \to X$ such that X agrees with each T_n on S_n. If $x \in S$, we define a p-seminorm on S by $|x| = \|T(x)\|_X$. Then, as observed, if $x \in S_n$ then $|x| \le \|x\|_n$. By assertion (2) of Proposition 2.1, for all $x \in S$, we have $\|T(x)\|_X = |x| \le \|x\|_w$. Since S

is dense in $L_p(w)$, T extends to a unique nonexpansive linear map from $L_p(w)$ to X. $\square$

If $I \in \Pi_n$, we shall say that $\frac{1}{|I|}1_I$ is a Π_n-*block*, and if $I \in \Pi$, $\frac{1}{|I|}1_I$ is called a *block*. Notice that if $b = \frac{1}{|I|}1_I$ is a Π_n-block, then for some $I_1, \dots, I_m \in \Pi_{n+1}$ we have $I = \cup_{k=1}^m I_k$, and since the intervals $I_1, \dots I_m$ have equal length, $|I_k|/|I| = 1/m$. Thus if we let $b_k = \frac{1}{|I_k|}1_{I_k}$, then

$$b = \frac{1}{|I|}1_I = \sum_{k=1}^m \frac{|I_k|}{|I|}\left(\frac{1}{|I_k|}1_{I_k}\right) = \frac{1}{m}\sum_{k=1}^m b_k.$$

In other words, every Π_n-block is a unique average of Π_{n+1}-blocks. In light of these comments we can give (without proof) the following restatement of Proposition 2.6:

Proposition 2.7. *Let B_n denote the Π_n-blocks of a space $L_p(w)$ and let $B = \cup_{n=0}^\infty B_n$. Suppose X is a p-Banach space and $\Phi\colon B \to X$ satisfies the following conditions:*

(1) *For every $b = \frac{1}{|I|}1_I \in B$,*

$$\Phi(b) \le \frac{w(I)}{|I|^p}.$$

(2) *Whenever $b \in B_n$ and $b = \frac{1}{m}\sum_{k=1}^m b_k$ with each $b_k \in B_{n+1}$, then*

$$\Phi(b) = \frac{1}{m}\sum_{k=1}^m \Phi(b_k).$$

Then there exists a unique nonexpansive $T \in L(L_p(w), X)$ such that $T(b) = \Phi(b)$ for each $b \in B$.

Notice that if $T \in L(L_p(w), Y)$ and $X = \overline{T(L_p(w))}$, then $X^* = \{0\}$ (since if $\lambda \in X^*$, then $\lambda \circ T \in L_p(w)^*$ and so $\lambda \circ T = 0$). For this reason we concentrate on operators from $L_p(w)$ to X with $X^* = \{0\}$. To see how one might map a space $L_p(w)$ into X note that $X^* = \{0\}$ if and only if, whenever $x \in X$ and $\varepsilon > 0$, there exist $x_1, \dots x_n \in X$ such that $\|x_k\|_X < \varepsilon$ for each k and $x = \frac{1}{m}\sum_{k=1}^m x_k$. If we have $x = \frac{1}{m}\sum_{k=1}^m x_k$ as above, we may then write each x_k as an average of small elements. Continuing with this we can produce a tree-like (actually a bush-like) construction with points on the lower branches tending to zero in norm. Mapping $L_p(w)$ into X amounts to mapping the blocks in $L_p(w)$ to the points in our tree-like structure so that averages are preserved and appropriately set for each block. This idea will be a central theme throughout the rest of the paper.

3. Uniform $L_p(w)$ spaces

We now define a special class of $L_p(w)$ spaces.

Definition 3.1. $L_p(w)$ is said to be *uniform* if the intervals in each Π_n have the same length and the same weight.

Note that in a general $L_p(w)$ space, if $I \in \Pi_n$, then the subintervals of I in Π_{n+1} have the same length, but if J is another interval in Π_n, the size and number of subintervals in J may be different from those in I. If $L_p(w)$ is uniform this irregularity does not occur. Indeed, if $|\Pi_n| = N_n$, each interval in Π_n has length $1/N_n$. Of course, N_{n+1} must be an integer multiple of N_n. Also if

$$\varepsilon_n = \left\| \frac{1}{|I|} 1_I \right\|_n = \frac{w(I)}{|I|^p}$$

for each $I \in \Pi_n$ (with the sequence $\langle \varepsilon_n \rangle$ is chosen so that $\lim \varepsilon_n = 0$), then

$$w(I) = \varepsilon_n |I|^p = \frac{\varepsilon_n}{N_n^p}.$$

Furthermore, if

$$x = \sum_{k=1}^{N_n} \alpha_k 1_{I_k} \in S_n,$$

then

$$\begin{aligned} \|x\|_n &= \sum_{k=1}^{N_n} |\alpha_k|^p w(I_k) = \sum_{k=1}^{N_n} |\alpha_k|^p \frac{\varepsilon_n}{N_n^p} \\ &= \varepsilon_n N_n^{1-p} \sum_{k=1}^{N_n} |\alpha_k|^p \frac{1}{N_n} = \varepsilon_n N_n^{1-p} \sum_{k=1}^{N_n} |\alpha_k|^p |I_k| = \varepsilon_n N_n^{1-p} \|x\|_p. \end{aligned}$$

Thus, with $C_n = \varepsilon_n N_n^{1-p}$, we have

$$\| \cdot \|_n = C_n \| \cdot \|_p.$$

Since $\langle S_n, \| \cdot \|_n \rangle$ is a stacked sequence, $\langle C_n \rangle$ is a nondescending sequence. It is easy to see that if the sequence $\langle C_n \rangle$ is bounded, the space is isomorphic to L_p. Szarvas [8] has shown that if $\langle C_n \rangle$ is unbounded then the uniform $L_p(w)$ space is not isomorphic to L_p. The inclusion map from the uniform L_p spaces into L_p is clearly nonexpansive. Szarvas showed that if $\langle C_n \rangle$ is unbounded, the inclusion map is a compact operator. Since L_p does not admit compact operators, these spaces cannot be isomorphic to L_p. Rowe [6] studied a special class of uniform $L_p(w)$ spaces in which the sequence $\langle C_n \rangle$ increases very rapidly. He showed that in this class of spaces, all compact convex sets are locally convex (i.e., can be affinely embedded into locally convex spaces) and that these spaces are robust. Note that by Proposition 2.4 the uniform $L_p(w)$ spaces are robust if the sequence $\langle C_n \rangle$ is unbounded.

