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ON THE BARBAN-DAVENPORT-HALBERSTAM THEOREM: XVIII

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Abstract. We consider sequences, of positive density C, of positive integers s that are postulated to have the property that

$$S(x;a,k)=\sum_{\substack{s\le x\\ s\equiv a,\bmod k}}1=f(a,k)x+O\left(x\log^{-A}x\right)$$

for any positive constant A. Let

$$G(x,Q)=\sum_{k\le Q}\sum_{0<a\le k}E^2(x;a,k)\quad (Q\le x),$$

where $E(x;a,k)=S(x;a,k)-f(a,k)x$.

Then previously we had shewn that

$$G(x,Q)=D\{1+o(1)\}x^2+O\left(x^2\log^{-A}x\right)\quad (Q/x\to 0).$$

When $D=0$, many particular examples of which are known, this formula supplies little information about $G(x,Q)$ and about how small it can be. The first result obtained in this paper is the lower bound

$$G(x;Q)\ge\frac{1}{12}\{3C-2C^2+\min{}^2(C,1-C)+o(1)\}Q^2+O(x^2\log^{-A}x)$$

that is best possible when $C=\frac{1}{2}$ or 1.

The other subject of the paper is the sum

$$G_\lambda(x,Q)=\sum_{k\le Q}\sum_{0<a\le\lambda k}E^2(x;a,k)\quad(\lambda<1)$$

and its connection with $G(x,Q)$. If $G(x,Q)/Q^2$ be bounded, it is demonstrated that the expected limiting equality of $G_\lambda(x,Q)$ and $\lambda G(x,Q)$ can be false. On the other hand, it is shewn that this equality holds in the appropriate sense for any sequence of Q for which $G(x,Q)/Q^2\to\infty$.

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1. Introduction

In articles[1] III,IX,X,XIV of this series and also in [3] we revealed various properties related to the subject of the title that were possessed by strictly increasing sequences (of positive density) of positive integers s obeying a primary condition of the type

$$S(x;a,k)=\sum_{\substack{s\le x\\ s\equiv a,\bmod k}}1=f(a,k)x+O\{\Delta_k(x)\} \tag{1}$$

for values of k that may be small compared with x. Save in XIV where the consequences of sharp forms of $\Delta_k(x)$ were explored, the remainder term was taken to be of the form $O(x\log^{-A}x)$ for any positive constant A in order to provide an environment analogous to that furnished by the prime numbers in the other members of the series. Hence, since all ancillary conditions previously imposed were shewn in X to be superfluous in the deduction of the main conclusions reached, we resume our researches on the sequences s under the sole assumption that they conform to

Criterion V. *For any positive constant A,*

$$S(x;a,k)=f(a,k)x+O\left(x\log^{-A}x\right),$$

where

$$f(0,1)=C>0. \tag{2}$$

Before we describe our present intentions, it is necessary to look back and briefly recapitulate the main findings in X, although the reader is referred to that paper, IX, and parts of XIV—or to our survey [8]—for a fuller appreciation of what has gone before. The first conclusion in X was that, if

$$E(x;a,k)=S(x;a,k)-f(a,k)x \tag{3}$$

and

$$G(x,Q)=\sum_{k\le Q}\sum_{0<a\le k}E^2(x;a,k)=\sum_{k\le Q}H(x,k),\ \text{say}, \tag{4}$$

then under Criterion V the sequence s possesses a theorem of Barban-Davenport-Halberstam type that asserts that

$$G(x,Q)=O(Qx)+O\left(x^2\log^{-A}x\right)\quad (Q\le x) \tag{5}$$

for any positive constant A; moreover it is then seen that in fact (5) and Criterion V are equivalent because $|E(x;a,k)|\le G^{\frac12}(x,Q)$ for $k\le Q<$

[1]We refer to these articles by the Roman numerals indicating their positions in the series; their full particulars are given in the list of references at the end.

$x \log^{-A} x$. From this and a further analytical argument, it was then shewn that the upper bound could be replaced by the more exact formula

$$G(x,Q) = D_1\{1+o(1)\}Qx + O\left(x^2 \log^{-A} x\right) \quad (Q/x \to 0) \tag{6}$$

of so called Barban-Montgomery type. There thus emerged two classes of sequences answering to the non-vanishing or vanishing of D_1, it having been shewn in IX by special examples that each actually existed. Although it is only for sequences in the first class that a genuine asymptotic formula immediately arises, many familiar sequences such as that of the square-free numbers (see, for example, Croft [1]) belong to the second class and can be shewn to possess formulae in which the formal main term in (6) is superseded by an expression such as

$$D_2\{1+o(1)\}Q^{1+\alpha}x^{1-\alpha} \quad (0<\alpha<1)$$

with $D_2 > 0$. In fact, such replacement formulae are always available when $f(a,k)$ has a sufficiently smooth and predictable behaviour, as was demonstrated by Vaughan [11] for a variety of sequences for which was postulated the asymptotic behaviour of a sum indirectly containing this function.

Even so, the occurrence of an explicit asymptotic formula for $G(x,Q)$ for sequences in the second class may be atypical because it was found in XIV that there are sequences in this class for which the term $o(1)$ in (6) fluctuates between $O(Q^{2-\epsilon}x^{\epsilon})$ and $Q^{1+\epsilon}x^{1-\epsilon}$ as $x \to \infty$ and Q/x varies. Thus, the goal of obtaining an asymptotic formula being in general illusory, we shall now follow the only credible alternative course of seeking a lower bound for $G(x,Q)$ when Q is not too small compared with x and shall shew that what we obtain is inextricably connected with our second theme to be shortly unfolded. This bound is reminiscent of Roth's well known lower bound in [10] (see, also, Montgomery [9]) but is not directly connected with it.

We prove in Theorem 1 below that, for any sequence of density $C > 0$ answering to Criterion V, we have

$$G(x,Q) \geq \frac{1}{12}\{3C - 2C^2 - \min^2(C, 1-C) + o(1)\}Q^2 + O\left(x^2 \log^{-A} x\right) \qquad (Q/x \to 0)$$

and then shew that for $C = \frac{1}{2}$ and 1 this is best possible in the sense that there is actually some sequence of such density for which the sign of equality may replace that of inequality. For other values of C it seems likely that the bound is capable of improvement; this, however, is as far as we take this particular matter apart from some associated results, since we have yet to evolve a fully satisfactory treatment and since we are anxious to proceed to our second subject.

We also take the already verified formula

$$G(x,Q)=\frac{1}{12}\{1+o(1)\}Q^2+o(x^2\log^{-A}x)$$

for the natural numbers and proceed to sketch a direct and simple proof of this to serve as a templet for the production of a counter example in future speculations. These concern such sums as

$$G_\lambda(x,Q)=\sum_{k\le Q}\sum_{0<a\le\lambda k}E^2(x;a,k)=\sum_{k\le Q}H_\lambda(x,k) \tag{7}$$

for $\lambda\le 1$ and the not unnatural conjective that

$$G_\lambda(x,Q)\sim\lambda G(x,Q)$$

over suitable domains of x, Q, and λ. But, in the special case $C=1$, our alternative method of calculating $G(x,Q)$ demonstrates that

$$G_\lambda(x,Q)\sim\frac{1}{12}\lambda(2-3\lambda+2\lambda^2)Q^2$$

so that there must be some limit to our expectations. Actually, as we shall see below, the abatement required is very slight because all we shall need is that

$$G(x,Q)/Q^2\to\infty\ . \tag{8}$$

In the meanwhile, however, an exact interpretation of the meaning to be attached to this condition will have been given in a prior discussion, which amongst other things will shew that its failure to hold is tantamount to the existence of an asymptotic formula of the type

$$G(x,Q)=D_2\{1+o(1)\}Q^2+O\left(x^2\log^{-A}x\right). \tag{9}$$

These remarks are the prelude to our second theme about whether the pattern of the distribution of $S(x;a,k)$—or, at least, its dispersion—be affected by a restriction in the residue classes, mod k, that are to be counted. We thus follow the precedent set in XI where we investigated the parallel question for primes and the effect on the Barban-Montgomery formula by a curtailment of the reduced residue classes appearing within it. Yet, although the basic structure of the already complicated method in XI is retained, the proof here is much more taxing than before because (i) we need to estimate some sums whose antecedents in XI had obvious values, (ii) the generality of $f(a,k)$ as compared with the specificity of $1/\phi(k)$ in XI places more demands on the analysis, and (iii) there is a need to produce sharp remainder terms when $D_1=0$ to cater for the possibility that $G(x,Q)$ may be very small compared with Qx. Indeed, to dilate on the last point, some care is required in order that $G_\lambda(x,Q)$, or its precursors in the analysis, can be realistically likened to a multiple of the elusively sized $G(x,Q)$ whenever condition (8) is imposed.

There are places where the exposition might have been slightly shortened by citing extracts from XI. But the overall treatment is sufficiently complicated for such a procedure to be a serious distraction for the reader, and therefore, apart from summarizing some of the results from X and XIV in a preface, we have produced a self-contained account in which we indicate from time to time the similarities and dissimilarities of our exposition to that of XI.

We should remark that we shall proceed indirectly to the sum $G_\lambda(x, Q)$ by way of an intermediate theorem for sums of the type

$$G^*(x, u; Q_1, Q_2) = \sum_{Q_1 < k \leq Q_2} k \sum_{0 < a \leq u} E^2(x; a, k),$$

which are themselves associated with situations where there is a truncation in the range of residue classes. In the same way, we would treat sums of the type

$$G^*(x, u_1, u_2; Q_1, Q_2) = G^*(x, u_2; Q_1, Q_2) - G^*(x, u_1; Q_1, Q_2),$$

whose assessments could lead to results on the sums

$$G_{\lambda,\mu}(x, Q) = \sum_{k \leq Q} \sum_{\mu k < a \leq (\lambda+\mu)k} E^2(x; a, k) \tag{10}$$

that we mention at the end.

Finally, the weak form of the remainder term in Criterion V—chosen, as said before, for comparability with the prime number theorem for arithmetical progressions—limits the usefulness of our results for large moduli k to those values that exceed $x \log^{-A} x$ for any chosen large A. Yet, if we sharpen our hypothesis on the remainder term as in XIV or if we consider known sequences for which the remainder term is definitely smaller, then we can obtain our results with a smaller lower bound for k.

2. Notation

Owing to the length of the memoir it is not feasible to lay down a completely consistent notation. But the meaning of all symbols should be clear from their context in view of the following guide.

The letters (adorned or unadorned with marks such as superscripts) a, b, c, l, r, and s are usually non-negative integers, although l is normally positive; k, d and m are positive integers; y, u, v, w are positive numbers; x is a positive number to be regarded as tending to infinity, all stated inequalities being valid when it takes sufficiently large values.

The letters B_i are specific constants whose values are immaterial to the investigation; A, A_1 are any positive absolute constants that need not be connected, while $A_2, A_3, \ldots$ are positive absolute constants whose association with each other and with A, A_1 will be plain from the text; ϵ is an arbitrarily small number that is not necessarily the same on each occasion.

The constants implied by the O-notation depend at most on ϵ and on those values of A, A_i that are relevant to each occasion. As usual $(a,b), [a,b]$ respectively denote the positive highest common factor and least common multiple of a and b when these are defined; $d(m)$ is the number of divisors of m.

3. Prologue

Assuming throughout that Criterion V is observed, we summarize those findings of IX,X, and XIV that we shall need in the sequel. First, letting

$$M(k) = \sum_{0<a\le k} f^2(a,k) \tag{11}$$

and defining the function $w(a,l)$ by

$$f(a,k) = \frac{1}{k}\sum_{l|k} w(a,l) \text{ or } w(a,l) = \sum_{d|l} \mu\left(\frac{l}{d}\right) df(a,d) \tag{12}$$

as in X, (10) and (12), we have

$$M(k) = \frac{1}{k}\sum_{l|k} N(l) = \frac{1}{k}\sum_{l|k} la_l, \quad \text{say,} \tag{13}$$

where initially we know that

$$N(l) = \sum_{0<a\le l} w(a,l)f(a,l) \tag{14}$$

by X, (16), (17), and (18). But also, as in XIV (22), we have

$$N(l) = \frac{1}{l}\sum_{0<a\le l} w^2(a,l) \ge 0 \tag{15}$$

and then that the series

$$\sum_{1=l}^{\infty} N(l) = \sum_{1=l}^{\infty} \frac{1}{l} \sum_{0<a\le l} w^2(a,l) \tag{16}$$

is (absolutely) convergent (see top of p. 6 in X). Moreover, if

$$F(s) = \sum_{k=1}^{\infty} \frac{M(k)}{k^{s+1}},$$

then

$$F(s) = \zeta(s+1)\Phi(s),$$

where the function

$$\Phi(s) = \sum_{l=1}^{\infty} \frac{a_l}{l^s}$$

is regular for $\sigma > -1$ and has a value

$$\Phi(-1) = C_1, \quad \text{say,} \tag{17}$$

at $s = -1$. Next, slightly rephrasing the derivation of the asymptotic formula in X through IX, we meet the sum

$$T^*(u) = \frac{1}{2} \sum_{l<u} \left(1 - \frac{l}{u}\right)^2 M(l) \tag{18}$$

and eventually express $G(x, Q)$ in terms of

$$I^*(u) = \Phi(0) \log u + \left\{\Phi'(0) + \left(\gamma - \frac{3}{2}\right) \Phi(0)\right\} + \frac{\Phi(-1)}{u} - 2T^*(u) \tag{19}$$

by means of the formula

$$\begin{aligned} G(x, Q) &= \{C - \Phi(-1)\}Qx + x^2 I^*(x/Q) + O\left(x^2 \log^{-A} x\right) \\ &= (C - C_1)Qx + x^2 I^*(x/Q) + O\left(x^2 \log^{-A} x\right). \end{aligned} \tag{20}$$

From this then follows (6) with

$$D_1 = C - C_1 \geq 0 \tag{21}$$

because it was shewn in IX (argument there used is valid for generalized case in X) that

$$I^*(u) = o(1/u) \tag{22}$$

as $u \to \infty$ (in XIV a better estimate was derived on the assumption of sharper bounds for $\triangle_k(x)$ in (1)).

We shall also need the trivial and related estimates

$$S(x; a, k) = O(x/k) + O(1), \quad f(a, k) = O(1/k) \tag{23}$$

that represent a slight extension of those stated in X (8). To these, for $d|k$, we then add the fresh and last relation

$$f(a, d) = \sum_{0<\lambda\leq k/d} f(a + \lambda d, k) \tag{24}$$

that follows from Criterion V and the equation

$$S(x; a, d) = \sum_{0<\lambda\leq k/d} S(x; a + \lambda d, k)$$

after dividing by x and letting $x \to \infty$.

4. Summation formulae and the lower bound for $G(x,Q)$

Certain summations based on the Euler-Maclaurin formula will be needed for the production of the lower bound for $G(x,Q)$ and in certain aspects of the later analysis. In preparation, let us introduce the (weighted) periodic Bernoulli functions $\phi_1(t),\phi_2(t),\phi_3(t)$ that are, respectively, defined as

$$\frac{1}{2}-t', \quad \frac{1}{2}t'-\frac{1}{2}t'^2-\frac{1}{12}\ , \quad \frac{1}{4}t'^2-\frac{1}{6}t'^3-\frac{1}{12}t'=\frac{1}{12}t'(1-t')(2t'-1)$$

for $t'=t-[t]$, using the first two of these to state the Euler-Maclaurin sum formula in the form

$$\sum_{a\le r\le b} g(r)=\int_a^b g(t)dt+g(b)\phi_1(b)+\frac{1}{2}g(a)-{}_a^b\,[g'(t)\phi_2(t)]$$
$$+\int_a^b g''(t)\phi_2(t)dt$$

when b exceeds an integer a and $g(t)$ is of an appropriate type. Then, taking

$$g(t)=\frac{1}{t}(v-t)^2$$

in the primary application so that

$$g'(t)=-\frac{v^2}{t^2}+1 \text{ and } g''(t)=\frac{2v^2}{t^3},$$

we deduce that, for $v\ge 1$,

$$\begin{aligned}\sum_{l<v}\frac{(v-l)^2}{l} &= \int_1^v \frac{(v-t)^2dt}{t}+\frac{1}{2}(v-1)^2+\frac{1}{12}(v^2-1)+2v^2\int_1^v\frac{\phi_2(t)dt}{t^3}\\ &= v^2\log v-2v(v-1)+\frac{1}{2}(v^2-1)+\frac{1}{2}(v-1)^2\\ &\quad +\frac{1}{12}(v^2-1)+2v^2\int_1^v\frac{\phi_2(t)dt}{t^3}\\ &= v^2\log v-\frac{11}{12}v^2+v-\frac{1}{12}+2v^2\int_1^v\frac{\phi_2(t)dt}{t^3}\\ &= v^2\log v+v^2\left(-\frac{11}{12}+2\int_1^\infty\frac{\phi_2(t)dt}{t^3}\right)\\ &\quad +v-\frac{1}{12}-2v^2\int_v^\infty\frac{\phi_2(t)dt}{t^3}\\ &= v^2\log v+v^2\left(-\frac{11}{12}+2\int_1^\infty\frac{\phi_2(t)dt}{t^3}\right)+v+O(1)\end{aligned} \tag{25}$$

in view of the preliminary inequality

$$\left|\int_v^\infty \frac{\phi_2(t)dt}{t^3}\right| < A' \int_v^\infty \frac{dt}{t^3} = O\left(\frac{1}{v^2}\right) \tag{26}$$

that we shall shortly need to improve. But, since

$$\sum_{l\le w} \frac{1}{l} = \log w + \gamma + O\left(\frac{1}{w}\right)$$

for $w \ge 1$, we also know through a double integration that

$$\sum_{l<v} \frac{(v-l)^2}{l} = v^2 \log v + \left(\gamma - \frac{3}{2}\right) v^2 + O(v \log 2v),$$

which when compared with the last line of (25) shews first that

$$-\frac{11}{12} + 2\int_1^\infty \frac{\phi_2(t)dt}{t^3} = \gamma - \frac{3}{2}$$

and then that

$$\begin{aligned}\sum_{l<v} \frac{(v-l)^2}{l} &= v^2 \log v + \left(\gamma - \frac{3}{2}\right) v^2 + v - \frac{1}{12} - 2v^2 \int_v^\infty \frac{\phi_2(t)dt}{t^3} \\ &= v^2 \log v + \left(\gamma - \frac{3}{2}\right) v^2 + v - \frac{1}{12} - 2v^2 K_1(v) \\ &= v^2 \log v + \left(\gamma - \frac{3}{2}\right) v^2 + v - K_2(v), \text{ say.}\end{aligned} \tag{27}$$

To harness this equation to the method of bounding $G(x,Q)$ from below we need to shew that

$$K_2(v) > \frac{1}{24} > 0. \tag{28}$$

Now it is readily seen that the maxima and minima of $\phi_3(t)$ occur when

$$t \equiv \frac{1}{2} \pm \frac{\sqrt{3}}{6}, \text{mod } 1,$$

with the implication that

$$|\phi_3(t)| \le \frac{1}{12}\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)\frac{\sqrt{3}}{3} = \frac{1}{12}\left(\frac{1}{4} - \frac{1}{12}\right)\frac{\sqrt{3}}{3} = \frac{1}{72}\,\frac{\sqrt{3}}{3}.$$

Placed in the equation

$$K_1(v) = \left[\frac{\phi_3(t)}{t^3}\right]_v^\infty + 3\int_v^\infty \frac{\phi_3(t)dt}{t^4}$$

obtained by partial integration, this inequality gives

$$K_1(v) > -\frac{1}{36}\,\frac{\sqrt{3}}{3}\,\frac{1}{v^3} > -\frac{1}{54v^3} > -\frac{1}{48v^3}$$

with the required result that

$$K_2(v) = \frac{1}{12} + 2v^2 K_1(v) > \frac{1}{12} - \frac{1}{24} = \frac{1}{24}$$

for $v \geq 1$; also, of course

$$K_2(v) = \frac{1}{12} + o(1) \tag{29}$$

as $v \to \infty$.