It is natural to ask for a representation of elements in a uniform $L_p(w)$ space as functions on $[0,1]$. Szarvas [8] showed that the inclusion map from a uniform $L_p(w)$ space to L_p is one-to-one. Thus the elements of $L_p(w)$ can be represented by equivalence classes of functions in L_p. It turns out, however, that if the sequence $\langle C_n \rangle$ is increasing suitably rapidly, the only continuous functions in $L_p(w)$ are the constant functions, though we shall not prove this here. Indeed, it is *not true* that if $x, y \in S$ and $|x| \leq |y|$ then $\|x\|_w \leq \|y\|_w$. For this reason, these spaces are a bit more pathological than they appear to be at first sight. We now prove a lemma that will be useful in this section as well as in the next section.

LEMMA 3.1. *Suppose X is a p-Banach space with trivial dual. Also suppose $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$. Then there is an integer M so that if N is an integer multiple of M, then for any k, $1 \leq k \leq N$, we have*

$$x_k = \frac{1}{N} \sum_{i=1}^{N} x_{ki}$$

with $\|x_{ki}\|_X < \varepsilon$, $1 \leq i \leq N$. Furthermore, if $y \in X$ is such that $\|y\| < \varepsilon$, then we may choose the elements $\langle x_{ki} \rangle$ so that $x_{11} = y$.

Proof. Since X has trivial dual, for each k, $1 \leq k \leq n$, we have

$$x_k = \frac{1}{M_k} \sum_{j=1}^{M_k} y_{kj}$$

with $\|y_{kj}\|_X < \varepsilon$. We let $M = \Pi_{k=1}^{n} M_k$. If N is a multiple of M, say $N = mM$, then for each k, we let $x_{k1}, \ldots x_{kN}$ be a finite sequence such that each y_{kj} is listed exactly N/M_k-many times. Thus

$$\frac{1}{N} \sum_{i=1}^{N} x_{ki} = \frac{1}{N} \sum_{j=1}^{M_k} \left(\frac{N}{M_k} y_{kj} \right) = \frac{1}{M_k} \sum_{j=1}^{M_k} y_{jk} = x_k.$$

To obtain the second part of the lemma we need only show that

$$x_1 = \frac{1}{M_1} \sum_{j=1}^{M_1} x_{1j}$$

with $\|x_{1j}\|_X < \varepsilon$ for each j, $1 \leq j \leq M_1$, and $x_{11} = y$. We then apply the above argument. Since X has trivial dual there exist $y_1, \ldots, y_{M_1}$ with $\|y_j\|_X < \delta/2$ such that

$$x_1 = \frac{1}{M_1} \sum_{j=1}^{M_1} y_j,$$

where $\delta = \varepsilon - \|y\|_X$. Now let $x_{11} = y$ and

$$x_{1j} = y_j + \frac{y_1 - y}{M_1 - 1}$$

if $2 \le j \le M_1$. Then, for $2 \le j \le M_1$,

$$\begin{aligned}\|x_{1j}\|_X &\le \|y_j\|_X + \left\| \frac{y_1 - y}{M_1 - 1} \right\|_X \\ &\le \|y_j\|_X + \|y_1\|_X + \|y\|_X < \frac{\delta}{2} + \frac{\delta}{2} + \|y\|_X = \varepsilon.\end{aligned}$$

Now

$$\begin{aligned}\frac{1}{M_1}\sum_{j=1}^{M_1} x_{1j} &= \frac{1}{M_1}\left(y + \sum_{j=2}^{M_1}\left(y_j + \frac{y_1 - y}{M_1 - 1}\right)\right) \\ &= \frac{1}{M_1}\left(y + (y_1 - y) + \sum_{j=2}^{M_1} y_j \right) = \frac{1}{M_1}\sum_{j=1}^{M_1} y_j = x_1. \qquad \square\end{aligned}$$

The following lemma is a standard (and easily proved) metric space result (see [3, p. 203]).

Lemma 3.2. *Suppose (X, d) is a metric space and $\langle K_n \rangle$ is a sequence of compact sets such that $K_n \subset K_{n+1}$ for each n. Also suppose there is a sequence of positive numbers $\langle \varepsilon_n \rangle$ such that $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and so that if $x \in K_{n+1}$, then $d(x, K_n) < \varepsilon_n$. Then $\cup_{n=0}^{\infty} K_n$ is totally bounded.*

Theorem 3.1. *If X is a p-Banach space with trivial dual, then there exists a uniform $L_p(w)$ space and a nonzero compact operator $T\colon L_p(w) \to X$.*

Proof. Let $\langle \varepsilon_n \rangle$ and $\langle \delta_n \rangle$ be positive sequences with $\delta_n \le \varepsilon_n$, $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \delta_n = 0$, and $\sum_{n=1}^{\infty} \delta_n/\varepsilon_n < \infty$. We shall define inductively a sequence $\langle N_n \rangle$ of positive integers such that, for each n, N_{n+1} is an integer multiple of N_n (with $N_0 = 1$). Corresponding to each N_n is the partition Π_n of $[0, 1]$ into N_n intervals of length $1/N_n$. Also, if $I \in \Pi_n$, then $w(I) = \varepsilon_n/N_n^p$ and $\|\cdot\|_n = C_n\|\cdot\|_p$ with $C_n = \varepsilon_n N_n^{1-p}$. We shall define $T_n\colon S_n \to X$ inductively so that each T_{n+1} extends T_n, each T_n is nonexpansive on $(S_n, \|\cdot\|_n)$, and if $x \in S_n$ with $\|x\|_n \le 1$, then $\|T_n(x)\|_X < \delta_n/\varepsilon_n$.

Let $a \in X$ such that $0 < \|a\|_X \le 1$, and let $T_0(1) = a$. Suppose that $N_0, N_1, \ldots, N_n$ and $T_0, T_1, \ldots, T_n$ have been defined satisfying the above conditions. Let $\Pi_n = \{I_1, I_2, \ldots, I_{N_n}\}$ and let

$$T_n\left(\frac{1}{|I_k|} 1_{I_k}\right) = x_k, \quad 1 \le k \le N_n.$$

By Lemma 3.1, there exists an integer M such that if N_n is an integer multiple of M then, for each k,

$$x_k = \frac{1}{N_n}\sum_{i=1}^{N_n} x_{ki} \quad \text{with} \quad \|x_{ki}\|_X < \delta_{n+1}.$$

Choose N_n large enough so that $N_n \geq (\varepsilon_n/\varepsilon_{n+1})^{1/(1-p)}$. We divide each interval I_k into intervals $I_{k1}, \ldots, I_{kN}$ each of length $1/(N_nN)$, and we let $N_{n+1} = N_nN$. We define

$$T_{n+1}\left(\frac{1}{|I_{ki}|}1_{I_{ki}}\right) = x_{ki}.$$

Thus T_{n+1} extends T_n. Also, since

$$\left\|\frac{1}{|I_{ki}|}1_{I_{ki}}\right\|_{n+1} = \varepsilon_{n+1}$$

and $\|x_{ki}\|_X < \delta_{n+1}$, if $x \in S_{n+1}$ and $\|x\|_{n+1} \leq 1$, then

$$\|T_{n+1}(x)\|_X \leq \frac{\delta_{n+1}}{\varepsilon_{n+1}}.$$

Thus, since $\delta_{n+1}/\varepsilon_{n+1} \leq 1$, T_{n+1} is nonexpansive on $(S_{n+1}, \|\cdot\|_{n+1})$. Also,

$$\begin{aligned} C_{n+1} &= \varepsilon_{n+1}N_{n+1}^{1-p} = \varepsilon_{n+1}N^{1-p}N_n^{1-p} \\ &\geq \varepsilon_n N_n^{1-p} = C_n, \end{aligned}$$

by our choice of N. We let T denote the common extension of $\langle T_n\rangle$ to $L_p(w)$. Let $B = \{x \in L_p(w) : \|x\|_w \leq 1\}$, and let $B_n = S_n \cap B$. Note that, since each S_n is finite dimensional, B_n is compact in $L_p(w)$ and $T(B_n)$ is compact in X. Also, $\cup_{n=0}^{\infty} B_n$ is dense in B. To demonstrate the compactness of T it suffices to show that $T(\cup_{n=0}^{\infty} B_n) = \cup_{n=0}^{\infty} T(B_n)$ is totally bounded. We apply Lemma 3.2 along with our assumption that $\sum_{n=1}^{\infty}\delta_n/\varepsilon_n < \infty$. Suppose $y \in T(B_{n+1})$, i.e., $y = T(x)$ with $x \in B_{n+1}$. Then $x = \sum_{k=0}^{n+1} x_k$ such that each $x_k \in S_k$ and $\|x\|_w = \sum_{k=0}^{n+1}\|x_k\|_k \leq 1$. In particular, $\|x_{n+1}\|_{n+1} \leq 1$ and $\sum_{k=0}^{n} x_k \in B_n$. Thus

$$d(y, T(B_n)) \leq \left\|T(x) - T\left(\sum_{k=0}^{n} x_k\right)\right\|_x = \|T(x_{n+1})\|_X \leq \frac{\delta_{n+1}}{\varepsilon_{n+1}}.$$

The compactness of T now follows. $\square$

Lemma 3.3. *Suppose X and Y are p-Banach spaces and $T \in L(X,Y)$. Further suppose that there exists $D \subset Y$ and $\lambda > 0$ satisfying the following conditions:*

(1) *The set $\left\{y/\|y\|_Y^{1/p} : y \in D\right\}$ is dense in the unit sphere of Y.*

(2) *For every* $y \in D$, *there exists* $x \in X$ *such that* $T(x) = y$ *and* $\|y\|_Y > \lambda \|x\|_X$.

Then T *is a surjection.*

Proof. Let

$$D_0 = \left\{ \frac{y}{\|y\|_Y^{1/p}} : y \in D \right\}.$$

Since $\overline{D}_0 = \{y \in Y : \|y\|_Y = 1\}$ and, by assumption (2), $D_0 \subset T(B_{1/\lambda})$, the closed unit ball in Y, we have $B_Y \subset \overline{T(B_{1/\lambda})}$. The lemma now follows from Theorem 1.4 in [3]. □

Theorem 3.2. *The class of uniform* $L_p(w)$ *spaces is projective in* T_p*; i.e., if* X *is a separable p-Banach space with trivial dual, then there exists a uniform* $L_p(w)$ *space such that* X *is a quotient of* $L_p(w)$.