Passing on to the question of a lower bound for $G(x, Q)$ when $Q = o(x)$, we need only consider the case where $D_1 = 0$ because otherwise the asymptotic formula (6) at once supplies, for any positive constant D_1', an inequality

$$G(x, Q) = \{D_1 + o(1)\}Qx + O\left(x^2 \log^{-A} x\right) > D_1' Q^2 + O\left(x^2 \log^{-A} x\right) \tag{30}$$

of the type we seek. In the situation we are now placed where $C = C_1$, the formula (20) takes the form

$$G(x, Q) = x^2 I^*(x/Q) + O\left(x^2 \log^{-A} x\right), \tag{31}$$

where we remind ourselves that $I^*(u)$ is given by formula (19) for $u \geq 1$. Found by a contour integration method, the latter formula is then to be utilized by alternatively evaluating the sum

$$2u^2 T^*(u) = \sum_{n \leq u} (u - n)^2 M(n)$$

by means of (13) and (27). Accordingly, we find that

$$\begin{aligned} 2u^2 T^*(u) &= \sum_{lm \leq u} \frac{(u - lm)^2 a_m}{l} = \sum_{m \leq u} m^2 a_m \sum_{l \leq u/m} \frac{(u/m - l)^2}{l} \\ &= \sum_{m \leq u} m^2 a_m \left\{ \frac{u^2}{m^2} \log \frac{u}{m} + \left(\gamma - \frac{3}{2}\right) \frac{u^2}{m^2} + \frac{u}{m} - K_2\left(\frac{u}{m}\right) \right\} \\ &= u^2 \sum_{m \leq u} a_m \left(\log \frac{u}{m} + \left(\gamma - \frac{3}{2}\right) + \frac{m}{u} \right) - \sum_{m \leq u} m^2 a_m K_2\left(\frac{u}{m}\right), \end{aligned}$$

whence, by (19),

$$u^2 I^*(u) = \sum_{m > u} a_m \left(\log \frac{u}{m} + \left(\gamma - \frac{3}{2}\right) + \frac{m}{u} \right) u^2 + \sum_{m \leq u} m^2 a_m K_2\left(\frac{u}{m}\right). \tag{32}$$

In both sums on the right of (32) the coefficients of a_m are positive because of (28) and because $\gamma > \frac{1}{2}$ and $\eta - 1 - \log \eta \geq 0$ for $\eta \geq 1$. Also

$$a_1 = N(1) = w^2(0, 1) = f^2(0, 1) = C^2$$

by (13), (15), (12), and (2). Therefore, taking $u_1 = u^{\frac{1}{2}}$ for convenience, letting $u \to \infty$, and first using a simple argument, we deduce from (29) and the non-negativity of a_m that

$$
\begin{aligned}
u^2 I^*(u) &\geq \frac{1}{12}\{1+o(1)\} \sum_{m \leq u_1} m^2 a_m \\
&\geq \frac{1}{12}\{1+o(1)\} \left(a_1 + 2 \sum_{1<m \leq u_1} m a_m \right) \\
&= \frac{1}{12}\{1+o(1)\} \left(-a_1 + 2 \sum_{m \leq u_1} m a_m \right) \\
&= \frac{1}{12}\{1+o(1)\} \left(-a_1 + 2\Phi(-1) + o(1) \right) \\
&= \frac{1}{12}\{1+o(1)\} \left(2C_1 - C^2 + o(1) \right) \\
&= \frac{1}{12}\{2C - C^2\} + o(1)
\end{aligned} \tag{33}
$$

after an appeal to (17). From this and (31) and then from (20) we thus establish the inequality

$$
G(x,Q) > \frac{1}{12}\{2C - C^2 + o(1)\}Q^2 + o\left(x^2 \log^{-A} x\right) \quad (x/Q \to \infty). \tag{34}
$$

There is thus a lower bound of definite order of magnitude Q^2 for $G(x,Q)$, a fact that will be invaluable when we come to discuss the behaviour of the analogues of $G(x,Q)$ that are defined over restricted ranges of summation. That this is best possible for the cases $C = 1$ and $C = \frac{1}{2}$ is easily seen by taking in turn the sequence of natural numbers and the sequence of odd numbers, for which the equations in §3 give successively

$$
\begin{gathered}
f(a,k) = 1/k; \quad w(a,k) = 1 \text{ or } 0 \text{ according as } k = 1 \text{ or } k > 1, \\
a_0 = C^2; \quad a_k = 0 \text{ if } k > 1; \quad C_1 = a_0 = 1 = C
\end{gathered}
$$

in the former instance and

$$
f(a,k) = \begin{cases} 1/2k, & \text{if } k \text{ odd}, \\ 1/k, & \text{if } k \text{ even and } a \text{ even}, \\ 0, & \text{if } k \text{ even and } a \text{ odd}; \end{cases}
$$

$$
\begin{gathered}
w(a,k) = 1 \text{ if } k = 1; \quad w(a,2) = \pm\frac{1}{2}; \quad w(a,k) = 0 \text{ if } k > 2; \\
a_1 = C^2; \quad a_2 = \frac{1}{8}; \quad a_k = 0 \text{ if } k > 2; \\
C_1 = a_1 + 2a_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} = C
\end{gathered}
$$

in the latter. Consequently, on consulting (29) and (32), we confirm that in these two cases we may replace the sign of inequality in (34) by that of equality.

Yet for other values of C the inequality is certainly not best possible because we can sharpen it by introducing an element into the analysis that for the sake of simplicity had been previously ignored when $C = C_1$. To implement this improvement, let us note that we can write

$$f(0,2) = \frac{1}{2}(C+b), f(1,2) = \frac{1}{2}(C-b)$$

where certainly

$$|b| \le C,\ 1-C \tag{35}$$

because $f(0,2)+f(1,2) = C$ and $0 \le f(a,2) \le \frac{1}{2}$. Hence, in this notation, we have $w(a,2) = \pm\, b$ and $a_2 = \frac{1}{2}b^2$ with the consequence that first

$$\begin{aligned} C = C_1 = \Phi(-1) &= C^2 + b^2 + \sum_{m\ge 3} m a_m \\ &= C^2 + b^2 + \sum_{3\le m\le u_1} m a_m + o(1) \end{aligned}$$

and then

$$\begin{aligned} \sum_{m\le u_1} m^2 a_m &= C^2 + 2b^2 + \sum_{3\le m\le u_1} m^2 a_m \ge C^2 + 2b^2 + 3\sum_{3\le m\le u_1} m a_m \\ &= C^2 + 2b^2 + 3(C - C^2 - b^2) + o(1) \\ &= 3C - 2C^2 - b^2 + o(1). \end{aligned}$$

Because of (35) and the first part of (33), this yields

$$u^2 I^*(u) > \frac{1}{12}\{3C - 2C^2 - \min{}^2(C, 1-C)\} + o(1),$$

and, as in the derivation of (34), we establish

THEOREM 1. *Let the sum $G(x,Q)$ be defined as in the Introduction. Then, for any sequence conforming to Criterion V, we have*

$$G(x,Q) \ge \frac{1}{12}\{D_3(C) + o(1)\}Q^2 + O\left(x^2 \log^{-A} x\right) \quad (x/Q \to \infty)$$

where

$$D_3(C) = \begin{cases} 5C - 3C^2 - 1, & \text{if } C \ge \frac{1}{2}, \\ 3C - 3C^2, & \text{if } C \le \frac{1}{2}. \end{cases}$$

The improvement over (34) represented by this theorem is moderately noticeable when the value of C is not close to 0, $\frac{1}{2}$, or 1, the coefficient of $\frac{1}{12}Q^2$

for $C = \frac{1}{4}$ being now, for example, $\frac{9}{16}$ in place of the previous $\frac{7}{16}$. Nevertheless, unless $C = \frac{1}{2}$ or 1, Theorem 1 is probably still not best possible and therefore remains a subject for a further study we cannot undertake here.

Our work so far is not unconnected with our future need to differentiate in the case $D_1 = 0$ between situations within ranges of the type

$$Q = o(x), \quad Q > x\log^{-A_1} x \tag{36}$$

where $G(x, Q)/Q^2$ is bounded or otherwise. First let us suppose, for a sequence answering to the condition $D_1 = 0$, that

$$\varliminf_{n\to\infty} u^2 I^*(u)$$

is finite so that there is a sequence of numbers u_2 tending to infinity for which

$$u_2^2 I^*(u_2) < 2E$$

for some constant E . Next, employing the values $u = u_2$ in (32) to find that

$$\sum_{m\leq u_2} m^2 a_m < 48E$$

in virtue of (28) and the non-negativity of $v - 1 - \log v$ for ≥ 1, we deduce that the series

$$\sum_{m=1}^{\infty} m^2 a_m$$

is convergent. Moreover, since the first sum on the right of (32) is then

$$O\left(u \sum_{m>u} m a_m\right) = O\left(\sum_{m>u} m^2 a_m\right) = o(1)$$

as $u \to \infty$, we infer by (29) that

$$\lim_{u\to\infty} u^2 I^*(u) = \frac{1}{12} \sum_{m=1}^{\infty} m^2 a_m$$

and incidentally uncover by a previous routine the asymptotic formula

$$G(x, Q) = \frac{1}{12}\left(\sum_{m=1}^{\infty} m^2 a_m + o(1)\right) Q^2 + O\left(x^2 \log^{-A} x\right)$$

of the type (9) mentioned in the Introduction. However, for sequences for which $D_1 = 0$ but which are not of the above type,

$$\lim_{u\to\infty} u^2 I^*(u) = \infty,$$

and we therefore conclude from formula (20) (with a sufficiently large value of A) that within a range (36) the quantity $G(x, Q)/Q^2$ related to a particular sequence is either always bounded or always unbounded; indeed, in the former case, it even tends to the limit $\frac{1}{12}\Phi(-2)$, while the set of sequences corresponding to the latter case can be augmented by those for which $D_1 > 0$.

Instances of the former phenomenon are easy to find for many densities C; for example, for $C = \frac{1}{3}$, we cite the set of natural numbers divisible by 3.

Lastly, since we alluded in the Introduction to our having isolated examples where $G(x, Q)$ oscillated in size between $Q^{2-\epsilon}x^{\epsilon}$ and $Q^{1+\epsilon}x^{1-\epsilon}$, we remark that the minima in meaningful fluctuations cannot go as low in order of magnitude as Q^2.

5. Constraints on the asymptotic formula for $G_\lambda(x, Q)$

Having divided the sequences appertaining to Criterion V into two categories, we go on to shew by means of an example that the expected relation between $G(x, Q)$ and $G_\lambda(x, Q)$ can be false for members of the first category for which $G(x, Q)/Q^2$ is bounded in the sense described above.

In this example we take the sequence of natural numbers and consider the attached asymptotic formula that stems from the discussion after (34), the proof of which for such a simple sequence seems unnecessarily circuitous in view of its reliance on the mechanisms of IX, X, and XIV. We therefore in this special case sketch a direct treatment of such a nature that it is also applicable to the sum $G_\lambda(x, Q)$.

First, for convenience, we may assume that x is an integer because, by an elementary identity in probability theory, we have in (4) that

$$\begin{aligned} H(x, k) &= \sum_{0<a\le k} \left(S(x; a, k) - \frac{x}{k} \right)^2 = \sum_{0<a\le k} \left(S([x]; a, k) - \frac{x}{k} \right)^2 \\ &= \sum_{0<a\le k} \left(S[x]; a, k) - \frac{[x]}{k} \right)^2 + \frac{(x-[x])^2}{k} \end{aligned}$$

and hence that

$$G(x, Q) = G([x], Q) + O\left(\sum_{k\le Q} \frac{1}{k} \right) = G([x], Q) + O(\log x)$$

when $Q \le x$. For any appropriate k, let us then write $x = b + sk$, where $0 \le b < k$, and thereby express $S(x; a, k)$ for $0 < a \le k$ as the number of solutions in non-negative integers r of the inequality

$$0 < a + rk \le b + sk,$$

inferring that

$$S(x; a, k) = \begin{cases} s+1, & \text{if } 0 < a \le b, \\ s, & \text{if } b < a \le k \end{cases}$$

and then that

$$S(x;a,k) - \frac{x}{k} = \begin{cases} 1 - b/k, & \text{if } 0 < a \leq b, \\ -b/k, & \text{if } b < a \leq k. \end{cases} \tag{37}$$

Therefore

$$H(x,k) = b\left(1 - \frac{b}{k}\right)^2 + (k-b)\frac{b^2}{k^2} = k\frac{b}{k}\left(1 - \frac{b}{k}\right), \tag{38}$$

in which b is determined uniquely by k and x.

The summation of $H(x,k)$ over values of k corresponding to the most influential segment of $G(x,Q)$ depends on the distribution of the numbers b/k, to come close to which we examine the fractions x/k and the intervals between them under the assumption that $Q = o(x)$ exceeds, say, $x^{\frac{7}{8}}$. For this purpose, let $Q_1 = [Q]$ and attempt to define the integers Q_s, r_s and the consequential set $\mathcal{S}_s$ of increasing positive fractions

$$\frac{x}{Q_s}, \frac{x}{Q_s - 1}, \ldots, \frac{x}{Q_s - r_s}$$

iteratively by demanding that the last member of $\mathcal{S}_s$ be the greatest fraction $x/(Q_s - r_s)$ for which

$$\frac{x}{Q_s - r_s} - \frac{x}{Q_s} \leq 1 \tag{39}$$

and by then defining Q_{s+1} to be $Q_s - r_s - 1$. For any subscript s we reach for which $Q_s > x^{\frac{3}{4}}$, it is possible to proceed to the set $\mathcal{S}_s$ and then to Q_{s+1} because (39) implies that

$$r_s = \frac{Q_s^2}{x + Q_s} = \frac{Q_s^2}{x}\left\{1 + O\left(\frac{Q_s}{x}\right)\right\} \tag{40}$$

with the corollary that

$$r_s = o(Q_s) < \frac{1}{3}Q_s; \tag{41}$$

also, since in this case $r_s \geq \frac{1}{2}x^{\frac{1}{2}}$, we must after a finite number of stages reach a first value $Q_t > \frac{1}{2}x^{\frac{3}{4}}$ for which Q_t is less than $x^{\frac{3}{4}}$. Having thus constructed the sets $\mathcal{S}_s$, we see that the numbers x/k in each one are approximately evenly distributed and intuit from (38) that

$$\sum_{Q_{s+1} < k \leq Q,} \frac{H(x,k)}{k} = \sum_{Q_{s+1} < k \leq Q_s} \left[\frac{x}{k}\right]\left(1 - \left[\frac{x}{k}\right]\right) = R_s, \tag{42}$$

is approximately equal to

$$(Q_s - Q_{s+1})\int_0^1 w(1-w)dw = \frac{1}{6}(Q_s - Q_{s+1}),$$

with an appropriate consequence for $G(x,Q)$.