Proof. Let $\langle \varepsilon_n \rangle$ be a positive sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$. Further let $\langle y_n \rangle$ be a sequence in X such that $\varepsilon_n/2 < \|y_n\|_X < \varepsilon_n$ for each n and such that $\left\{ y_n/\|y_n\|_X^{1/p} : n = 1, 2, \dots \right\}$ is dense in the unit sphere of X. As in the proof of Theorem 3.1 we define $\langle N_n \rangle$ and $\langle T_n \rangle$ inductively so that each $T_n \colon S_n \to X$ is nonexpansive and each T_{n+1} extends T_n. We further insist that for each n there exists $I \in \Pi_n$ such that $T_n\left(\frac{1}{|I|} 1_I\right) = y_n$, where $\left\| \frac{1}{|I|} 1_I \right\|_n = \varepsilon_n$. Suppose $N_0, \dots, N_n$ and $T_0, T_1, \dots T_n$ have been defined. Further suppose $\Pi_n = \{I_1, I_2, \dots, I_{N_n}\}$. We let $x_k = T_n\left(\frac{1}{|I_k|} 1_{I_k}\right)$, $1 \le k \le N_n$. By Lemma 3.1 there exists an integer M such that if N is an integer multiple of M, then there exists $x_{k1}, \dots x_{kN} \in X$ such that $\|x_{ki}\|_X < \varepsilon_{n+1}$ and $x_k = \frac{1}{N} \sum_{k=1}^N x_{ki}$. We may further insist that $x_{11} = y_{n+1}$. We choose N large enough so that $N \ge (\varepsilon_n/\varepsilon_{n+1})^{1/(1-p)}$, and let $N_{n+1} = N N_n$. Each $I_k \in \Pi_n$ is divided into intervals $I_{k1}, \dots, I_{kN}$, and we define $T_{n+1}\left(\frac{1}{|I_{ki}|} 1_{I_{ki}}\right) = x_{ki}$. Since $\left\| \frac{1}{|I_{ki}|} 1_{I_{ki}} \right\|_n = \varepsilon_{n+1}$ and $\|x_{ki}\|_X < \varepsilon_{n+1}$, $T_{n+1} \colon S_{n+1} \to X$ is nonexpansive.

We let T denote the common extension of $\langle T_n \rangle$ to a nonexpansive linear map on $L_p(w)$. By our construction, for each $n = 1, 2, \dots$ there exists $I \in \Pi_n$ such that

$$\left\| \frac{1}{|I|} 1_I \right\|_w \le \left\| \frac{1}{|I|} 1_I \right\|_n = \varepsilon_n$$

and $T\left(\frac{1}{|I|} 1_I\right) = y_n$ with $\|y_n\|_X \ge \varepsilon/2$. By Lemma 3.3, it follows that T is a surjection. □

4. Biuniform unbalanced $L_p(w)$ spaces

We now introduce the biuniform unbalanced $L_p(w)$ spaces. This will turn out to be another projective class in T_p. These spaces have the property that their quotient by the constant functions (a one dimensional subspace) is rigid. As a consequence of this fact, we shall show that every space in T_p is a quotient of a rigid space in T_p.

We now define the biuniform unbalanced $L_p(w)$ spaces.

Definition 4.1. Suppose that $\langle R_2, R_3, \dots \rangle$ is a sequence of intervals in Π such that $R_n \in \cup_{k=0}^{n-1} \Pi_k$ and suppose that $\langle A_n \rangle$ and $\langle B_n \rangle$ are sequences of positive numbers such that $A_2 \le B_2 \le A_3 \le B_3 \le \cdots$. Further suppose that $L_p(w)$ satisfies the following conditions:

(1) $\Pi_1 = \{[0,1/2], [1/2,1]\}$ with $w([0,1/2]) = w([1/2,1])$.
(2) For $n \ge 2$, there exist integers p_n and q_n such that for $I \in \Pi_n$, if $I \subset R_n$ then $|I| = 1/p_n$ and if $I \subset R_n^c$ then $|I| = 1/q_n$.
(3) If $x \in S_n$ with $n \ge 2$, then

$$\|x\|_n = A_n \|1_{R_n} x\|_p + B_n \|1_{R_n^c} x\|_p.$$

Then the space $L_p(w)$ is said to be *biuniform*.

Condition (1) is not really essential, but is included for the sake of neatness. Notice that if $p_n = q_n$ and $A_n = B_n$ for all $n \ge 2$, then $L_p(w)$ is a uniform space.

Definition 4.2. Suppose that $L_p(w)$ is biuniform. We say that $L_p(w)$ is *unbalanced biuniform* if it satisfies

(4) for each $I \in \Pi, R_n = I$ for infinitely many n,

and there exists a positive decreasing sequence $\langle \varepsilon_n \rangle$ with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

(5) for $n \ge 2$, $\mathrm{co}(B_{4\varepsilon_n} \cap S_{n-1}) \subset B_{\varepsilon_{n-1}} \cap S_{n-1}$,
(6) $A_n = \varepsilon_n p_n^{1-p} \ge 2B_{n-1}$ for $n \ge 2$,
(7) $B_n = \varepsilon_n q_n^{1-p} \ge \left(\frac{A_n+1}{\varepsilon_{n-1}}\right) B_{n-1}$ for $n \ge 2$.

Notice that if $I \in \Pi_n$ and $I \subset R_n$, then $|I| = 1/p_n$ and

$$\left\| \frac{1}{|I|} 1_I \right\|_n = A_n \left\| \frac{1}{|I|} 1_I \right\|_p = \frac{A_n}{p_n^{1-p}} = \varepsilon_n,$$

by condition (6). Similarly, from condition (7), if $I \subset R_n^c$, then

$$\left\| \frac{1}{|I|} 1_I \right\|_n = \varepsilon_n.$$

The imbalance comes from condition (7), where B_n is necessarily much larger than A_n (and consequently q_n is much larger than p_n). Henceforth, we let C denote the constant functions in $L_p(w)$, i.e., $C = \{c1 : c \in \mathbf{R}\}$.

THEOREM 4.1. *If X is a separable p-Banach space with trivial dual, then there exists an unbalanced biuniform space $L_p(w)$ such that X is a quotient of $L_p(w)/C$.*

Proof. Recall that $\Pi_0 = \{[0,1]\}$ and $\Pi_1 = \{[0,1/2],[1/2,1]\}$. Select $a \in X$ such that $\|a\|_X \le 1/2$. Define $T_0(1) = 0$ (so that $T_0(C) = \{0\}$) and $T_1\left(1_{[0,1/2]}\right) = a = -T_1\left(1_{[1/2,1]}\right)$. Also $\|\cdot\|_0 = \|\cdot\|_1 = \|\cdot\|_p$. The selection of the sequence $\langle R_n\rangle$ will be accomplished as follows: Let $\langle L_n\rangle$ be a sequence of intervals with rational endpoints such that every interval with rational endpoints appears in the sequence infinitely many times. For $n \ge 2$, we let $R_n = L_{k_n}$, where k_n is the least integer such that $L_{k_n} \in \cup_{j=0}^{n-1}\Pi_j$ and $k_n \notin \{k_j : j < n\}$. Thus, as the sequence $\langle \Pi_n\rangle$ is constructed, we automatically obtain $\langle R_n\rangle$ with condition (4) satisfied. Now let $\{z_n : n = 2,3,\dots\}$ denote a dense sequence in $\{x \in X : \|x\|_X = 1\}$. Suppose that ε_k, p_k, q_k and nonexpansive T_k have been selected for all $k < n$ (so that Π_k, S_k, R_k, A_k, B_k, and $\|\cdot\|_k$ have been determined for $k < n$). In the following we take $p_1 = 2$. (Note that there are no intervals in Π_1 of length $1/q_1$.) We then let

$$\{I_{11}, I_{21}, \dots I_{r1}\} = \{I \in \Pi_{n-1} : I \subset R_n \text{ and } |I| = 1/p_{n-1}\}$$

and

$$\{I_{12}, I_{22}, \dots, I_{s2}\} = \{I \in \Pi_{n-1} : I \subset R_n \text{ and } |I| = 1/q_{n-1}\}.$$

We further select ε_n with $0 < \varepsilon_n < \min\{1/n, \varepsilon_{n-1}\}$ such that $\mathrm{co}(B_{4\varepsilon_n} \cap S_{n-1}) \subset B_{\varepsilon_{n-1}} \cap S_{n-1}$. This is possible since S_{n-1} is finite dimensional.

We let

$$x_{ki} = T_{n-1}\left(\frac{1}{|I_{ki}|}1_{I_{ki}}\right)$$

with $1 \le k \le r$ if $i = 1$, and $1 \le k \le s$ if $i = 2$. By Lemma 3.1 there exists an integer M such that

$$x_{ki} = \frac{1}{M}\sum_{j=1}^{M} y_{kij}$$

with $\|y_{kij}\|_X \le \varepsilon_n$. Furthermore, we may insist that $y_{111} = (\varepsilon_n/2)^{1/p} z_n$ (so that $\|y_{111}\|_X = \varepsilon_n/2$). Now let p_n be an integer multiple of $p_{n-1}q_{n-1}M$ chosen large enough so that $\varepsilon_n p_n^{1-p} \ge 2B_{n-1}$. Notice that if $[0,1]$ is partitioned into intervals of length $1/p_n$ then the resulting partition refines Π_{n-1}. Each I_{k1} (with $1 \le k \le r$) is divided into p_n/p_{n-1}-many intervals $I_{k11}, \dots I_{k1l}$ with $l = p_n/p_{n-1}$. We list each of the elements, $y_{k11}, \dots, y_{k1M}$, p_n/Mp_{n-1}-many times in a finite sequence, $x_{k11}, \ldots, x_{k1l}$, and define

$$T_n\left(\frac{1}{|I_{k1j}|}1_{I_{k1j}}\right) = x_{k1j}.$$

Thus

$$\begin{aligned} T_{n-1}\left(\frac{1}{|I_{k1}|}1_{I_{k1}}\right) &= x_{k1} = \frac{1}{M}\sum_{j=1}^{M} y_{k1j} = \frac{1}{l}\sum_{j=1}^{l} x_{k1j} \\ &= \frac{1}{l}\sum_{j=1}^{l} T_n\left(\frac{1}{|I_{k1j}|}1_{k1j}\right) = \frac{p_n}{l}\sum_{j=1}^{l} T_n(1_{I_{k1j}}) \\ &= \frac{p_n}{l}T_n(1_{I_{k1}}) = p_{n-1}T_n(1_{I_{k1}}) = T_n\left(\frac{1}{|I_{k1}|}1_{Ik1}\right). \end{aligned}$$

Thus T_n (as defined so far) extends T_{n-1}. We define T_n similarly for the characteristic functions of the remaining intervals in R_n of length $1/p_n$. Note that for a subinterval I_{kij},

$$A_n\left\|\frac{1}{|I_{kij}|}1_{I_{kij}}\right\|_p = \varepsilon_n p_n^{1-p}\frac{|I_{kij}|}{|I_{kij}|^p} = \varepsilon_n p_n^{1-p}\frac{p_n^p}{p_n} = \varepsilon_n.$$

Also

$$\left\|T_n\left(\frac{1}{|I_{kij}|}1_{I_{kij}}\right)\right\|_X = \|x_{kij}\|_X \le \varepsilon_n.$$

Note also that for some x_{k1j} we have $x_{k1j} = y_{111} = (\varepsilon_n/2)^{1/p} z_n$ with $\|y_{111}\|_X = \varepsilon_n/2$. Thus, when the induction is completed, the conditions of Lemma 3.3 will be met so that the resulting T is a quotient map.