Let us now be rather more precise and take into account the effect of the length

$$I_{k,s} = \frac{x}{k(k-1)} = \frac{x}{Q_s^2}\left\{1 + O\left(\frac{r_s}{Q_s}\right)\right\}$$

of an interval that is either between x/k and $x/(k-1)$ in $\mathcal{S}_s$, or between x/k and the first member $x/(k-1)$ of $\mathcal{S}_{s+1}$. Then, since

$$r_s I_{k,s} = \frac{Q_s^2}{x}\left\{1 + O\left(\frac{Q_s}{x}\right)\right\}\frac{x}{Q_s^2}\left\{1 + O\left(\frac{Q_s}{x}\right)\right\} = 1 + O\left(\frac{Q_s}{x}\right)$$

in virtue of (40), we may affect the summand in the intermediate constituent of (42) by a factor $r_s I_{k,s}$ provided that we add a compensating term

$$O\left(\frac{r_s Q_s}{x}\right) = o(r_s)$$

or, in other words,

$$r_s \sum_{Q_{s+1} < k \leq Q_s} I_{k,s}\left[\frac{x}{k}\right]\left(1 - \left[\frac{x}{k}\right]\right) = R_s + o(r_s) \tag{43}$$

is an approximation to

$$r_s \int_{x/Q_s}^{x/Q_{s+1}} [t](1-[t])dt \tag{44}$$

by the theory of Riemann integration. It being readily seen that the difference between (43) and (44) is $O(1)$, we deduce from (42) that

$$\begin{aligned}
\sum_{Q_{s+1} < k \leq Q_s} H(x,k) &= r_s\{Q_s + O(r_s)\}\int_{x/Q_s}^{x/Q_{s+1}} [t](1-[t])dt + o(r_s Q_s) \\
&= r_s\{Q_s + O(r_s)\}\left\{\int_0^1 w(1-w)dw + O\left(\frac{1}{r_s}\right)\right\} + o(r_s Q_s) \\
&= r_s Q_s \int_0^1 w(1-w)dw + o(r_s Q_s) \\
&= \frac{1}{12}\{Q_s^2 - (Q_s - r_s - 1)^2 + o\{Q_s^2 - (Q_s - r_s - 1)^2\},
\end{aligned}$$

whence

$$\begin{aligned}
G(x,Q) - G(x,Q_t) &= G(x,Q_1) - G(x,Q_t) \\
&= \frac{1}{12}\{1 + o(1)\}(Q_1^2 - Q_t^2) = \frac{1}{12}\{1 + o(1)\}Q^2
\end{aligned}$$

by the previous choice of Q_t. Since trivially $G(x,Q_t) = O(Q_t^2)$, we then conclude that

$$G(x,Q) = \frac{1}{12}\{1 + o(1)\}Q^2$$

under the aforementioned conditions on Q and x.

We thus recoup the formula in a form that reflects the fact that the natural numbers obey the formula in Criterion V with a much more accurate remainder term. But, from our present point of view, the advantage of the new approach is that it also enables us to produce an asymptotic formula for $G_\lambda(x, Q)$.

In outlining the treatment of $G_\lambda(x, Q)$, we assume that λ is any fixed constant less than 1 and go back to the formula (37). Then, since the residue a, mod k, in the conditions of summation pertaining to the definition of $H_\lambda(x, k)$ in (7) is subject to the inequality $0 < a \le \lambda k$, it is natural to consider the cases (i) $b \ge \lambda k$, (ii) $b < \lambda k$ separately. In case (i) the inequality $0 < a \le b$ holds so that $E(x; a, k) = 1 - b/k$, whereas in case (ii)

$$E(x; a, k) = \begin{cases} 1 - b/k, & \text{if } 0 < a \le b, \\ -b/k, & \text{if } b < a \le k, \end{cases}$$

the upshot being that $H_\lambda(x, k)/k$ equals

$$\lambda\left(1 - \frac{b}{k}\right)^2 \text{ or } \frac{b}{k}\left(1 - \frac{b}{k}\right)^2 + \left(\lambda - \frac{b}{k}\right)\frac{b^2}{k^2}$$

according as $b \ge \lambda k$ or $b < \lambda k$. We therefore expect the previous argument to demonstrate that

$$\sum_{Q_{s+1} < k \le Q_s,} \frac{H_\lambda(x, k)}{k}$$

is asymptotic to

$$\begin{aligned}(Q_s - Q_{s+1})&\left(\lambda\int_\lambda^1 (1 - w)^2 dw + \int_0^\lambda \{w(1 - w)^2 + (\lambda - w)w^2\} dw\right)\\ &= (Q_s - Q_{s+1})\left(\frac{1}{3}\lambda(1 - \lambda)^3 + \int_0^\lambda (w - 2w^2 + \lambda w^2) dw\right)\\ &= \frac{1}{6}\lambda(2 - 3\lambda + 2\lambda^2)(Q_s - Q_{s+1}),\end{aligned}$$

wherefore, summing over s, we foresee that

$$G_\lambda(x, Q) \sim \frac{1}{12}\lambda(2 - 3\lambda + 2\lambda^2)Q^2.$$

It being routine to make this conclusion rigorous, we conclude that

$$G_\lambda(x, Q) \not\sim \lambda G(x, Q)$$

in the situation delineated above.

Therefore, of the two situations described in the antepenultimate paragraph of §4, the former where $G(x, Q)/Q^2$ is bounded is one in which we cannot always assert that $G_\lambda(x, Q)$ can be likened to $\lambda G(x, Q)$. Consequently, in our forthcoming investigation of $G_\lambda(x,Q)$, we shall ultimately assume that the latter situation obtains and, for the sake of clarity, define it again through the condition that

$$(45)_1 \qquad G(x,Q)/Q^2 \to \infty$$

for all values of x and Q satisfying

$$(45)_2 \qquad Q = o(x), \quad Q > x \log^{-A_1} x$$

for any positive constant A_1.

However, until we arrive at §16, we shall neither need to assume this hypothesis nor even that Q and x conform to $(45)_2$, a similar comment being apposite for the symbol Q_2 that replaces Q in $G^*(x,u;Q_1,Q_2)$.

6. Lemmata based on sum formulae

The narrative in XI was interspersed with various estimates for sums that for the most part were proved by contour integration. Now needing parallel estimates of greater delicacy in the estimation of $G_\lambda(x,Q)$, we eschew the previous methods that no longer readily meet our requirements and instead use the formulae of §§3 and 4 to establish at once a number of lemmata.

First, as an enhancement of Lemma 2 in XI, we have

Lemma 1. *If $0 < h < 1$ and $y \geq 1$, then*

$$\begin{aligned}\frac{1}{2}\sum_{l<y}\frac{(y-l)^2}{l} - \frac{1}{2}\sum_{l<y-h}\frac{(y-h-l)^2}{l} &= \frac{1}{2}\{y^2\log y - (y-h)^2\log(y-h)\} \\ &\quad + \frac{1}{2}B_1\{y^2-(y-h)^2\} + \frac{1}{2}h + O\left(\frac{h}{y}\right)\end{aligned}$$

where $B_1 = \gamma - \frac{3}{2}$.

Dismissing the case where $y - h \leq 1$ because then $y < 2$ and the left-side of the formula is $O(h^2)$, we see otherwise from (27) that the proposed result is correct provided that

$$y^2K_1(y) - (y-h)^2K_1(y-h) = O(h/y),$$

which estimate follows from the mean-value theorem and the equation

$$\frac{d}{dv}v^2K_1(v) = 2vK_1(v) - \frac{v^2\phi_2(v)}{v^3} = O\left(\frac{1}{v}\right) \quad (v > 1)$$

that stems from the continuity of $\phi_2(t)$ and from an inequality for $K_1(v)$ latent in §4.

Secondly, for the proof of an analogue of Lemma 3 in XI and also for our later analysis, we need the auxiliary

LEMMA 2. *As* $u \to \infty$,

$$\frac{d}{du}\left(u^2 I^*(u)\right) = O(1).$$

From its genesis through a twice performed integration, $u^2 I^*(u)$ has a continuous first differential coefficient that may be calculated even at integral values of u by differentiating both series in (32) term by term (consider right-hand continuity). Indeed, by the last two lines of (27),

$$\begin{aligned}\frac{d}{du}(u^2 I^*(u)) = 2u \sum_{m>u} a_m \left(\log\frac{u}{m} + (\gamma - \frac{3}{2}) + \frac{m}{u}\right) + \sum_{m>u} (u-m) a_m \\ - 4u \sum_{m\le u} a_m K_1\left(\frac{u}{m}\right) - 2u^2 \sum_{m \le u} \frac{a_m}{m} K_1'\left(\frac{u}{m}\right)\end{aligned}$$

and then, by (26) and (16),

$$\begin{aligned}\frac{d}{du}(u^2 I^*(u) &= O\left(\sum_{m>u} m a_m\right) + O\left(\frac{1}{u}\sum_{m\le u} m^2 a_m\right) \\ &= O\left(\sum_{m=1}^{\infty} m a_m\right) = O(1),\end{aligned}$$

which equality is what was asserted.

We then have

LEMMA 3. *Let* $T^*(v)$ *be defined as in* (18). *Then, for* $0 < h < 1$ *and* $y \ge 1$, *we have*

$$\begin{aligned}y^2 T^*(y) - (y-h)^2 T^*(y-h) = \frac{1}{2}\Phi(0)\{y^2 \log y - (y-h)^2 \log(y-h)\} \\ + \frac{1}{2} B_2\{y^2 - (y-h)^2\} + O(h),\end{aligned}$$

where $B_2 = \Phi'(0) + (\gamma - \frac{3}{2})\Phi(0)$.

It being enough to consider the case where $y - h > 1$ as in the proof of Lemma 1, we need to shew that

$$y^2 I^*(y) - (y-h)^2 I^*(y-h) = O(h)$$

on account of (19). But the left-side of this is

$$h\left\{\frac{d}{du}(u^2 I^*(u))\right\}_{u=y-\theta h} \qquad (0 < \theta < 1),$$

which is $O(h)$ by Lemma 2.

Lastly, for the unweighted version

$$T(v) = \sum_{l \le v} M(l)$$

of the sum $T^*(v)$, we require an elementary estimate that is more accurate than those stated in IX and X. This is given in

Lemma 4. *For $y \ge 1$, we have*

$$\sum_{l \le y} M(l) = \Phi(0) \log y + \Phi'(0) + \gamma \Phi(0) + O\left(\frac{\log 2y}{y}\right).$$

By (13) and the convergence at $s = -1$ of the Dirichlet's series defining $\Phi(s)$, the left-side of the stated formula equals

$$\begin{aligned}
\sum_{km \le y} \frac{a_k}{m} &= \sum_{k \le y} a_k \sum_{m \le y/k} \frac{1}{m} = \sum_{k \le y} a_k \left\{ \log \frac{y}{k} + \gamma + O\left(\frac{k}{y}\right) \right\} \\
&= \Phi(0) \log y + \Phi'(0) + \gamma \Phi(0) + O\left(\sum_{k > y} a_k \log 2k \right) + O\left(\frac{1}{y} \sum_{k \le y} k a_k \right) \\
&= \Phi(0) \log y + \Phi'(0) + \gamma \Phi(0) + O\left(\frac{\log 2y}{y} \sum_{k=1}^{\infty} k a_k \right) \\
&= \Phi(0) \log y + \Phi'(0) + \gamma \Phi(0) + O\left(\frac{\log 2y}{y} \right),
\end{aligned}$$

as proposed.

7. Initial analysis of $G^*(x, u; Q_1, Q_2)$

The primary object $G_\lambda(x, Q)$ of our second study is approached through the medium of the sum

$$G^*(x, u; Q_1, Q_2) = \sum_{Q_1 < k \le Q_2} k \sum_{0 < a \le u} E^2(x; a, k),$$

to which the major part of the investigation is devoted under the assumptions[2]

$$u < Q_1, \quad x \log^{-A_1} x < Q_1 < Q_2 \le \frac{1}{2} x. \tag{46}$$

Here we have already initiated a convention to the effect that the insertion of a superscript asterisk in a given notation for a sum over k means that its summand is to be affected by a weight k, an understanding that facilitates our moving to and fro between weighted and unweighted sums during our analysis

[2]The upper bound $\frac{1}{2}x$ instead of x is laid down for convenience rather than necessity; this point, however, is anyway ultimately irrelevant because of our use of (45).

of one of the entities into which $G^*(x, u; Q_1, Q_2)$ is now split by means of (3). The resulting decomposition being given by

$$
\begin{aligned}
(47)\quad G^*(x, u; Q_1, Q_2) &= \sum_{Q_{<}k\le Q_2} k \sum_{0<a\le u} \Big(f^2(a,k)x^2 \\
&\qquad - 2xf(a,k)S(x;a,k) + S^2(x;a,k) \Big) \\
&= x^2 \sum_{Q_1<k\le Q_2} k \sum_{0<a\le u} f^2(a,k) \\
&\qquad - 2x \sum_{Q_1<k\le Q_2} k \sum_{0<a\le u} f(a,k)S(x;a,k) \\
&\qquad + \sum_{Q_1<k\le Q_2} k \sum_{0<a\le u} S^2(x;a,k) \\
&= x^2 G_1^*(x,u;Q_1,Q_2) - 2xG_2^*(x,u;Q_1,Q_2) + G_3^*(x,u;Q_1,Q_2), \quad \text{say},
\end{aligned}
$$

we then go on in the following Sections to estimate the sums therein according to ascending order of difficulty.

8. The sum $G_1^*(x, u; Q_1, Q_2)$

The sum $G_1^*(x, u; Q_1, Q_2)$ differs from its analogues in earlier papers of the series in that its treatment is neither immediate nor completely obvious, the basis of its estimation being a comparison between the sum

$$
G_1^\dagger(x,u;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} k^2 \sum_{0<a\le u} f^2(a,k)
$$

and the sum

$$
G_1^\$(x;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} k \sum_{0<a\le k} f^2(a,k)
$$

that is a weighted version of a sum appearing implicitly in X (see (9) therein).

First, for any number Q_2' lying between Q_1 and Q_2, (12) implies that

$$
\begin{aligned}
G_1^\dagger(x,u;Q_1,Q_2') &= \sum_{Q_1<k\le Q_2'} \sum_{0<a\le u} \sum_{l_1|k;l_2|k} w(a,l_1)w(a,l_2) \\
&= \sum_{Q_1<k\le Q_2'} \sum_{l_1|k;l_2|k} \sum_{0<a\le u} w(a,l_1)w(a,l_2),
\end{aligned}
$$

the innermost sum in which equals

$$
(48)\qquad \sum_{\substack{0<b\le l_1\\ 0<c\le l_2}} w(b,l_1)w(c,l_2)H(b,c;l_1,l_2;u)
$$

where $H(b,c;l_1,l_2;u)$ is the number of solutions of the simultaneous congruences

$$a \equiv b, \bmod l_1, \quad a \equiv c, \bmod l_2,$$

satisfying the inequality $0 < a \le u$. Hence, since $H(b,c;l_1,l_2,l_2;u)$ equals

$$\frac{u}{[l_1,l_2]} + O(1)$$

or zero according as $b-c \equiv 0$, mod (l_1,l_2), or otherwise,[3]

(49)

$$\begin{aligned}
G_1^{\dagger}(x,u;Q_1,Q_2') &= \sum_{Q_1<k\le Q_2'} \sum_{[l_1,l_2]|k} \sum_{\substack{0<b\le l_1\\ 0<c\le l_2\\ b-c\equiv 0,\ \mathrm{mod}\ (l_1,l_2)}} w(b,l_1)w(c,l_2)\\
&\qquad \times \left(\frac{u}{[l_1,l_2]} + O(1)\right)\\
&= \sum_{[l_1,l_2]\le Q_2'} \sum_{\substack{0<b\le l_1\\ 0<c\le l_2\\ b-c\equiv 0,\ \mathrm{mod}\ (l_1,l_2)}} w(b,l_1)w(c_1l_2)\\
&\qquad \times \left(\frac{u}{[l_1,l_2]} + O(1)\right)\left(\frac{Q_2'-Q_1}{[l_1,l_2]} + O(1)\right)\\
&= u(Q_2'-Q_1)\sum_{[l_1,l_2]\le Q_2'} \frac{1}{[l_1,l_2]^2} \sum_{\substack{0<b\le l_1\\ 0<c\le l_2\\ b-c\equiv 0,\ \mathrm{mod}\ (l_1,l_2)}} w(b,l_1)w(c,l_2)\\
&\qquad + O\left((u+Q_2')\sum_{[l_1,l_2]\le Q_2} \frac{1}{[l_1,l_2]} \sum_{\substack{0<b\le l_1\\ 0<c\le l_2\\ b-c\equiv 0,\ \mathrm{mod}\ (l_1,l_2)}} |w(b,l_1)||w(c,l_2)|\right)\\
&= u(Q_2'-Q_1)\sum\nolimits_A + O\left(Q_2'\sum\nolimits_B\right), \text{ say.}
\end{aligned}$$

The evaluation of $\sum_A$ need not yet be pursued although a treatment is possible by using our method of estimating $\sum_B$, to begin which we set $l_1 = l_1'd, l_2 = l_2'd$, where $(l_1',l_2') = 1$. Then the square of the sum over a, b, and c

[3]In what immediately follows it is helpful to note that the O term depends only on l_1 and l_2 when u is given.