The same procedure is carried out for the characteristic functions of intervals in Π_{n-1} that are in R_n^c. The intervals are subdivided into intervals of length $1/q_n$, with q_n chosen large enough so that $\varepsilon_n q_n^{1-p} \ge (A_{n+1}/\varepsilon_{n-1})\, B_{n-1}$. Also, T_n is extended so that T_n is nonexpansive with respect to $\|\cdot\|_n$. $\square$

Notice that in any unbalanced biuniform space, if $I, J \in \Pi$ and $I \subset J$, then $I \in \Pi_m$ and $J \in \Pi_n$ with $m \ge n$. Thus

$$\left\|\frac{1}{|I|}1_I\right\|_m = \frac{w(I)}{|I|^p} = \varepsilon_m,$$

so that $w(I) = \varepsilon_m |I|^p \le \varepsilon_n |J|^p = w(J)$. In other words, if $I \subset J$, then $w(I) \le w(J)$. Notice that for any $I, J \in \Pi$ we have either $I \subset J$, $J \subset I$, or $I \cap J = \emptyset$. Consequently $w(I \cap J) \le w(J)$ (we define $w(\emptyset) = 0$). Thus if $x = \sum_{k=1}^{n} \alpha_k 1_{I_k} \in S$, then

$$\|1_I x\|_w \le \sum_{k=1}^{n} \alpha_k w(I \cap I_k) \le \sum_{k=1}^{n} \alpha_k w(I_k).$$

Therefore $\|1_I x\|_w \leq \|x\|_w$ for all $x \in S$. Hence multiplication by 1_I is a nonexpansive operator on S and extends to a nonexpansive operator on $L_p(w)$. We denote this operator acting on $x \in L_p(w)$ by $1_I x$. If E is a finite union of intervals in Π we define $1_E x$ for $x \in L_p(w)$ similarly. The operator 1_E may not be nonexpansive, however. Note that if $I \in \Pi$, then $\|1_{I^c} x\|_w = \|x - 1_I x\|_w \leq \|x\|_w + \|1_I x\|_w \leq 2\|x\|_w$. Also if E is a union of intervals in Π_n, then $\|1_E x\|_k \leq \|x\|_k$ if $x \in S_k$ with $k \geq n$, i.e., 1_E is nonexpansive with respect to the norms $\|\cdot\|_k$ when $k \geq n$. (Note that $\|\cdot\|_k$ is just a weighted $l_p^{m_k}$ norm with $m_k = |\Pi_k|$.) Thus 1_E is nonexpansive on $L_p(w)$ with respect to $|||\cdot|||_n$, by the same argument as in Proposition 2.6. We record this in the following proposition:

Proposition 4.1. *Suppose $L_p(w)$ is an unbalanced biuniform space. Then we have:*

1. *If $I \in \Pi$, then for every $x \in L_p(w)$, $\|1_I x\|_w \leq \|x\|_w$ and $\|1_{I^c} x\|_w \leq 2\|x\|_w$.*
2. *If E is a finite union of intervals in Π_n, then for every $x \in L_p(w)$, $|||1_E x|||_n \leq |||x|||_n$.*

Theorem 4.2. *Suppose that $L_p(w)$ is biuniform unbalanced. Also suppose that $T \in L(L_p(w), L_p(w)/C)$ and that $I \in \Pi$. Then there exists $x_0 \in L_p(w)$ such that $1_I x_0 = x_0$ and $T(1_I) = q(x_0)$ where $q(x) = x + C$ is the quotient map from $L_p(w)$ to $L_p(w)/C$.*

Proof. We let $H = I^c$ and we shall show that there exists $x_0 \in L_p(w)$ such that $T(1_I) = q(x_0)$ with $1_H x_0 = 0$ so that $1_I x_0 = x_0$. To simplify matters we may suppose that $\|1_I\|_w < 1$. Without loss of generality we may assume that T is nonexpansive. Since $\|1_I\|_w < 1$, we have $\|T(1_I)\| < 1$. Hence we may select $x \in L_p(w)$ with $\|x\|_w < 1$ so that $T(1_I) = q(x)$. Now let $E = \{n : R_n = I\}$. Since $L_p(w)$ is robust, by conditions (6) and (7) and Proposition 2.4, there exists a sequence $\langle a_k \rangle$ with each $a_k \in S_k$ so that

$$x = \sum_{k=0}^{\infty} a_k \quad \text{and} \quad \|x\|_w = \sum_{k=0}^{\infty} \|a_k\|_k < 1.$$

For each $n \in E$, we let

$$y_n = \sum_{k=0}^{n-1} a_k \quad \text{and} \quad z_n = \sum_{k=n}^{\infty} a_k.$$

Thus $x = y_n + z_n$ with $y_n \in S_{n-1}$ such that

1. $\|y_n\| + |||z_n|||_n = \|x\|_w < 1,$
2. $\lim_{n \in E} |||z_n|||_n = 0.$

For fixed $n \in E$, we have

$$1_I = 1_{R_n} = \frac{1}{N_n} \sum_{k=1}^{N_n} N_n 1_{I_k},$$

where $I_k \in \Pi_n$, $|I_k| = 1/p_n$, and $N_n = |R_n| p_n$. Now

$$\|N_n 1_{I_k}\|_w \le \|N_n 1_{I_k}\|_n = \| \, |R_n| p_n 1_{I_k}\|_n \le \|p_n 1_{I_k}\|_n = \left\| \frac{1}{|I_k|} 1_{I_k} \right\|_n = \varepsilon_n.$$

Since T is nonexpansive, we have $T(N_n 1_{I_k}) = q(u_k)$ with $\|u_k\|_w < 2\varepsilon_n$. Now $u_k = v_k + w_k$, where $v_k \in S_{n-1}$ and $\|v_k\|_w + |||\, w_k \,|||_n = \|u_k\|_w < 2\varepsilon_n$. Noting that $I = R_n \in \cup_{j=0}^{n-1} \Pi_j$ and $H = I^c$, $v_k \in S_{n-1}$, we have, by Proposition 4.1, $\|1_H v_k\|_w \le 2\|v_k\|_w < 4\varepsilon_n$. Now let

$$v = \frac{1}{N_n} \sum_{k=1}^{N_n} v_k \quad \text{and} \quad w = \frac{1}{N_n} \sum_{k=1}^{N_n} w_k.$$