in the consequent equation

$$\sum\nolimits_B = \sum_{\substack{dl_1' l_2' \le Q_2' \\ (l_1', l_2')=1}} \frac{1}{dl_1' l_2'} \quad \sum_{0<a\le d} \sum_{\substack{b\equiv c\equiv a, \bmod d \\ 0<b\le dl_1'; 0<c\le dl_2'}} |w(b, dl_1')||w(c, dl_2')|$$

does not exceed

$$\left(\sum_{0<a\le d} \sum_{\substack{b\equiv a, \bmod d \\ 0<b\le dl_1'}} w^2(b, dl_1') \sum_{\substack{c\equiv a, \bmod d \\ 0<c\le dl_2'}} 1 \right)$$

$$\times \left(\sum_{0<a\le d} \sum_{\substack{c\equiv a, \bmod d \\ 0<c\le dl_2'}} w^2(c, dl_2') \sum_{\substack{b\equiv a, \bmod d \\ 0<b\le dl_1'}} 1 \right)$$

$$= l_1' l_2' \sum_{0<b\le dl_1'} w^2(b, dl_1') \sum_{0<c\le dl_2'} w^2(c, dl_2')$$

by the Cauchy-Schwarz inequality. Therefore

$$\sum\nolimits_B \le \sum_{dl_1' l_2' \le Q_2'} \frac{1}{dl_1'^{\frac{1}{2}} l_2'^{\frac{1}{2}}} \left(\sum_{0<b\le dl_1'} w^2(b, dl_1') \sum_{0<c\le dl_2'} w^2(c, dl_2') \right)^{\frac{1}{2}},$$

whence, by symmetry,

$$\sum\nolimits_B \le 2 \sum_{dl_1' l_2' \le Q_2'} \frac{1}{dl_1'^{\frac{1}{2}} l_2'^{\frac{1}{2}}} \sum_{0<b\le dl_1'} w^2(b, dl_l') \tag{50}$$

$$= 2 \sum_{dl_1' \le Q_2'} \frac{1}{dl_1'^{\frac{1}{2}}} \sum_{0<b\le dl_1'} w^2(b, dl_1') \sum_{l_2' \le Q_2'/dl_1'} \frac{1}{l_2'^{\frac{1}{2}}}$$

$$= O\left(Q_2'^{\frac{1}{2}} \sum_{dl_1' \le Q_2'} \frac{1}{d^{\frac{3}{2}} l_1'} \sum_{0<b\le dl_1'} w^2(b, d_1 l_1') \right)$$

$$= O\left(Q_2'^{\frac{1}{2}} \sum_{l\le Q_2'} \frac{d(l)}{l} \sum_{0<b\le l} w^2(b, l) \right)$$

$$= O\left(Q'^{\frac{1}{2}+\epsilon} \sum_{l\le Q_2'} \frac{1}{l} \sum_{0<b\le l} w^2(b, l) \right) = O\left(Q'^{\frac{1}{2}+\epsilon} \right)$$

in virtue of (16). Combined with (49), this then yields the preliminary estimate

$$u(Q_2' - Q_1) \sum\nolimits_A + O\left(Q_2'^{\frac{3}{2}+\epsilon} \right) \tag{51}$$

for $G_1^\dagger(x,u;Q_1,Q_2')$.

But it is plain that $G_1^\$(x;Q_1,Q_2')$, whose value we already essentially know from X, can be calculated in comparable terms by a slight adjustment to the above argument. Indeed, by suppressing a factor k but substituting k for u as a limit of summation, we find that $G_1^\$(x;Q_1,Q_2')$ is a sum over k,l_1,l_2 of

$$\frac{1}{k}\sum_{\substack{0<b\le l_1\\0<c\le l_2}} w(b,l_1)w(c,l_2)H(b,c;l_1,l_2;k)$$

that is an analogue of (48), wherein now for $b-c\equiv 0$, mod (l_1,l_2) we have

$$H(b,c;l_1,l_1,l_2;k)=\ \frac{k}{[l_1,l_2]}$$

because $l_1|k$ and $l_2|k$. Therefore, following through the reasoning that led to (49) and (50), we easily discover the equation

$$G_1^\$(x;Q_1,Q_2')=(Q_2'-Q_1)\sum\nolimits_A+O\left(\sum\nolimits_B\right)=(Q_2'-Q_1)\sum\nolimits_A+O\left(Q_2^{\frac{1}{2}+\epsilon}\right)$$

that is a counterpart of (51).

Consequently, as we might expect,

$$\begin{aligned}G_1^\dagger(x,u;Q_1,Q_2')&=uG_1^\$(x;Q_1,Q_2')+O\{u+Q_2')Q'^{\frac{1}{2}+\epsilon}\}\\&=uG_1^\$(x;Q_1,Q_2')+O(Q'^{\frac{3}{2}+\epsilon}),\end{aligned}$$

or, in other words, the sum

$$G_1^\ddagger(x,u;Q_1,Q_2')=\sum_{Q_1<k\le Q_2'}k\left\{k\sum_{0<a\le u}f^2(a,k)-u\sum_{0<a\le k}f^2(a,k)\right\}$$

is $O(Q_2'^{\frac{3}{2}+\epsilon})$. Hence, since then

$$\begin{aligned}\sum_{Q_1<k\le Q_2}\left(k\sum_{0<a\le u}f^2(a,k)-u\sum_{0<a\le k}f^2(a,k)\right)&=\int_{Q_1}^{Q_2}\frac{dG_1^\ddagger(x,u;Q_1,v)}{v}\\&=\left[\frac{G_1^\ddagger(x,u;Q_1,v)}{v}\right]_{Q_1}^{Q_2}+\int_{Q_1}^{Q_2}\frac{G_1^\ddagger(x,u;Q_1,v)}{v^2}\\&=O\left(Q_2^{\frac{1}{2}+\epsilon}\right)+O\left(\int_{Q_1}^{Q_2}\frac{dv}{v^{\frac{1}{2}-\epsilon}}\right)=O\left(Q_2^{\frac{1}{2}+\epsilon}\right),\end{aligned}$$

we infer from (47) and (11) that

$$G_1^*(x,u;Q_1,Q_2) = u \sum_{Q_1<k\le Q_2} \sum_{0<a\le k} f^2(a,k) + O\left(Q_2^{\frac{1}{2}+\epsilon}\right)$$
$$= u \sum_{Q_1<k\le Q_2} M(k) + O\left(Q_2^{\frac{1}{2}+\epsilon}\right).$$

The sum over $M(k)$ being determined through Lemma 4, we thus attain our goal by concluding that

$$G_1^*(x,u;Q_1,Q_2) = u\Phi(0)\log\frac{Q_2}{Q_1} + O\left(\frac{u\log Q_1}{Q_1}\right) + O\left(Q_2^{\frac{1}{2}+\epsilon}\right) \tag{52}$$
$$= u\Phi(0)\log\frac{Q_2}{Q_1} + O\left(Q_2^{\frac{1}{2}+\epsilon}\right)$$

in view of (46).

9. Estimation of $G_2^*(x,u;Q_1,Q_2)$

The treatment of $G_2^*(x,u;Q_1,Q_2)$ has something in common with its namesake in XI, although the transition from the previously occurring primes to the general sequence of numbers s inevitably entails the appearance of fresh techniques.

First, akin to the dissection in XI (11), there is the decomposition expressed by

(53)
$$G_2^*(x,u;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} k \sum_{0<a\le u} f(a,k)S(u;a,k)$$
$$+ \sum_{Q_1<k\le Q_2} \sum_{0<a\le u} f(a,k)\{S(x;a,k) - S(u;a,k)\}$$
$$= G_2^{\$}(u;Q_1,Q_2) + G_2^{\dagger}(x,u;Q_1,Q_2), \text{say,}$$

whose first constituent is given by

$$G_2^{\$}(u;Q_1,Q_2) = \sum_{Q_1<k\le Q_2} k \sum_{s\le u} f(s,k) \tag{54}$$

because the variables of summation therein conform to the condition $a \le u \le Q_1 < k$ by (46). This is a weighted variant of the sum

$$\sum_{k\le Q}\sum_{s\le u} f(s,k) \tag{55}$$

that has the semblance of the sum

$$\sum_{k\le Q} \Psi_k(x) = \sum_{k\le Q}\sum_{s\le x} f(s,k)$$

we estimated in X (20) as

$$x\sum_{k\le Q}M(k)+O\left(\frac{x}{\log^A x}\right). \tag{56}$$

But here $u<Q$ so that, using the notation of X for ease of explanation, we must consider how (56) can be extended when the previously imposed condition $Q\le x$ is relaxed. The first change needed being that the first item in X (8) must be augmented by

$$S(x;a,k)=O(1)\quad (k>x)$$

as in (23), we see that for $l>x$ the formula for $W(x,l)$ must also contain the term

$$O\left(\sum_{0<a\le l}|w(a,l)|\right),$$

which creates the extra component

$$\begin{aligned}O\left\{\left(\sum_{m\le Q}\frac{1}{m}\right)\left(\sum_{x<l\le Q}\frac{1}{l}\sum_{0<a\le l}|w(a,l)|\right)\right\}&\\ =O\left(\log Q\sum_{l\le Q}\frac{1}{l}\sum_{0<a\le l}|w(a,l)|\right)&\end{aligned}$$

in X (17) when it is placed in X (14). Since, for any $\xi>1$,

$$\begin{aligned}&\left(\sum_{l\le \xi}\frac{1}{l}\sum_{0<a\le l}|w(a,l)|\right)^2\\ &\qquad\le\left(\sum_{l\le \xi}\frac{1}{l}\sum_{0<a\le l}1\right)\left(\sum_{l\le \xi}\frac{1}{l}\sum_{0<a\le l}|w(a,l)|^2\right)=O(\xi)\end{aligned} \tag{57}$$

by the Cauchy-Schwarz inequality and (16), we must add $O(Q^{\frac{1}{2}}\log Q)$ to the right of (56), whence (55) is seen to be

$$u\sum_{k\le Q}M(k)+O\left(\frac{u}{\log^A x}\right)+O\left(Q^{\frac{1}{2}}\log Q\right)=u\sum_{k\le Q}M(k)+O\left(\frac{x}{\log^A x}\right)$$

after a reversion to the notation of the present paper. Thus, knowing that

$$\sum_{k\le Q}\left(\sum_{s\le u}f(s,k)-uM(k)\right)=O\left(\frac{u}{\log^A x}\right)\quad (Q\le Q_2)$$

and deducing that

$$\sum_{Q_1<k\le Q_2} k\left(\sum_{s\le u} f(s,k) - uM(k)\right) = O\left(\frac{uQ_2}{\log^A x}\right)$$

by partial summation, we infer from (54) and Lemma 4 that

$$\begin{aligned} G_2^{\$}(u;Q_1,Q_2) &= u\sum_{Q_1<k\le Q_2} M(k) + O\left(\frac{uQ_2}{\log^A x}\right) \\ &= u(Q_2-Q_1)\Phi(0) + O(u\log Q_2) + O\left(uQ_2\log^{-A} x\right) \\ &= u(Q_2-Q_1)\Phi(0) + O\left(x^2\log^{-A} x\right). \end{aligned} \tag{58}$$

We draw near to the second constituent $G_2^{\dagger}(x,u;Q_1,Q_2)$ of $G_2^{*}(x,u;Q_1,Q_2)$ by way of the sums

$$J_2^{\dagger}(x,u;Q) = G_2^{\dagger}(x,u;Q,x)$$

for

$$u<Q, \quad x\log^{-A_1} x < Q \le \frac{1}{2}x, \tag{59}$$

between which and $G_2^{\dagger}(x,u;Q_1,Q_2)$ there is the relation

$$G_2^{\dagger}(x,u;Q_1,Q_2) = J_2^{\dagger}(x,u;Q_1) - J_2^{\dagger}(x,u;Q_2) \tag{60}$$

implied by (53). Next, a suitable parameter

$$\xi = \log^{2A+2} x \tag{61}$$

having been introduced, the sum $J_2^{\dagger}(x,u;Q)$ is then itself split into two pieces by expressing (12) as

$$\begin{aligned} kf(a,k) &= \sum_{d|k} w(d) = \sum_{\substack{d|k\\ d>\xi}} w(d) + \sum_{\substack{d|k\\ d\le\xi}} w(d) \\ &= kf_1(a,k) + kf_2(a,k), \text{ say,} \end{aligned} \tag{62}$$

and using the decomposition

$$J_2^{\dagger}(x,u;Q) = J_{2,1,}^{\dagger}(x,u;Q) + J_{2,2}^{\dagger}(x,u;Q), \tag{63}$$

where the sums

$$J_{2,i}^{\dagger}(x,u;Q) = \sum_{Q<k\le x} k \sum_{0<a\le u} f_i(a,k)\{S(x;a,k) - S(u;a,k)\} \tag{64}$$

are ripe for assessment.

By (64) and (62),

$$\begin{aligned}
J_{2,1}^{\dagger}(x,u;Q) &= O\left(x\sum_{k\le x}\frac{1}{k}\sum_{0<a\le x}\sum_{\substack{d\mid k\\ d>\xi}}|w(a,d)|\right)\\
&= O\left(x\sum_{\xi<d\le x}\sum_{0<a\le x}|w(a,d)|\sum_{\substack{k\le x\\ k\equiv 0,\bmod d}}\frac{1}{k}\right)\\
&= O\left(x\log x\sum_{\xi<d\le x}\frac{1}{d}\sum_{0<a\le x}|w(a,d)|\right)\\
&= O\left(x^2\log x\sum_{d>\xi}\frac{1}{d^2}\sum_{0<a\le d}|w(a,d)|\right)
\end{aligned}$$

owing to the periodicity of $w(a,d)$, mod d. The square of the sum in the last line in this does not exceed

$$\left(\sum_{\substack{d>\xi\\ 0<a\le d}}\frac{1}{d^3}\right)\left(\sum_{d>\xi}\frac{1}{d}\sum_{0<a\le d}|w(a,d)|^2\right) = O\left(\sum_{d>\xi}\frac{1}{d^2}\right) = O\left(\frac{1}{\xi}\right)$$

by the Cauchy-Schwarz inequality and (16), whence we arrive at the estimate

$$J_{2,1}^{\dagger}(x,u;Q) = O\left(\frac{x^2\log x}{\xi^{\frac{1}{2}}}\right) = O\left(\frac{x^2}{\log^A x}\right) \tag{65}$$

for the less important part of $J_2^{\dagger}(x,u;Q)$.

If we regard $J_{2,2}^{\dagger}(x,u;Q)$ as a triple iterated sum in which the innermost sum is over values of s for which $u < s \le x$, then the conditions of summation therein are tantamount to the constraints

$$0 < s-a = lk, \quad a+lQ < s \le x, \quad l < x/Q, \quad 0 < a \le u \tag{66}$$

on a, k, s, and a positive integer l with the implication that not only $a \le u$ but also $a \le x - lQ$. Hence, bringing in the definition of $f_1(a,k)$ in (62) so that the first item in (66) implies that $s - a \equiv 0$, mod ld, we infer that

$$J_{2,2}^{\dagger}(x,u;Q) = \sum_{d\le\xi}\sum_{l<x/Q}\sum_{0<a\le v_l} w(a,d)\sum_{\substack{a+lQ<s\le x\\ s\equiv a,\bmod ld}} 1,$$

in which

$$v_l = \min(u, x-lQ) \tag{67}$$

and the innermost sum is

$$(x-a-lQ)f(a,ld)+O\left(x\log^{-A_1-2A-1}x\right)$$

by Criterion V. Since the remainder term here is seen to induce a contribution to $J^{\dagger}_{2,2}(x,u;Q)$ of

$$\begin{aligned}
&O\left(\frac{x^2}{Q\log^{A_1+2A+1}x}\sum_{d\le\xi}\sum_{0<a\le x}|w(a,d)|\right)\\
&\quad=O\left(\frac{x^3}{Q\log^{A_1+2A+1}x}\sum_{d\le\xi}\frac{1}{d}\sum_{0<a\le d}|w(a,d)|\right)\\
&\quad=O\left(\frac{x^3\xi^{\frac12}}{Q\log^{A_1+2A+1}x}\right)=O\left(\frac{x^2\log^{A_1+A+1}x}{\log^{A_1+2A+1}x}\right)=O\left(\frac{x^2}{\log^{A}x}\right)
\end{aligned}$$

by (57), (59), and (61), the estimate

$$\begin{aligned}
J^{\dagger}_{2,2}(x,u;Q)&=\sum_{d\le\xi}\sum_{l<x/Q}\sum_{0<a\le v_l}w(a,d)f(a,ld)(x-a-lQ)\\
&\qquad+O\left(\frac{x^2}{\log^{A}x}\right)\\
&=\sum_{d\le\xi}\sum_{l<x/Q}\sum\nolimits_{l,d}+O\left(\frac{x^2}{\log^{A}x}\right)\\
&=\sum\nolimits_{c}+O\left(x^2\log^{-A}x\right),\ \text{say,}
\end{aligned}\tag{68}$$

then emerges for our consideration.

Next

$$\begin{aligned}
\sum\nolimits_{l,d}&=\sum_{0<b\le d}w(b,d)\sum_{\substack{0<b_1\le ld\\ b_1\equiv b,\ \mathrm{mod}\ d}}f(b_1,ld)\sum_{\substack{0<a\le v_l\\ a\equiv b_1,\ \mathrm{mod}\ dl}}(x-a-lQ)\\
&=\sum_{0<b\le d}w(b,d)\sum_{\substack{0<b_1\le ld\\ b_1\equiv b,\ \mathrm{mod}\ d}}f(b_1,ld)\left(\frac{(x-lQ-\frac12 v_l)v_l}{dl}+O(x)\right)
\end{aligned}$$

$$= \frac{(x - lQ - \frac{1}{2}v_l)v_l}{dl} \sum_{0<b\le d} w(b,d) \sum_{\substack{0<b_1\le ld \\ b_1\equiv b, \bmod d}} f(b_1, ld)$$
$$+ O\left(x \sum_{0<b\le d} |w(b,d)| \sum_{\substack{0<b_1\le ld \\ b_1\equiv b, \bmod d}} f(b_1, ld) \right)$$
$$= \frac{(x - lQ - \frac{1}{2}v_l)v_l}{dl} \sum_{0<b\le d} w(b,d) f(b,d) + O\left(x \sum_{0<b\le d} |w(b,d)| f(b,d) \right)$$
$$= \frac{(x - lQ - \frac{1}{2}v_l)v_l N(d)}{dl} + O\left(\frac{x}{d} \sum_{0<b\le d} |w(b,d)| \right)$$

by (24), (14), and (23). The effect of the remainder term here being to donate

$$O\left(\frac{x^2}{Q} \sum_{d\le\xi} \frac{1}{d} \sum_{0<b\le d} |w(b,d)| \right) = O\left(\frac{x^2 \xi^{\frac{1}{2}}}{Q} \right) = O\left(x \log^{A+A_1+1} x \right) \tag{69}$$
$$= O\left(x^2 \log^{-A} x \right)$$

to $\sum_C$ because of (57) and (46), we must then sum the first term over l to get

$$\frac{N(d)}{d} \left(\frac{1}{2} \sum_{l\le(x-u)/Q} \frac{u(2x - 2lQ - u)}{l} + \frac{1}{2} \sum_{(x-u)/Q<l<x/Q} \frac{(x - lQ)^2}{l} \right)$$
$$= Q^2 \frac{N(d)}{d} \left(\frac{1}{2} \sum_{l<x/Q} \frac{(x/Q - l)^2}{l} - \frac{1}{2} \sum_{l<(x-u)/Q} \frac{((x-u)/Q - l)^2}{l} \right)$$

on availing ourselves of the definition of v_l in (67). This, by Lemma 1, is

$$\frac{Q^2 N(d)}{2d} \left\{ \frac{x^2}{Q^2} \log \frac{x}{Q} - \frac{(x-u)^2}{Q^2} \log \frac{x-u}{Q} \right.$$
$$\left. + \frac{B_1 x^2}{Q^2} - \frac{B_1 (x-u)^2}{Q^2} + \frac{u}{Q} + O\left(\frac{u}{x} \right) \right\}$$
$$= \frac{N(d)}{2d} \left\{ x^2 \log \frac{x}{Q} - (x-u)^2 \log \frac{x-u}{Q} + B_1 x^2 \right.$$
$$\left. - B_1 (x-u)^2 + uQ + O\left(\frac{uQ^2}{x} \right) \right\},$$

and we therefore first deduce from (68) and (69) that

$$J^\dagger_{2,2}(x,u;Q) = \frac{1}{2}\left\{x^2\log\frac{x}{Q} - (x-u)^2\log\frac{x-u}{Q} + B_1x^2 \right.$$
$$\left. -B_1(x-u)^2 + uQ + O\left(\frac{uQ^2}{x}\right)\right\}\sum_{d\le\xi}\frac{N(d)}{d} + O\left(\frac{x^2}{\log^A x}\right)$$

and then from (63) and (65) that

(70)
$$J^\dagger_{2}(x,u;Q) = \frac{1}{2}\left\{x^2\log\frac{x}{Q} - (x-u)^2\log\frac{x-u}{Q} + B_1x^2 \right.$$
$$\left. -B_1(x-u)^2 + uQ + O\left(\frac{uQ^2}{x}\right)\right\}\sum_{d\le\xi}\frac{N(d)}{d} + O\left(\frac{x^2}{\log^A x}\right)$$

because $N(d)$ is non-negative.

Finally, since

$$\sum_{d\le\xi}\frac{N(d)}{d} = \Phi(0) + O\left(\frac{1}{\xi}\sum_{d>\xi}N(d)\right) = \Phi(0) + O\left(\frac{1}{\xi}\right),$$

(60) and (70) yield

$$\begin{aligned} G^\dagger_2(x,u;Q_1,Q_2) &= \frac{1}{2}\left\{\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} - \frac{1}{2}u(Q_2-Q_1)\right. \\ &\qquad \left. +O\left(\frac{uQ_2^2}{x}\right)\right\}\sum_{d\le\xi}\frac{N(d)}{d} + O\left(\frac{x^2}{\log^A x}\right) \\ &= \frac{1}{2}\Phi(0)\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} - \frac{1}{2}u(Q_2-Q_1)\Phi(0) \\ &\qquad + O\left(\frac{uQ_2^2}{x}\right) + O\left(\frac{xu\log x}{\xi}\right) + O\left(\frac{x^2}{\log^A x}\right), \end{aligned}$$

from which we gain the requisite equation

$$\begin{aligned} G^*_2(x,u;Q_1,Q_2) &= \frac{1}{2}\Phi(0)\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} + \frac{1}{2}u(Q_2-Q_1)\Phi(0) \\ &\qquad + O\left(\frac{uQ_2^2}{x}\right) + O\left(\frac{x^2}{\log^A x}\right) \end{aligned} \tag{71}$$

with the aid of (61), (53), and (58).

10. The earlier analysis of $G_3(x,u;Q_1,Q_2)$

We reach some of the harder parts of the analysis now that we meet the sum $G^*_3(x,u;Q_1,Q_2)$ that is contained in (47). To mollify the resistance it offers we first move over to the unweighted sum $G_3(x,u;Q_1,Q_2)$ given by

$$G_3(x,u;Q_1,Q_2) = \sum_{Q_1<k\leq Q_2} \sum_{0<a\leq u} S^2(x;a,k)$$

in accordance with the conventions laid down at the beginning of §7, then letting

$$J_3(x,u;Q) = G_3(x,u;Q,x) \tag{72}$$

when Q is still either Q_1 or Q_2 as in (46) so that

$$G_3(x,u;Q_1,Q_2) = J_3(x,u;Q_1) - J_3(x,u;Q_2) \tag{73}$$

much as in (60). Then the square $S^2(x;a,k)$ in $J_3(x,u;Q)$ is equal to the number of solutions of the conditions

$$s_1 - a = l_2 k, \qquad s_2 - a = l_1 k, \qquad a \leq s_1, s_2 \leq x \tag{74}$$

in numbers s_1, s_2 and non-negative integers l_1, l_2, since the stipulation (59) implies that positive numbers congruent to a, mod k, are not less than a. These solutions fall into six mutually exclusive categories characterized by the features

$$\begin{array}{lll} \text{(i) } s_1 = s_2 = a; & \text{(iv) } s_1 = s_2 > a, & l_1 = l_2 > 0; \\ \text{(ii) } s_1 = a,\ s_2 > a; & \text{(v) } 0 < l_2 < l_1; & \\ \text{(iii) } s_1 > a,\ s_2 = a; & \text{(vi) } 0 < l_1 < l_2; & \end{array} \tag{75}$$

from which it is deduced that

$$J_3(x,u;Q) = J_3^{\S}(x,u;Q) + 2J_3^{\ddagger}(x,u;Q) + J_3^{\$}(x,u;Q) + 2J_3^{\dagger}(x,u;Q) \tag{76}$$

where $J_3^{\S}$, $J_3^{\ddagger}$, $J_3^{\$}$, $J_3^{\dagger}$ are, respectively, the contributions of categories (i), (ii), (iv), (v) to J_3 and are the name-children of the entities appearing in the analogous equation (26) in XI. Of these, the final is not only the most difficult but is actually harder to treat than before, while in partial compensation the penultimate one becomes easier because there is no longer a weighting factor (corresponding to $\log^2 p$) to be taken into account.

The sum $J_3^{\S}(x,u;Q)$ can be eliminated from the work at once, since (75) (i), (74), and (73) imply that

$$J_3^{\S}(x,u;Q_1) - J_3^{\S}(x,u;Q_2) = \sum_{Q_1<k\leq Q_2} \sum_{s\leq u} 1 = O(uQ_2) \tag{77}$$

by Criterion V. But each of the other sums deserves an individual Section before the assessment of the last one is initiated, where it is to be understood that in future the symbol l, with or without subscript, will denote a positive integer.

11. Estimation of $J_3^{\ddagger}(x, u; Q)$

By (75) (ii), (73), and (74), $J_3^{\ddagger}(x, u; Q)$ is the number of solutions in s_1, s_2, and l of the conditions

$$s_1 \le u, \qquad s_2 \equiv s_1, \text{ mod } l, \qquad l < x/Q, \qquad s_1 + lQ < s_2 \le x$$

because in particular the stipulation that $k > Q$ is equivalent to $l < x/Q$ in the present setting. Therefore, having been reminded of the definition of v_l in (67), we have

$$\begin{aligned} J_3^{\ddagger}(x, u; Q) &= \sum_{l < x/Q} \sum_{0 < s_1 \le v_l} \sum_{\substack{s_1 + lQ < s_2 \le x \\ s_2 \equiv s_1, \text{ mod } l}} 1 \\ &= \sum_{l < x/Q} \sum_{0 < s_1 \le v_l} \left\{ (x - lQ - s_1) f(s_1, l) + O\left(\frac{x}{\log^{A+A_1} x} \right) \right\} \\ &= \sum_{l < x/Q} \sum_{0 < s_1 \le v_l} (x - lQ - s_1) f(s_1, l) + O\left(\frac{x^3}{Q \log^{A+A_1} x} \right) \\ &= J_3^{\ddagger\ddagger}(x, u; Q) + O\left(\frac{x^2}{\log^{A} x} \right), \text{ say,} \end{aligned}$$

by Criterion V and (59). But, by reasoning based yet again on Criterion V and on partial summation, the inner sum in $J_3^{\ddagger\ddagger}(x, u; Q)$ is

$$\begin{aligned} \sum_{0 < b \le l} f(b, l) \sum_{\substack{0 < s_1 \le v_l \\ s_1 \equiv b, \text{ mod } l}} (x - lQ - s_1) \\ &= v_l \left(x - lQ - \frac{1}{2} v_l \right) \sum_{0 < b \le l} f^2(b, l) \\ &\qquad + O\left(\frac{x^2}{\log^{A+A_1} x} \sum_{0 < b \le l} f(b, l) \right) \\ &= v_l \left(x - lQ - \frac{1}{2} v_l \right) M(l) + O\left(\frac{x^2}{\log^{A+A_1} x} \right) \end{aligned}$$

on account of (11). Therefore, by (59) again,

$$\begin{aligned}
J_3^{\ddagger}(x,u;Q) &= \sum_{l<x/Q} v_l \left(x - lQ - \frac{1}{2}v_l\right) M(l) + O\left(\frac{x^2}{\log^A x}\right) \\
&= \frac{1}{2} \sum_{l \le (x-u)/Q} u(2x - 2lQ - u)M(l) \\
&\qquad + \frac{1}{2} \sum_{(x-u)/Q < l < x/Q} (x - lQ)^2 M(l) + O\left(\frac{x^2}{\log^A x}\right) \\
&= Q^2 \left(\frac{1}{2} \sum_{l<x/Q} \left(\frac{x}{Q} - l\right)^2 M(l) - \frac{1}{2} \sum_{l<(x-u)/Q} \left(\frac{x-u}{Q} - l\right)^2 M(l) \right) \\
&\qquad + O\left(\frac{x^2}{\log^A x}\right),
\end{aligned}$$

from which is taken the equation

$$\begin{aligned}
J_3^{\ddagger}(x,u;Q) = \frac{1}{2}\Phi(0)\left(x^2 \log \frac{x}{Q} - (x-u)^2 \log \frac{x-u}{Q}\right) + \frac{1}{2}B_2\left\{x^2 - (x-u)^2\right\} \\
+ O(uQ) + O\left(\frac{x^2}{\log^A x}\right)
\end{aligned}$$

by means of Lemma 3.

We thus conclude that

$$\begin{aligned}
J_3^{\ddagger}(x,u;Q_1) - J_3^{\ddagger}(x,u;Q_2) = \frac{1}{2}\Phi(0)\left\{x^2 - (x-u)^2\right\} \log \frac{Q_2}{Q_1} \\
+ O(uQ_2) + O\left(x^2 \log^{-A} x\right).
\end{aligned} \tag{78}$$

12. Estimation of $J_3^{\$}(x,u;Q)$

The assessment of $J_3^{\$}$ may be truncated because it shares several features with that of earlier sums. Being the number of solutions in a, s, and l of the conditions

$$a \le u, \qquad s \equiv a, \bmod l, \qquad l < x/Q, \qquad a + lQ < s \le x$$

by (75) (iv), it equals

$$\begin{aligned}
&\sum_{l<x/Q} \sum_{0<a\le v_l} \sum_{\substack{a+lQ<s\le x \\ s\equiv a, \bmod l}} 1 \\
&\qquad = \sum_{l<x/Q} \sum_{0<a\le v_l} (x - lQ - a) f(a,l) + O\left(\frac{x^2}{\log^A x}\right),
\end{aligned}$$

in the last line of which the inner sum equals

$$\begin{aligned}\sum_{0<b\le l} f(b,l) \sum_{\substack{0<a\le v_l\\ a\equiv b,\ \mathrm{mod}\ l}} (x-lQ-a) &= \frac{1}{2}\sum_{0<b\le l} f(b,l)\left\{\frac{v_l(2x-2lQ-v_l)}{l}+O(x)\right\}\\ &= \frac{1}{2}f(0,1)\frac{v_l(2x-2lQ-v_l)}{l}+O(x)\\ &= \frac{1}{2}C\frac{v_l(2x-2lQ-v_l)}{l}+O(x)\end{aligned}$$

by (2). Therefore, by Lemma 1,

$$\begin{aligned}J_3^{\$}(x,u;Q) &= \frac{1}{2}CQ^2\left(\sum_{l<x/Q}\frac{(x/Q-l)^2}{l}-\sum_{l<(x-u)/Q}\frac{((x-u)/Q-l)^2}{l}\right)\\ &\quad + O\left(x\log^{A_1}x\right)+O\left(\frac{x^2}{\log^A x}\right)\\ &= \frac{1}{2}C\left(x^2\log\frac{x}{Q}-(x-u)^2\log\frac{x}{Q}\right)+\frac{1}{2}CB_1\left\{x^2-(x-u)^2\right\}\\ &\quad + O(uQ)+O\left(x^2\log^{-A}x\right),\end{aligned}$$

whence

$$J_3^{\$}(x,u;Q_1)-J_3^{\$}(x,u;Q_2)=\frac{1}{2}C\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} + O(uQ_2)+O\left(x^2\log^{-A}x\right). \tag{79}$$

13. Estimation of $J_3^{\dagger}(x,u;Q)$—the preliminary stages and the initial application of the circle method

The earlier part of the treatment of $J_3^{\dagger}(x,u;Q)$ may be described succinctly because it follows that of its counterpart in §8 in XI, although a new and seemingly thorny trail must be blazed as soon as we arrive at the singular series in the application of the circle method.

First, if in the conditions of summation defining $J_3^{\dagger}(x,u;Q)$ we set $l_1=l_1'\delta$, $l_2=l_2'\delta$, where

$$(l_1',l_2')=1, \tag{80}$$

then the first two items in (74) are tantamount to the pair

$$s_1\equiv s_2\equiv a,\ \mathrm{mod}\ \delta, \tag{81}$$

and

$$l_1' \{(s_1 - a)/\delta\} = l_2' \{(s_2 - a)/\delta\},$$

the last of which may be restated as

$$l_1' s_1 - l_2' s_2 - l_3' a = 0, \qquad (l_3' = l_2' - l_1'). \tag{82}$$

Secondly, the conditions related to k translate into the implication

$$l_2 < l_1 < x/Q \tag{83}$$

and the two sets of requirements

$$a + l_1 Q < s_2 \leq x, \qquad a + l_2 Q < s_1 \leq x,$$

the second one of which is implied by the first when (83) (or (82)) is in place. We can therefore complete the first phase of the calculation by deducing that

$$J_3^{\dagger}(x, u; Q) = \sum_{\delta < x/Q} \sum_{\substack{l_2' < l_1' < x/(Q\delta) \\ (l_1', l_2') = 1}} P(x, u, Q\delta l_1'; l_1', l_2'; \delta), \tag{84}$$

the inner summand in which is defined by letting $\Theta = \Theta_{\delta, l_1', l_2'}$ indicate the conjunction of (81) and (82) and by then setting

$$P(x, u; T, l_1', l_2'; \delta) = \sum_{\substack{\Theta \\ 0 < a \leq u; a + T < s_2 \leq x}} 1 \quad (x \log^{-A_1} x < T \leq x) \tag{85}$$

as a sum over the variables a, s_1, s_2.

Much as in IX, the next stage is to enlist the circle method to estimate $P(x, u, T; l_1', l_2'; \delta)$, although this still cannot be easily applied directly because of the connection between the domains of summations over each of s_1, s_2, a. We therefore, for any values of t_1, t_2 such that $0 < t_1 < t_2 \leq x$, introduce the associated sums

$$P_1(x, t_1, t_2; l_1', l_2'; \delta) = \sum_{\substack{\Theta \\ 0 < a \leq t_1; t_2 < s_2 \leq x}} 1$$

and consider the sums $P_2(x, t_1, t_2; l_1', l_2'; b, \delta)$ that appear in the dissection

$$\begin{aligned} P_1(x, t_1, t_2; l_1', l_2'; \delta) &= \sum_{0 < b \leq d} \sum_{s_1 \equiv s_2 \equiv a \equiv b, \bmod \delta} \\ &= \sum_{0 < b \leq d} P_2(x, t_1, t_2; l_1', l_2'; b, \delta) \end{aligned} \tag{86}$$

made possible by (81). Then, the exponential sums

$$
\begin{aligned}
g_1(\theta) = g_{1,b}(\theta) = g_{1,b,\delta}(\theta) &= \sum_{\substack{s_1 \le x \\ s_1 \equiv b, \bmod \delta}} e^{2\pi i l_1' s_1 \theta}, \\
g_2(\theta) = g_{2,b}(\theta) = g_{2,b,\delta}(\theta) &= \sum_{\substack{t_2 < s_2 \le x \\ s_2 \equiv b, \bmod \delta}} e^{-2\pi i i l_2' s_2 \theta}, \\
g_3(\theta) = g_{3,b}(\theta) = g_{3,b,\delta}(\theta) &= \sum_{\substack{0 < a \le t_1 \\ a \equiv b, \bmod \delta}} e^{-2\pi i l_3' a \theta}
\end{aligned} \tag{87}
$$

having been brought into play, the treatment of $P_2(x, t_1, t_2; l_1', l_2'; b, \delta)$ commences by expressing it as

$$\int_0^1 g_1(\theta) g_2(\theta) g_3(\theta) d\theta \tag{88}$$

in the usual way.