Since $1_H v \in \operatorname{co}(B_{4\varepsilon_n} \cap S_{n-1})$, we have $\|1_H v\|_w < \varepsilon_{n-1}$. Also

$$\begin{aligned} |||\, w \,|||_n &= \left|\left|\left| \frac{1}{N_n} \sum_{k=1}^{N_n} w_k \right|\right|\right|_n \\ &\le \frac{1}{N_n^p} \sum_{k=1}^{N_n} |||\, w_k \,|||_n \le \frac{N_n}{N_n^p} (2\varepsilon_n) \\ &= 2N_n^{1-p} \varepsilon_n \le 2p_n^{1-p} \varepsilon_n, \end{aligned}$$

since $N_n = |R_n| p_n \le p_n$. Since $q(x) = \frac{1}{N_n} \sum_{k=1}^{N_n} q(u_k)$, we have $q(x) = q(v + w)$; i.e., for some constant c_n, $v + w = c_n 1 + x = c_n 1 + y_n + z_n$ with $v, y_n \in S_{n-1}$. Thus $c_n 1 + y_n - v = w - z_n \in S_{n-1}$. So $1_H(c_n 1 + y_n - v) = 1_H(w - z_n) \in S_{n-1}$. Now, recalling that $H = R_n^c$, we have

$$\begin{aligned} B_n \|1_H(c_n 1 + y_n - v)\|_p &= \|1_H(c_n 1 + y_n - v)\|_n \le |||\, 1_H(c_n 1 + y_n - v) \,|||_n \\ &= |||\, 1_H(w - z_n) \,|||_n \le |||\, 1_H w \,|||_n + |||\, 1_H z_n \,|||_n \\ &\le |||\, w \,|||_n + |||\, z_n \,|||_n \le 2p_n^{1-p} \varepsilon_n + 1, \end{aligned}$$

where the next to last inequality follows from Proposition 4.1. Hence

$$\|1_H(c_n 1 + y_n - v)\|_p \le \frac{2p_n^{1-p} \varepsilon_n + 1}{B_n}.$$

Thus, since $1_H(c_n 1 + y_n - v) \in S_{n-1}$,

$$\begin{aligned}\|1_H(c_n 1 + y_n - v)\|_w &\le \|1_H(c_n 1 + y_n - v)\|_{n-1} = B_{n-1}\|1_H(c_n 1 + y_n - v)\|_p \\ &\le \frac{B_{n-1}}{B_n}(2p_n^{1-p}\varepsilon_n + 1) \le \frac{\varepsilon_{n-1}}{A_n + 1}(2p_n^{1-p}\varepsilon_n + 1) \\ &= \varepsilon_{n-1}\frac{(2p_n^{1-p}\varepsilon_n + 1)}{(p_n^{1-p}\varepsilon_n + 1)} \le 2\varepsilon_{n-1}.\end{aligned}$$

Hence

$$\begin{aligned}\|1_H(c_n 1 + y_n)\|_w &\le \|1_H(c_n 1 + y_n - v)\|_w + \|1_H v\|_w \\ &< 2\varepsilon_{n-1} + \varepsilon_{n-1} = 3\varepsilon_{n-1}.\end{aligned}$$

Applying this to all $n \in E$, we obtain

$$\lim_{n \in E} \|1_H(c_n 1 + y_n)\|_w = 0.$$

Since $\lim_{n\in E} |\!|\!| z_n |\!|\!|_n = 0$, we have

$$\lim_{n \in E} \|1_H(c_n 1 + x)\|_w = 0.$$

Clearly the sequence $\langle c_n \rangle$ is bounded. So by passing to an infinite subset of E, if necessary, we may assume that, for some constant c, $\lim_{n\in E} c_n = c$. Consequently,

$$1_H(c1 + x) = 0.$$

Setting

$$x_0 = c1 + x \in q(x) = T(1_I).$$

completes the proof. □

DEFINITION 4.3. A p-Banach space X is *rigid* if, whenever $T \in L(X)$ there is a constant c such that for every $x \in X$, $T(x) = cx$, i.e., $T = cI$.

THEOREM 4.3. *If $L_p(w)$ is an unbalanced biuniform space and C denotes the constant functions, then $L_p(w)/C$ is rigid.*

Proof. Suppose $T \in L(L_p(w)/C)$. Let $q \colon L_p(w) \to L_p(w)/C$ be the quotient map and let $T_0 = T \circ q$ so that $T_0 \colon L_p(w) \to L_p(w)/C$. For any positive integer m let $\Pi_m = \{I_1, I_2, \dots, I_m\}$. By Theorem 4.2 we have, for each k, $1 \le k \le m$, $T_0(1_{I_k}) = q(x_k)$, where $1_{I_k} x_k = x_k$ Now

$$0 = T_0(1) = \sum_{k=1}^{m} T(1_{I_k}) = q_0\left(\sum_{k=1}^{m} x_k\right).$$

Hence $\sum_{k=1}^{m} x_k = c1$ for some constant c. Also it is easily verified that $1_{I_k} x_j = 0$ if $j \ne k$. Thus,

$$c1_{I_k} = 1_{I_k}(c1) = 1_{I_k}\left(\sum_{j=1}^{m} x_j\right) = 1_{I_k} x_k = x_k,$$

i.e., $T_0(1_{I_k}) = cq(1_{I_k})$. Thus for $x \in S_n$, $T(q(x)) = T_0(x) = cq(x)$. Since the partitions $\langle \Pi_n \rangle$ are increasing (in the refinement sense), the constant c is the same for all Π_n. Hence, for $x \in S$ we have $T(q(x)) = cq(x)$ and thus $T = cI$. □