The division of the region of integration into Farey arcs and the consequential definition of the minor arcs is best described by reproducing almost verbatim the account given of the corresponding matters in XI, §8. Assuming throughout that

$$l_2' < l_1' < (\log^{A_1} x)/\delta \le \log^{A_1} x \qquad \text{and} \qquad \delta \le \log^{A_1} x \tag{89}$$

in conformity with (59), we choose a *sufficiently large absolute constant* A_2 and use a Farey dissection of order $M = x \log^{-A_2} x$ that has the property that each θ in the range of integration belongs to one and only one arc, mod 1 (apart from the end points) of the form

$$|\theta - h/k| \le \vartheta_{h,k}/(Mk), \tag{90}$$

where $k \le M$, $0 < h \le k$, $(h, k) = 1$, and $1/2 \le \vartheta_{h,k} \le 1$. Next, by (87), $g_3(\theta) = O(1/\|\delta l_3' \theta\|)$ so that

$$|g_3(\theta)| > A_3 x \log^{-A_2} x \tag{91}$$

only when $\delta l_3' \theta = m + \psi$ for some (non-negative) integer m and for $|\psi| < \frac{1}{2} x^{-1} \log^{A_2} x$, namely, only when

$$\theta = \frac{m}{\delta l_3'} + \phi \qquad \text{and} \qquad |\phi| < \frac{\log^{A_2} x}{2x\delta l_3'} = \frac{1}{2M\delta l_3'}$$

and hence certainly only when θ lies within an arc (90) for which $k|\delta l_3'$. All such arcs are then dilated to form the set $\mathfrak{M}$ of major arcs

$$|\theta - h/k| < 1/M, \qquad k|\delta l_3', \tag{92}$$

which are non-intersecting because here $k \le \log^{A_1} x$ by (89); the complement of $\mathfrak{M}$ in the range of integration then forms the set $\mathfrak{m}$, on which (91) is false. The effect of $\mathfrak{m}$ on the integral (88) is then determined to be

$$\begin{aligned}\int_{\mathfrak{m}} g_1(\theta)g_2(\theta)g_3(\theta)d\theta &= O\left\{\frac{x}{\log^{A_2} x}\left(\int_0^1 |g_1(\theta)|^2 d\theta\right)^{\frac{1}{2}}\left(\int_0^1 |g_2(\theta)|^2\right)^{\frac{1}{2}}\right\}\\ &= O\left(\frac{x}{\log^{A_2} x}\sum_{s\le x} 1\right) = O\left(\frac{x^2}{\log^{A_2} x}\right),\end{aligned}$$

the cumulative contribution of $\mathfrak{m}$ to $P_1(x, t_1, t_2; l_1', l_2', \delta)$ via (86) being

$$O\left(\frac{x^2\delta}{\log^{A_2} x}\right) = O\left(\frac{x^2}{\log^{A_2-A_1} x}\right) = O\left(\frac{x^2}{\log^{A_4} x}\right) \tag{93}$$

for $A_4 = A_2 - A_1$.

Still following for a short time longer the flow in XI, we almost treat matters on the major arcs as we would for a binary form and disengage the function $g_3(\theta)$ from the integral by performing the summation over a outside the integral sign. Accordingly the integral of $g_1(\theta)g_2(\theta)g_3(\theta)$ over $\mathfrak{M}$ is expressed as

$$\sum_{\substack{0<a\le t_1\\ a\equiv b,\ \mathrm{mod}\ \delta}} \int_{\mathfrak{M}} g_{1,b}(\theta)g_{2,b}(\theta)e^{-2\pi i l_3' a\theta}d\theta,$$

to which the contribution of the arc centred on h/k is

$$\begin{aligned}&\sum_{\substack{0<a\le t_1\\ a\equiv b,\ \mathrm{mod}\ \delta}} e^{-2\pi i h a l_3'/k}\int_{-1/M}^{1/M} g_{1,a}(h/k+\phi)g_{2,a}(h/k+\phi)e^{-2\pi i l_3' a\phi}d\phi \qquad (94)\\ &\qquad = \sum_{\substack{0<a\le t_1\\ a\equiv b,\ \mathrm{mod}\ \delta}} e^{-2\pi i h a l_3'/k} I^{(1)}_{l_3', a}, \text{ say.}\end{aligned}$$

But now we must begin to part company with the previous treatment for two reasons. The first is that here it is convenient to deduce from (86) that the total impact of the above arc on $P_1(x, t_1, t_2; l_1', l_2'; \delta)$ is

$$\sum_{0<a\le t_1} e^{-2\pi i h a l_3'/k} I^{(1)}_{l_3', a}; \tag{95}$$

the second is that, since we no longer have an earlier authority for formulae for the exponential sums in the integrand of $I^{(1)}_{l_3', a}$, we are obliged to continue by investigating sums of the type

$$g(h/k+\phi, y) = g_a(h/k+\phi, y) = \sum_{\substack{s\le y\\ s\equiv a,\ \mathrm{mod}\ \delta}} e^{2\pi i s \mathrm{l}(h/k+\phi)}$$

for $0 \le y \le x$, $(h,k)=1$, and an integer l of magnitude less than $\log^{A_1} x$.

Beginning with the case where $\phi = 0$, we have

$$\begin{aligned} g(h/k, y) &= \sum_{0<c\le k} e^{2\pi i c \mathrm{l} h/k} \sum_{\substack{s\le y \\ s\equiv a, \bmod \delta \\ s\equiv c, \bmod k}} 1 \\ &= \sum_{\substack{0<c\le k \\ (\delta,k)|(a-c)}} e^{2\pi i c \mathrm{l} h/k} \sum_{\substack{s\le y \\ s\equiv c^*, \bmod [\delta,k]}} 1, \end{aligned}$$

where c^* is the unique root, $\bmod [\delta,k]$, of the congruences

$$r \equiv a, \bmod \delta, \quad r \equiv c, \bmod k, \tag{96}$$

when these are compatible. Therefore, by Criterion V and (96) and then by (92),

(97)

$$\begin{aligned} g(h/k, y) &= \sum_{\substack{0<c\le k \\ c\equiv a, \bmod (\delta,k)}} e^{2\pi i c \mathrm{l} h/k} \left\{ y f(c^*, [\delta,k]) + O\left(\frac{x}{\log^{2A_1+A_2+A_6} x}\right)\right\} \\ &= y \sum_{\substack{0<c\le k \\ c\equiv a, \bmod (\delta,k)}} f(c^*, [\delta,k]) e^{2\pi i c \mathrm{l} h/k} + O\left(\frac{kx}{\log^{2A_1+A_2+A_6} x}\right) \\ &= y \Upsilon_a(h/k, \mathrm{l}) + O\left(\frac{x}{\log^{A_1+A_2+A_6} x}\right), \text{ say,} \end{aligned}$$

in which evidently

$$\Upsilon_a(h/k, \mathrm{l}) = O\left(\sum_{\substack{0<c\le k \\ c\equiv a, \bmod (\delta,k)}} \frac{1}{[\delta,k]} \right) = O\left(\frac{k}{(\delta,k)[\delta,k]}\right) = O(1). \tag{98}$$

From this and an application of partial summation, we then deduce that

(99)

$$\begin{aligned} g(h/k+\phi, y) &= \int_0^y e^{2\pi i \mathfrak{l}\phi z} d\left\{g(h/k,z)\right\} \\ &= \Upsilon_a(h/k,\mathfrak{l})\int_0^y e^{2\pi i\mathfrak{l}\phi z}dz + \int_0^y e^{2\pi i\mathfrak{l}\phi z} d\left\{O\left(\frac{x}{\log^{A_1+A_2+A_6} x}\right)\right\} \\ &= \Upsilon_a(h/k,\mathfrak{l})\int_0^y e^{2\pi i\mathfrak{l}\phi z}dz + O\left(\frac{x}{\log^{A_1+A_2+A_6} x}\right) \\ &\qquad + O\left(\frac{xy|\mathfrak{l}||\phi|}{\log^{A_1+A_2+A_6} x}\right) \\ &= \Upsilon_a(h/k,\mathfrak{l})\int_0^y e^{2\pi i\mathfrak{l}\phi z}dz + O\left(\frac{x}{\log^{A_6} x}\right) \end{aligned}$$

whenever $|\phi| \le 1/M = (\log^{A_2} x)/x$ as in the integral in (94), to which we now return.

Let us now set

$$\upsilon_1(\phi) = \int_0^x e^{2\pi i l_1'\phi z_1} dz_1, \qquad \upsilon_2(\phi) = \int_{t_2}^x e^{-2\pi i l_2'\phi z_2} dz_2,$$

where evidently $\upsilon_1(\phi)$, $\upsilon_2(\phi)$ are both $O(x)$ and $O(1/|\phi|)$. Then, by (94), (99) and (98),

$$\begin{aligned} (100) \qquad I^{(1)}_{l_3',a} &= \Upsilon_a(h/k,l_1')\Upsilon_a(h/k,-l_2')\int_{-1/M}^{1/M} \upsilon_1(\phi)\upsilon_2(\phi)e^{-2\pi i a l_3'\phi}d\phi \\ &\qquad + O\left(\frac{x^2}{M\log^{A_6} x}\right) \\ &= \Upsilon_a(h/k,l_1')\Upsilon_a(h/k,-l_2')\int_{-\infty}^{\infty} \upsilon_1(\phi)\upsilon_2(\phi)e^{-2\pi i a l_3'\phi}d\phi \\ &\qquad + O(M) + O\left(\frac{x^2}{M\log^{A_6} x}\right) \\ &= \Upsilon_a(h/k,l_1')\Upsilon_a(h/k,-l_2')\int_{-\infty}^{\infty} \upsilon_1(\phi)\upsilon_2(\phi)e^{-2\pi i a l_3'\phi}d\phi \\ &\qquad + O\left(\frac{x}{\log^{A_4} x}\right) \\ &= \Upsilon_a(h/k,l_1')\Upsilon_a(h/k,-l_2') I^{(2)}_{l_1',l_2',a} + O(x\log^{-A_4} x), \text{ say,} \end{aligned}$$

so long as A_6 in (97) be chosen large enough (in fact it suffices to take $A_6 = 2A_2 - A_1$). Next, on being interpreted as a double integral with variables of integration z_1, z_2, the product $\upsilon_1(\phi)\psi_2(\phi)$ in the integrand of $I^{(2)}_{l_1',l_2',a}$ is transformed by the substitution

$$Z_1 = l'_1 z_1 - l'_2 z_2, \qquad Z_2 = z_2$$

of absolute modulus l'_1 so that it becomes a Fourier transform of the type

$$\frac{1}{l'_1}\int_{-\infty}^{\infty} \Phi(Z_1) e^{2\pi i Z_1 \phi} dZ_1.$$

Consequently, since the limits t_2 and x for z_2 imply that $0 \le z_1 \le x$ when both $l'_1 z_1 - l'_2 z_2 - l'_3 a = l'_1(z_1 - a) - l'_2(z_2 - a)$ and $a \le t_1 \le t_2$, we see that

$$I^{(2)}_{l'_1, l'_2, a} = \frac{1}{l'_1}\Phi(al'_3) = \frac{x - t_2}{l'_1}$$

after a short examination, whence

$$I^{(1)}_{l'_3, a} = \Upsilon_a(h/k, l'_1)\Upsilon_a(h/k, l'_2)\frac{x - t_2}{l'_1} + O\left(\frac{x}{\log^{A_4} x}\right) \tag{101}$$

by (100).

The effect of the remainder term above on the proceedings is easily settled. Indeed, on a single arc (92), its influence is

$$O\left(\frac{t_1 x}{\log^{A_4} x}\right) = O\left(\frac{x^2}{\log^{A_4} x}\right)$$

by (95), wherefore its total contribution to $P_1(x, t_1, t_2; l'_1, l'_2; \delta)$ via the major arcs is inferred to be

$$\begin{aligned} O\left(\frac{x^2}{\log^{A_4} x}\sum_{k \mid \delta l'_3} \phi(k)\right) &= O\left(\frac{x^2 \delta l'_3}{\log^{A_4} x}\right) \\ &= O\left(\frac{x^2}{\log^{A_4 - A_1} x}\right) = O\left(\frac{x^2}{\log^{A_7} x}\right) \end{aligned} \tag{102}$$

by (92) and a summation over the appropriate values of h and k.

As for the explicit term in (101), its placement in (95) followed by a summation over all values of h and k answering to the major arcs produces the element

$$\begin{aligned} &P_3(x, t_1, t_2; l'_1, l'_2; \delta) \\ &\quad = \frac{x - t_2}{l'_1}\sum_{k \mid \delta l'_3}\sum_{0 < a \le t_1}\sum_{\substack{0 < h \le k \\ (h,k)=1}} \Upsilon_a(h/k, l'_1)\Upsilon_a(h/k, -l_2) e^{-2\pi i h a l'_3/k} \\ &\quad = \frac{x - t_2}{l'_1}\sum_{k \mid \delta l'_3}\sum_{0 < a \le t_1} Y(a, k) \\ &\quad = \frac{x - t_2}{l'_1} P_4(x, t_1; l'_1, l_2; \delta), \text{ say,} \end{aligned} \tag{103}$$

in $P_1(x, t_1, t_2; l_1', l_2'; \delta)$. The sum $P_4(x, t_1; l_1', l_2'; \delta)$ in this is essentially a composition of singular series but, not being amenable to the methods of XI owing to the generality of the function $f(a, l)$, warrants a Section to itself for its treatment.

14. Estimations of $P_4(x, t_1; l_1', l_2'; \delta)$

First, by (103) and (97),

(104)
$$\begin{aligned} Y(a,k) &= \sum_{\substack{0<h\leq k \\ (h,k)=1}} \sum_{\substack{0<c_1,c_2\leq k \\ c_1\equiv c_2\equiv a, \bmod (\delta,k)}} e^{2\pi i(l_1'c_1 - l_2'c_2 - l_3'a)h/k} f(c_1^*, [\delta,k]) f(c_2^*, [\delta,k]) \\ &= \sum_{\substack{0<c_1,c_2\leq k \\ c_1\equiv c_2\equiv a, \bmod (\delta,k)}} f(c_1^*, [\delta,k]) f(c_2^*, [\delta,k]) \sum_{\substack{0<h\leq k \\ (h,k)=1}} e^{2\pi i(l_1'c - l_2'c_2 - l_3'a)h/k}, \end{aligned}$$

in which the inner sum is the Ramanujan sum

$$\mathrm{c}_k(l_1'c_1 - l_2'c_2 - l_3'a) = \sum_{\substack{d|k \\ d|(l_1'c_1 - l_2'c_2 - l_3'a)}} \mu\left(\frac{k}{d}\right) d.$$

Hence, by transforming the variables of outer summation in (104) into numbers c_1^*, c_2^* implicitly defined by (96), we deduce that

$$\begin{aligned} Y(a,k) &= \sum_{d|k} \mu\left(\frac{k}{d}\right) d \sum_{\substack{0<c_1^*,c_2^*\leq[\delta,k] \\ c_1^*\equiv c_2^*\equiv a, \bmod \delta \\ l_1'c_1^* - l_2'c_2^* - l_3'a\equiv 0, \bmod d}} f(c_1^*, [\delta,k]) f(c_2^*, [\delta,k]) \\ &= \sum_{d|k} \mu\left(\frac{k}{d}\right) d \sum\nolimits^{(1)}_{a,k,d,\delta}, \text{ say,} \end{aligned}$$

and then gain through (103) the equation

(105)
$$\begin{aligned} P_4(x, t_1; l_1', l_2'; \delta) &= \sum_{k|\delta l_3'} \sum_{d|k} \mu\left(\frac{k}{d}\right) d \sum_{0<a\leq t_1} \sum\nolimits^{(1)}_{a,k,d,\delta} \\ &= \sum_{k|\delta l_3'} \sum_{d|k} \mu\left(\frac{k}{d}\right) d \sum\nolimits^{(2)}_{t_1,k,d,\delta}, \text{ say,} \end{aligned}$$

whose further development depends on a close examination of the conditions of summation in $\sum^{(1)}_{a,k,d,\delta}$.