Theorem 4.4. *If X is a separable p-Banach space with trivial dual, then there is an unbalanced biuniform space $L_p(w)$ such that if $Y = L_p(w)$ or $Y = L_p(w)/C$ then $L(X, Y) = \{0\}$ and X is a quotient of Y. In particular, we have:*

(1) *Every $X \in T_p$ is a quotient of a rigid space in T_p.*
(2) *T_p contains no projective spaces.*

Proof. First we observe that there is an unbalanced biuniform space $L_p(w)$ such that if $Y = L_p(w)/C$, there is a quotient map Q from Y onto X such that $\operatorname{Ker} Q \neq \{0\}$, i.e., Q is not an isomorphism. To see this, let $L_p(w_0)$ be an unbalanced biuniform space with a quotient map q_1 from $L_p(w_0)$ to X such that $q_1(1) = \{0\}$. Let $L_p(w)$ be an unbalanced biuniform space with a quotient map q_2 from $Y = L_p(w)/C$ onto $L_p(w_0)$, and set $Q = q_1 q_2$. Since $\operatorname{Ker} q_1 \neq \{0\}$, we have $\operatorname{Ker} Q \neq \{0\}$.

Now let $T \in L(X, Y)$. Then $TQ \in L(Y)$, so that $TQ = cI$ for some constant c. Since $\operatorname{Ker} Q \neq \{0\}$, we have $c = 0$, i.e., $TQ = 0$. Since Q is onto, $T = 0$.

Now suppose $T \in L(X, L_p(w))$. Let q denote the quotient map from $L_p(w)$ to $L_p(w)/C$ so that $qT \in L(X, L_p(w)/C)$. Then $qT = 0$, so T maps X to the one dimensional space C. Since $X^* = \{0\}$, it follows that $T = 0$.

(1) follows directly from Theorem 4.1 and Theorem 4.3. (2) follows since if $X \in T_p$ there exists $Y = L_p(w)$ an unbalanced biuniform space such that not only is Y not a quotient of X but $L(X, Y) = \{0\}$. □

The author at one point had conjectured (but was unable to prove) that, given any $X \in T_p$, there is a uniform space $L_p(w)$ such that $L(X, L_p(w)) = \{0\}$. This has now been proved by Szarvas [8].

In [7] Sisson showed that there exists a rigid space which admits compact operators. This result motivated most of the results of this paper. Specifically, the author was led to suspect that every space in T_p is a quotient of a rigid space in T_p. This now turns out to be the case. Note that Sisson's Theorem is a consequence of this fact. Let $X \in T_p$ be any space admitting a compact operator T to a space $Z \in T_p$. There is a quotient map Q from a rigid space $Y \in T_p$ to X. Thus TQ is a compact operator from the rigid space Y to Z.

The final theorem of this paper is motivated by the result in [4] mentioned in the introduction. Namely, there exists a collection of subspaces $\{X_\alpha : \alpha \in [0,1]\}$ of L_p so that if $\alpha \neq \beta$, then

$$L(X_\alpha, X_\beta) = L(X_\beta, X_\alpha) = \{0\}.$$

This suggests that, in a rather strange sense, T_p is a very wide class of spaces. Theorem 4.5 will show that T_p is also a rather tall class of spaces. First we shall prove a simple lemma.

LEMMA 4.1. *If $\langle X_n \rangle$ is a sequence of spaces in T_p, there exists a space Y of the form $Y = L_p(w)$ or $Y = L_p(w)/C$ with $L_p(w)$ an unbalanced biuniform space such that*

(1) *each X_n is a quotient of Y,*
(2) *$L(X_n, Y) = \{0\}$.*

Proof. Let $X = \sum_{n=1}^{\infty} X_n$ denote the l_1-sum of the spaces $\langle X_n \rangle$, i.e.,

$$X = \left\{ \langle x_n \rangle : \text{for each } n,\, x_n \in X_n, \|\langle x_n \rangle\|_X = \sum_{n=1}^{\infty} \|x_n\|_{X_n} < \infty \right\}.$$

It is easily seen that $X \in T_p$. Thus there exists Y as above with a quotient map Q from Y onto X such that $L(X, Y) = \{0\}$. Let $P_n \colon X \to X_n$ denote the projection defined by $P_n(\langle x_k \rangle) = x_n$. Then $P_n Q$ is a projection from Y onto X_n. Also, if $T \in L(X_n, Y)$, then $TP_n \in L(X, Y)$. But then $TP_n = 0$ so that $T = 0$. □

THEOREM 4.5. *Let Λ denote the first uncountable ordinal. There exists a family $\{Y_\alpha : \alpha \in \Lambda\}$ such that $Y_\alpha = L_p(w_\alpha)$ or $Y_\alpha = L_p(w_\alpha)/C$, where each $L_p(w_\alpha)$ is an unbalanced biuniform space such that*

(1) *if $\alpha < \beta$, then $L(Y_\alpha, Y_\beta) = \{0\}$,*
(2) *if $\alpha < \beta$, then Y_α is a quotient of Y_β.*

Proof. The spaces $\{Y_\alpha : \alpha \in \Lambda\}$ are constructed by transfinite induction. Let $\beta \in \Lambda$. If β is the first ordinal, let Y_β be any space as above. Otherwise, if $\beta \in \Lambda$, $\{Y_\alpha : \alpha < \beta\}$ is at most countable. By Lemma 4.1 there exists Y_β as above such that for each $\alpha < \beta$, $L(Y_\alpha, Y_\beta) = \{0\}$ and each Y_α is a quotient of Y_β. □

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