Setting

$$c_1^* = a + e_1, \qquad c_2^* = a + e_2,$$

for a given value of a in the above conditions of summation, we require that

$$e_1 \equiv 0, \bmod \delta, \qquad e_2 \equiv 0, \bmod \delta, \tag{106}$$

and

$$l_1' e_1 \equiv l_2' e_2, \bmod d, \tag{107}$$

because from $l_3' = l_1' - l_2'$ in (82) we infer that $l_1' c_1^* - l_2' c_2^* - l_3' a = l_1'(c_1^* - a) - l_2'(c_2^* - a)$. To solve (107) let $(l_1', d) = d_1$ and $(l_2', d) = d_2$ with the initial consequence that we may write

$$l_1' = d_1 l_1'', \qquad l_2' = d_2 l_2'', \tag{108}$$

where

$$d_1 d_2 | d, \qquad (l_1', d_2) = (l_2', d_1) = (l_1'', d/d_1 d_2) = (l_2'', d/d_1 d_2) = 1 \tag{109}$$

by (80). Then, since $d_1 | l_2' e_2$ and $d_2 | l_1' e_1$ in (107), we may also say that $e_1 = d_2 e_1'$, $e_2 = d_1 e_2''$ so that $l_1' e_1' \equiv l_2'' e_2'$, mod $d/d_1 d_2$, the incongruent solutions of which, mod $d/d_1 d_2$, are provided by

$$e_1' \equiv l_2'' \rho, \bmod d/d_1 d_2, \quad e_2' \equiv l_1'' \rho, \bmod d/d_1 d_2$$

for $0 \le \rho < d/d_1 d_2$. The corresponding solutions of (107) being

$$e_1' \equiv l_2' \rho, \bmod d/d_1 d_2, \quad e_2' \equiv l_1' \rho, \bmod d/d_1 d_2, \tag{110}$$

by (105), we then need to select those values of ρ for which these determinations are compatible with (106) and thus arrive at the simultaneous conditions

$$\delta_1 | l_2' \rho, \qquad \delta_2 | l_1' \rho$$

on putting

$$\delta_1 = (\delta, d/d_1), \qquad \delta_2 = (\delta, d/d_2). \tag{111}$$

Since $d_1 | l_1'$ and $d_2 | l_2'$, these conditions imply and are implied by

$$\delta_3 | \rho$$

where

$$\delta_3 = (\delta, d/d_1 d_2), \tag{112}$$

and we may thus restrict attention to the solutions of (110) that are of the form

$$e_1 \equiv l_2' \delta_3 \sigma, \bmod d/d_1, \qquad e_2 \equiv l_1' \delta_3 \sigma, \bmod d/d_2, \tag{113}$$

for $0 \le \sigma < d/d_1 d_2 \delta_3$.

Our path from this initial result will be eased if we record at once some simple relations between the entities δ, d, d_1, d_2, and those defined in (111) and (112). The first is that $d/d_1 d_2 \delta_3$ is a divisor of both $d/d_1 \delta_1$ and $d/d_1 \delta_2$, the second that δ_3 is a divisor of both δ_1 and δ_2, while the third is the slightly less obvious

$$\delta_1\delta_2 = (\delta, d)\delta_3,$$

which implies that

$$[\delta, d][\delta, d/d_1d_2] = [\delta, d/d_1][\delta, d/d_2]. \tag{114}$$

Also $[\delta, d]/[\delta, d/d_1]$ and $[\delta, d]/[\delta, d/d_2]$ are co-prime integers.

Let us write $\delta_1^\dagger = \delta/\delta_1$, $\delta_2^\dagger = \delta/\delta_2$ to assist us in the calculation of the simultaneous solutions of (106) and (107). Then the first part of (106) demands that

$$e_1 = \delta e_1^\dagger = \delta_1\delta_1^\dagger e_1^\dagger$$

for some integer $e_1^\dagger$ that must then satisfy the congruence

$$\delta_1^\dagger e_1^\dagger \equiv (l_2'\delta_3/\delta_1)\sigma, \bmod d/d_1\delta,$$

stemming from (113), where the co-primality of $\delta_1^\dagger$ and $d/d_1\delta_1$ deduced from (111) means that there is a number $\overline{\delta}_1^\dagger$ such that $\delta_1^\dagger\overline{\delta}_1^\dagger \equiv 1$, mod $d/d_1\delta_1$. Hence

$$e_1^\dagger \equiv (l_2'\delta_3/\delta_1)\overline{\delta}_1^\dagger\sigma, \bmod d/d_1\delta_1,$$

and thus

$$e_1 \equiv (l_2'\delta_3/\delta_1)\overline{\delta}_1^\dagger\delta\sigma \equiv \sigma V_1, \bmod [\delta, d/d_1], \text{ say,} \tag{115}$$

the corresponding solution in e_2 being

$$e_2 \equiv (l_1'\delta_3/\delta_2)\overline{\delta}_2^\dagger\delta\sigma \equiv \sigma V_2, \bmod [\delta, d/d_2], \text{ say,} \tag{116}$$

where $\delta_2^\dagger\overline{\delta}_2^\dagger \equiv 1$, mod $d/d_2\delta_2$. But for these formulae to participate profitably in the expression to be unfolded for $\sum_{a,k,d,\delta}$ it is needful to compare V_1 and V_2 to the modulus $[\delta, d/d_1d_2]$, to which end we write $\delta_3^\dagger = \delta/\delta_3$ and define $\overline{\delta}_3^\dagger$, mod $d/d_1d_2\delta_3$, by $\delta_3^\dagger\overline{\delta}_3^\dagger \equiv 1$, mod $d/d_1d_2\delta_3$. Then, interpreting $\overline{\delta}_1^\dagger$ as the multiplicative inverse of $\delta_1^\dagger$, mod $d/d_1d_2\delta_3$, we move from the equality $\delta_3^\dagger = (\delta_1/\delta_3)\delta_1^\dagger$ to the congruence

$$\overline{\delta}_1^\dagger \equiv (\delta_1/\delta_3)\overline{\delta}_3^\dagger, \bmod d/d_1d_2d_3,$$

so that

$$V_1 \equiv l_2'\delta\overline{\delta}_3^\dagger, \bmod [\delta, d/d_1d_2],$$

and also

$$V_2 \equiv l_1'\delta\overline{\delta}_3^\dagger, \bmod [\delta, d/d_1d_2]$$

in like manner. If now we appeal to the conditions $k|l_3'\delta$ and $d|k$ in the summations in (105), we have $[\delta,k]|l_3'\delta$ and conclude that

$$V_2 - V_1 \equiv l_3'\delta\bar{\bar{\delta}}_3^{\dagger} \equiv 0, \bmod [\delta, d/d_1d_2], \tag{117}$$

by using (82).

Having elicited the simultaneous solutions of (106) and (107) as (115) and (116) and then taking their incongruent determinations, mod $[\delta,k]$, we revert to (105) to discover that

$$\begin{aligned}\sum\nolimits^{(1)}_{a,k,d,\delta} = \sum_{0\le\sigma<d/d_1d_2\delta_3}\ \sum_{0\le r<[\delta,k]/[\delta,d/d_1]} &\sum_{0\le s<[\delta,k]/[\delta,d/d_2]} f(a+\sigma V_1 - r[\delta,d/d_1],[\delta,k]) \\ &\times f(a+\sigma V_2 + s[\delta,d/d_2],[\delta,k])\end{aligned}$$

and then get

$$\sum\nolimits^{(2)}_{t_1,k,d,\delta} = \left(\frac{t_1}{[\delta,k]} + O(1)\right)\sum\nolimits^{(2)}_{k,k,d,\delta}, \tag{118}$$

whither we come by the periodicity of $f(b,[\delta,k])$, mod $[\delta,k]$, as a function of b. Next because of the presence of the residue a in the argument in the first factor in the summand of $\sum^{(2)}_{k,k,d,\delta}$ quâ a quadruple sum over a, r, s, σ, this argument can attain any value j, mod $[\delta,k]$, whence

$$\begin{aligned}\sum\nolimits^{(2)}_{k,k,d,\delta} = \sum_{0<j\le[\delta,k]} f(j,[\delta,k]) \sum_{0\le\sigma<d/d_1d_2\delta_3} &\sum_{0\le r<[\delta,k]/[\delta,d/d_1]}\ \sum_{0\le s<[\delta,k]/[\delta,d/d_1]} f(j+\sigma(V_2-V_1) \\ &+ r[\delta,d/d_1] + s[\delta,d/d_2],[\delta,k]).\end{aligned} \tag{119}$$

Moreover, by a double application of principle (24) enunciated in §3, the innermost sum equals

$$\begin{aligned}&f(j+\sigma(V_2-V_1)+r[\delta,d/d_1],[\delta,d/d_2]) \\ &\quad= \sum_{0\le s<[\delta,d]/[\delta,d/d_2]} f(j+\sigma(V_2-V_1)+r[\delta,d/d_1]+s[\delta,d/d_2],[\delta,d]),\end{aligned}$$

its sum over the given range of r being

(120)

$$\sum_{\substack{0\le r<[\delta,k]/[\delta,d/d_1]\\ 0\le s<[\delta,d]/[\delta,d/d_2]}} f(j+\sigma(V_2-V_1)+r[\delta,d/d_1]+s[\delta,d/d_2],[\delta,d])$$
$$=\frac{[\delta,k]}{[\delta,d]}\sum_{\substack{0\le r<[\delta,d]/[\delta,d/d_1]\\ 0\le s<[\delta,d]/[\delta,d/d_2]}} f(j+\sigma(V_2-V_1)+r[\delta,d/d_1]+s[\delta,d/d_2],[\delta,d]).$$

In this, by (114) and neighbouring comments,

$$\frac{r[\delta,d/d_1]+s[\delta,d/d_2]}{[\delta,d/d_1d_2]}=\frac{[\delta,d/d_1][\delta,d/d_2]}{[\delta,d][\delta,d/d_1d_2]}\left(r\frac{[\delta,d]}{[\delta,d/d_2]}+s\frac{[\delta,d]}{[\delta,d/d_1]}\right)$$
$$=\left(r\frac{[\delta,d]}{[\delta,d/d_2]}+s\frac{[\delta,d]}{[\delta,d/d_1]}\right)$$

runs through all residues, to the modulus

$$[\delta,d]^2/[\delta,d/d_2][\delta,d/d_1]=[\delta,d]/[\delta,d/d_1d_2],$$

whereupon we infer that (120) equals

$$\frac{[\delta,k]}{[\delta,d]}\sum_{0\le q<[\delta,d]/[\delta,d/d_1d_2]} f(j+\sigma(V_2-V_1)+q[\delta,d/d_1d_2],[\delta,d])$$
$$=\frac{[\delta,k]}{[\delta,d]}f(j+\sigma(V_2-V_1),[\delta,d/d_1d_2])=\frac{[\delta,k]}{[\delta,d]}f(j,[\delta,d/d_1d_2])$$

in virtue of (117). Hence, completing the two outer summations over σ and j in (119), we conclude that

$$\sum\nolimits^{(2)}_{k,k,d,\delta}=\frac{[\delta,k]d}{[\delta,d]d_1d_2\delta_3}\sum_{0<j\le[\delta,k]} f(j,[\delta,d/d_1d_2])f(j,[\delta,k])$$
$$=\frac{[\delta,k]d}{[\delta,d]d_1d_2\delta_3}\sum_{0<j_1\le[\delta,d/d_1d_2]} f(j,[\delta,d/d_1d_2])$$
$$\sum_{\substack{0<j\le[\delta,k]\\ j\equiv j_1,\bmod [\delta,d/d_1d_2]}} f(j,[\delta,k])$$
$$=\frac{[\delta,k]d}{[\delta,d]d_1d_2\delta_3}\sum_{0<j_1\le[\delta,d/d_1d_2]} f^2(j_1,[\delta,d/d_1d_2])$$
$$=\frac{[\delta,k]d}{[\delta,d]d_1d_2\delta_3}M([\delta,d/d_1d_2])$$

in the notation of (11).

Everything is in place for the construction of the estimate we need for $P_4(x,t_1,t_2;l_1',l_2';\delta)$. By (105), (118), and the last equation above,

$$P_4(x,t_1,t_2;l_1',l_2';\delta) = t_1 \sum_{k|\delta l_3'} \sum_{d|k} \mu\left(\frac{k}{d}\right) \frac{d^2}{[\delta,d]d_1d_2\delta_3} M([\delta,d/d_1d_2]) + O\left(\sum_{k|\delta l_3'} [\delta,k] \sum_{d|k} \frac{d^2}{[\delta,d]d_1d_2\delta_3} M([\delta,d/d_1d_2] \right),$$

the remainder term in which we infer to be

$$O\left(\delta \sum_{k|\delta l_3'} k \sum_{d|k} 1\right) = O\left\{\delta(\delta l')^{1+\epsilon}\right\} = O\left(\log^{3A_1} x\right)$$

from relations (11) and (23). On the other hand, the explicit term is

$$\begin{aligned} t_1 \sum_{drs=\delta l_3'} & \frac{d^2}{[\delta,d]d_1d_2\delta_3} M([\delta,d/d_1d_2])\mu(r) \\ &= t_1 \sum_{d|\delta l_3'} \frac{d^2}{[\delta,d]d_1d_2\delta_3} M([\delta,d/d_1d_2]) \sum_{rs=\delta l_3'/d} \mu(r) \\ &= t_1 \left(\frac{d^2}{[\delta,d]d_1d_2\delta_3} M([\delta,d/d_1d_2]) \right)_{d=\delta l_3'} \\ &= t_1 l_3' M(\delta l_3') \end{aligned}$$

because when $d=\delta l_3'$ we have (i) $[\delta,d]=\delta l_3'$, (ii) $d_1d_2\delta_3 = d_1d_2(\delta,d/d_1d_2) = (\delta d_1d_2,d) = (\delta d_1d_2,\delta l_3') = \delta$ by the co-primality of $l_1'l_2'$ and l_3', and (iii) $[\delta,d/d_1d_2] = [\delta,(\delta/d_1d_2)l_3'] = \delta l_3'$. Therefore

$$P_4(x,t_1;l_1',l_2';\delta) = t_1 l_3' M(\delta l_3') + O\left(\log^{3A_1} x\right) \tag{121}$$

on the continuing assumption that (89) holds.

15. Estimate for $P(x,u,T;l_1',l_2';\delta)$ and return to $J_3^\dagger(x,u;Q)$

From (103) and (121) we deduce that

$$P_3(x,t_1,t_2;l_1',l_2';\delta) = t_1(x-t_2)\frac{l_3'M(\delta l_3')}{l_1'} + O\left(x\log^{3A_1} x\right)$$

and then from (102) and (93) that

$$P_1(x,t_1,t_2;l_1',l_2';\delta) = t_1(x-t_2)\frac{l_3'M(\delta l_3')}{l_1'} + O\left(\frac{x^2}{\log^{A_8} x}\right), \tag{122}$$

at which point we rejoin the road traversed in XI, §8 in order to draw out the behaviour of $P(x,u,T;l_1',l_2';\delta)$.

Making this part of the journey in a slightly more economical manner than before, we introduce the integrals

$$P_5(x,u,T;l_1',l_2';\delta) = P_5(x,u,T) = \int_T^x P(x,u,T_1)dT_1$$

and

$$\int_0^{x-T} P_1(x,\min(u,t),t+T)dt,$$

which are identical because, by (85) and the definition following it, the former is

$$\begin{aligned}\int_T^x \sum_{\substack{\Theta\\ 0<a\le u; a+T_1<s_2\le x}} 1\cdot dT_1 &= \sum_{\substack{\Theta\\ 0<a\le u; a+T<s_2\le x}} \int_T^{s_2-a} dT_1\\ &= \sum_{\substack{\Theta\\ 0<a\le u; a+T<s_2\le x}} (s_2-a-T)\end{aligned}$$

and the latter

$$\begin{aligned}\int_0^{x-T} \sum_{\substack{\Theta\\ 0<a\le u\\ a\le t; t+T<s_2\le x}} 1\cdot dt &= \sum_{\substack{\Theta\\ 0<a\le u\\ a+T<s_2\le x}} \int_a^{s_2-T} dt\\ &= \sum_{\substack{\Theta\\ 0<a\le u; a+T<s_2\le x}} (s_2-a-T).\end{aligned}$$

Therefore, by the estimate (122) just obtained,

$$\begin{aligned}P_5(x,u,T) &= \frac{l_3'M(\delta l_3')}{l_1'}\int_0^{x-T}\min(u,t)(x-T-t)dt\\ &\qquad + O\left(\frac{x^2(x-T)}{\log^{A_8} x}\right)\\ &= \frac{l_3'M(\delta l_3')}{l_1'} I(x,u,T) + O\left(\frac{x^2(x-T)}{\log^{A_8} x}\right),\end{aligned} \tag{123}$$

where, for $u > x-T$,

$$I(x,u,T) = \int_0^{x-T} t(x-T-t)dt = \frac{1}{6}(x-T)^3$$

but where, for $u < x - T$,

$$\begin{aligned}
I(x,u,T) &= \int_0^u t(x-T-t)dt + \int_u^{x-T} u(x-T-t)dt \\
&= \int_0^{x-T} t(x-T-t)dt - \int_u^{x-T} (t-u)(x-T-t)dt \\
&= \int_0^{x-T} t(x-T-t)dt - \int_0^{x-u-T} t'(x-u-T-t')dt' \\
&= \frac{1}{6}(x-T)^3 - \frac{1}{6}(x-u-T)^3.
\end{aligned}$$

Since it is readily verified that $I(x,u,T)$ is twice differentiable with respect to T with a second derivative that is $O(x-T)$, the sum $P(x,u,T)$ can now be calculated from the estimate (123) for its Césaro mean by means of a 'de la Vallée Poussin differentiation'. If $0 < H < x - T, T$, then the inequality

$$\begin{aligned}
\frac{1}{H}\{P_5(x,u,T) - P_5(x,u,T+H)\} &\leq P(x,u,T) \\
&\leq \frac{1}{H}\{P_5(x,u,T-H) - P_5(x,u,T)\}
\end{aligned}$$

with (11) and (23) implies that

$$P(x,u,T) = -\frac{l_3' M(\delta l_3')}{l_1'}\frac{\partial}{\partial T}I(x,u,T) + O(Hx) + O\left(\frac{x^2(x-T)}{H\log^{A_8} x}\right).$$

Hence, the choice of H as $(x-T)\log^{-\frac{1}{2}A_8} x$ being legitimate in the light of the condition stated in (85), we conclude that

$$P(x,u,T) = \frac{1}{2}\frac{l_3' M(\delta l_3')}{l_1'} I_1(x,u,T) + O\left(\frac{x^2}{\log^{\frac{1}{2}A_8} x}\right), \tag{124}$$

in which

$$I_1(x,u,T) = \begin{cases} (x-T)^2, & \text{if } u \geq x-T, \\ (x-T)^2 - (x-u-T)^2, & \text{if } u < x-T. \end{cases}$$

Equipped with (124), we are at long last able to develop (84), remarking on account of the condition in (82) that l_3' can replace l_2' in the summatory conditions and thus obtaining the equation

$$J_3^\dagger(x,u;Q) = \sum_{\delta < x/Q} \sum_{l_3' < l_1' < x/Q\delta} \left\{ \frac{1}{2}\frac{l_3' M(\delta l_3')}{l_1'} I_1(x,u,Q\delta l_1') + O\left(\frac{x^2}{\log^{A_9} x}\right)\right\}$$

because the number $Q\delta l_1'$ therein exceeds $x\log^{-A_1} x$ by (59). Then, if we take a short cut instead of the previous unnecessarily roundabout route, we reach this Section's conclusion by shewing that

(125)

$$\begin{aligned} J_3^{\dagger}(x,u;Q) &= \sum_{\delta<x/Q}\sum_{l_3'<l_1'<x/Q\delta}\left\{\frac{1}{2}\frac{\delta l_3'}{\delta l_1'}M(\delta l_3')I_1(x,u,Q\delta l_1')+O\left(\frac{x^2}{\log^{A_9} x}\right)\right\} \\ &= \frac{1}{2}\sum_{l_3<l_1<x/Q}\left\{\frac{l_3M(l_3)}{l_1}I_1(x,u,Ql_1)+O\left(\frac{x^2}{\log^{A_9} x}\right)\right\} \\ &= \frac{1}{2}\sum_{l_3<l_1<x/Q}\frac{l_3M(l_3)}{l_1}I_1(x,u,Ql_1)+O\left(\frac{x^2}{\log^{A_9-A_1} x}\right) \\ &= J_3^{\dagger\dagger}(x,u;Q)+O\left(\frac{x^2}{\log^{A} x}\right), \text{ say,} \end{aligned}$$

in the knowledge that l_1', l_3' indicated co-prime integers, it being clear that any value of A may be taken by our having chosen A_2 and then A_6 sufficiently large.

16. Estimations of $J_3^{\dagger\dagger}(x,u;Q)$ and $G_3(x,u;Q_1,Q_2)$

Again following for a while the road taken in XI (§9), we discern the sum

$$U(v)=\frac{1}{2}\sum_{l_3<l_1<v}\frac{(v-l_1)^2 l_3M(l_3)}{l_1}$$

that is present in the relation

$$J_3^{\dagger\dagger}(x,u;Q)=Q^2\{U(y)-U(y-h)\}, \tag{126}$$

where $y=x/Q$, $h=u/Q$ and where therefore

$$h<1, \quad y-h>1 \tag{127}$$

by (59). An investigation of $U(v)$ must then follow, the principal difficulty being to take advantage of the smoothing element that is latent in the sum.

Dropping the subscript from l_3 to lighten the notation, we have

$$U(v)=\frac{1}{2}\sum_{l<v}lM(l)\sum_{l<l_1<v}\frac{(v-l_1)^2}{l_1}=\frac{1}{2}\sum_{l<v}lM(l)V(v,l), \text{ say,} \tag{128}$$

and

$$\frac{1}{2}V(v,l)=\frac{1}{2}V_1(v,v)-\frac{1}{2}V_1(v,l), \tag{129}$$

where

$$\frac{1}{2}V_1(v,w)=\frac{1}{2}\sum_{l\le w}\frac{(v-l)^2}{l}$$

and

$$R_2(v) = \frac{1}{2}V_1(v,v) - \frac{1}{2}v^2\log v - \frac{1}{2}\left(\gamma - \frac{3}{2}\right)v^2 - \frac{1}{2}v = O(1) \tag{130}$$

by our basic summation formula (27). Next, for any positive integer w, we also have the formula

$$\begin{aligned}\frac{1}{2}V_1(v,w) &= \frac{1}{2}v^2\sum_{l\le w}\frac{1}{l} - v\sum_{l\le w}1 + \frac{1}{2}\sum_{l\le w}l\\ &= \frac{1}{2}v^2\sum_{l\le w}\frac{1}{l} - vw + \frac{1}{4}w^2 + \frac{1}{4}w,\end{aligned}$$

whence first, on comparing these two formulae for $v = w$, we find that

$$\sum_{l\le w}\frac{1}{l} = \log w + \gamma + \frac{1}{2w} + \frac{2R_2(w)}{w^2}$$

and then that

$$\frac{1}{2}V_1(v,w) = \frac{1}{2}v^2\left(\log w + \gamma + \frac{1}{2w} + \frac{2R_2(w)}{w^2}\right) - vw + \frac{1}{4}w^2 + \frac{1}{4}w.$$

Therefore, by this, (129), and (130), we secure an equation

$$\begin{aligned}\frac{1}{2}V(v,l) = \frac{1}{2}v^2\log\frac{v}{l} - \frac{3}{4}v^2 + vl - \frac{1}{4}l^2 - \frac{v^2}{4l}&\\ + \frac{1}{2}v - \frac{1}{4}l + R_2(v) - \frac{v^2R_2(l)}{l^2}&\end{aligned} \tag{131}$$

that is the analogue of the equation following (83) in XI. But, in exploiting this formula, we must again strike out on an entirely new path because our previous method is not sufficiently sensitive to meet our new needs.

We orientate ourselves by turning toward the identities

$$l\left(\frac{1}{2}v^2\log\frac{v}{l} - \frac{3}{4}v^2 + vl - \frac{1}{4}l^2\right) = \frac{3}{4}\int_l^v (t-l)^2dt - \frac{v^2}{4}\int_l^v\frac{(t-l)^2dt}{t^2} \tag{132}$$

and

$$l\left(\frac{v^2}{4l} - \frac{1}{2}v + \frac{1}{4}l\right) = \frac{1}{4}(v-l)^2. \tag{133}$$

The former is not merely a felicitous accident but has an explicable genesis within the present scene, although its treatment is most expeditiously conducted by verifying that its right side is equal to

$$\frac{1}{4}(v-l)^3 - \frac{v^2}{4}\left[t - 2l\log t - \frac{l^2}{t}\right]_t^v.$$

At any rate, whatever its origin, (132) is taken with (128), (131), and (133) to give

$$\begin{aligned}U(v) &= \frac{3}{4}\sum_{l<v} M(l)\int_l^v (t-l)^2 dt - \frac{v^2}{4}\sum_{l<v} M(l)\int_l^v \frac{(t-l)^2}{t^2}dt\\ &\quad - \frac{1}{4}\sum_{l<v}(v-l)^2 M(l) + R_2(v)\sum_{l<v} lM(l) - v^2\sum_{l<v}\frac{R_2(l)M(l)}{l}\\ &= \frac{3}{4}\int_1^v \sum_{l<t}(t-l)^2 M(l)dt - \frac{v^2}{4}\int_1^v \frac{1}{t^2}\sum_{l<t}(t-l)^2 M(l)dt\\ &\quad - \frac{1}{4}\sum_{l<v}(v-l)^2 M(l) + R_2(v)\sum_{l<v} lM(l) - v^2\sum_{l<v}\frac{R_2(l)M(l)}{l}\end{aligned}$$

in the first place. Hence, by (130) and the definition of $T^*(t)$ in (18),

(134)

$$\begin{aligned}U(v) &= \left\{\frac{3}{2}\int_1^v t^2 T^*(t)dt - \frac{v^2}{2}\int_1^v T^*(t)dt\right\} - \frac{1}{2}v^2 T^*(v)\\ &\quad + \left\{R_2(v)\sum_{l<v} lM(l) + v^2\sum_{l\geq v}\frac{R_2(l)M(l)}{l}\right\} - v^2\sum_{l=1}^{\infty}\frac{R_2(l)M(l)}{l}\\ &= U_1(v) - U_2(v) + U_3(v) + B_3 v^2, \text{ say,}\end{aligned}$$

the first three constituents in which will be subject to the differencing process of (126).

We readily confirm that

$$\begin{aligned}U_3(y) - U_3(y-h) = \{R_2(y) - R_2(y-h)\}&\sum_{l<y-h} lM(l)\\ &+ \left\{y^2 - (y-h)^2\right\}\sum_{l\geq y-h}\frac{R_2(l)M(l)}{l}\\ &+ \sum_{y-h\leq l<y} lM(l)\left\{R_2(y) - \frac{y^2 R_2(l)}{l^2}\right\},\end{aligned}$$

the first two terms in which are

$$O\left(\frac{h}{y}\sum_{l<y} 1\right) = O(h) \text{ and } O\left(hy\sum_{l\geq y-h}\frac{1}{l^2}\right) = O(h)$$

by (23), (127), (130), and Lemma 1. Also, since $h<1$, the third term is zero unless the interval $[y-h, y)$ contain a single integer $l = y - \theta h$ with $0<\theta\leq 1$, in which case, by Lemma 1 and (130) again, this term is merely

$$O\left\{R_2(y)-R_2(l)\right\}+O\left(\frac{hR_2(l)}{l}\right)=O\left(\frac{h}{y}\right).$$

Thus

$$U_3(y)-U_3(y-h)=O(h). \tag{135}$$

The next constituent in (134) to be differenced is the subject of Lemma 3, which states that

$$\begin{aligned} U_2(y)-U_2(y-h)=\frac{1}{4}\Phi(0)\left\{y^2\log y-(y-h)^2\log(y-h)\right\} \\ +\frac{1}{4}B_2\left\{y^2-(y-h)^2\right\}+O(h). \end{aligned} \tag{136}$$

The most important part of $U(v)$ is $U_1(v)$. By (19) (see Lemma 3 for value of B_2), this equals

$$\begin{aligned}
&\frac{3}{4}\int_1^v\left\{\Phi(0)t^2\log t+B_2t^2+\Phi(-1)t\right\}dt\\
&\qquad-\frac{v^2}{4}\int_1^v\left(\Phi(0)\log t+B_2+\frac{\Phi(-1)}{t}\right)dt\\
&\qquad+\frac{v^2}{4}\int_1^v I^*(t)dt-\frac{3}{4}\int_1^v t^2I^*(t)dt\\
&=\frac{3}{4}\,{}_1^v\!\left[\frac{1}{3}\Phi(0)t^3\log t-\frac{1}{9}\Phi(0)t^3+\frac{1}{3}B_2t^3+\frac{1}{2}\Phi(-1)t^2\right]\\
&\qquad-\frac{v^2}{4}\,{}_1^v\!\left[\Phi(0)t\log t-\Phi(0)t+B_2t+\Phi(-1)\log t\right]\\
&\qquad+\frac{v^2}{4}\int_1^v I^*(t)dt-\frac{3}{4}\int_1^v t^2I^*(t)dt\\
&=\frac{1}{6}\Phi(0)v^3-\frac{1}{4}\Phi(-1)v^2\log v+\left\{\frac{v^2}{4}\int_1^v I^*(t)dt-\frac{3}{4}\int_1^v t^2I^*(t)\right\}\\
&\qquad+B_3v^2+B_4\\
&=\frac{1}{6}\Phi(0)v^3-\frac{1}{4}\Phi(-1)v^2\log v+U_4(v)+B_3v^2+B_4,\ \text{say,}
\end{aligned}$$

wherefore

$$\begin{aligned} U_1(y)-U_1(y-h)=\frac{1}{6}\Phi(0)\left\{y^3-(y-h)^3\right\}\\ -\frac{1}{4}\Phi(-1)\left\{y^2\log y-(y-h)^2\log(y-h)\right\}\\ +U_4(y)-U_4(y-h)+B_3\left\{y^2-(y-h)^2\right\}. \end{aligned} \tag{137}$$

The difference $U_4(y) - U_4(y-h)$ represents an indispensable aspect of the analysis that the formulation of equation (132) was designed to uncover. Requiring some delicacy in its treatment, especially as we shall need to differentiate between the cases where D_1 in (6) is zero and non-zero, it can at this stage only undergo an initial metamorphosis in preparation for further work when the sum $G_3^*(x, u; Q_1, Q_2)$ is considered in the next Section. Accordingly, deploying (22) and Lemma 2 with their implications regarding $I^*(t)$ and its derivatives, we merely transform $U_4(y) - U_4(y-h)$ into

$$hU_4'(y) + \frac{1}{2}h^2 U_4''\{y - \theta(h, y)\},$$

where

$$0 < \theta(h, y) < h \tag{138}$$

in conformity with our earlier suppositions embedded in (127). Hence

$$\begin{aligned} U_4(y) - U_4(y-h) &= \frac{1}{2}h\left(y\int_1^y I^*(t)dt - y^2 I^*(y)\right) \\ &\quad + \frac{1}{4}h^2\left(\int_1^{y_1} I^*(t)dt - y_1 I^*(y_1) - y_1^2\frac{dI^*(y_1)}{dy_1}\right)_{y_1 = y - \theta(h,y)} \\ &= \frac{1}{2}h\left(y\int_1^y I^*(t)dt - y^2 I^*(y)\right) \\ &\quad + \frac{1}{4}h^2\int_1^{y-\theta(h,y)} I^*(t)dt + O(h^2). \end{aligned} \tag{139}$$

We are ready to regress to (126) by way of (134), (135), (136), (137), and (139), which imply that

$$\begin{aligned} U(y) - U(y-h) &= \frac{1}{6}\Phi(0)\{y^3 - (y-h)^3\} \\ &\quad + \frac{1}{4}\{\Phi(0) - \Phi(-1)\}\{y^2\log y - (y-h)^2\log(y-h)\} \\ &\quad + \frac{1}{2}h\left(y\int_1^t I^*(t)dt - y^2 I^*(y)\right) \\ &\quad + \frac{1}{4}h^2\int_1^{y-\theta(h,y)} I^*(t)dt + B_5\{y^2 - (y-h)^2\} + O(h). \end{aligned} \tag{140}$$

Therefore, taking Q in (126) to be Q_1 and Q_2 with the corresponding values of y and h, we deduce that

$$\begin{aligned}
J_3^{\dagger\dagger}(x,u;Q_1) - J_3^{\dagger\dagger}(x,u;Q_2) &= \frac{1}{6}\Phi(0)\left(\frac{1}{Q_1}-\frac{1}{Q_2}\right)\left\{x^3-(x-u)^3\right\} \\
&\quad - \frac{1}{4}\{\Phi(0)+\Phi(-1)\}\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} + \frac{1}{2}ux\int_{x/Q_2}^{x/Q_1} I^*(t)dt \\
&\quad - \frac{1}{2}ux^2\left\{\frac{1}{Q_1}I^*\left(\frac{x}{Q_1}\right) - \frac{1}{Q_2}I^*\left(\frac{x}{Q_2}\right)\right\} \\
&\quad + \frac{1}{4}u^2\int_{x/Q_2-\theta(x/Q_2,u/Q_2)}^{x/Q_1-\theta(x/Q_1,u/Q_1)} I^*(t)dt + O(uQ_2)
\end{aligned}$$

because the terms corresponding to the penultimate item in (140) annihilate themselves. From this then flows

$$\begin{aligned}
(141)\quad & J_3^{\dagger\dagger}(x,u;Q_1) - J_3^{\dagger\dagger}(x,u;Q_2) \\
&= \frac{1}{6}\Phi(0)\left(\frac{1}{Q_1}-\frac{1}{Q_2}\right)\left\{x^3-(x-u)^3\right\} \\
&\qquad - \frac{1}{4}\{\Phi(0)+\Phi(-1)\}\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} \\
&\qquad + \frac{1}{2}u\left[x\int_{x/Q_2}^{x/Q_1} I^*(t)dt - x^2\left\{\frac{1}{Q_1}I^*\left(\frac{x}{Q_1}\right) - \frac{1}{Q_2}I^*\left(\frac{x}{Q_2}\right)\right\}\right] \\
&\qquad + O\left(u^2\log\frac{2Q_2}{Q_1}\right) + O(uQ_2) \\
&= \frac{1}{6}\Phi(0)\left(\frac{1}{Q_1}-\frac{1}{Q_2}\right)\left\{x^3-(x-u)^3\right\} \\
&\qquad - \frac{1}{4}\left\{\Phi(0)+\Phi(-1)\right\}\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} \\
&\qquad + \frac{1}{2}uG_4(x;Q_1,Q_2) + O(uQ_2), \text{ say,}
\end{aligned}$$

through (22) and (138) and then through the inequalities $u < Q_1$ and $\log 2Q_2/Q_1 = O(Q_2/Q_1)$.

The peak represented by $G_3(x,u;Q_1,Q_2)$ has at last been almost crested because (73), (76), (77), (78), (79), and (141) altogether provide us with

$$
\begin{aligned}
(142) \qquad G_3(x,u;Q_1,Q_2) &= O(Q_2u) + \Phi(0)\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} \\
&\quad + \frac{1}{2}C\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} \\
&\quad + \frac{1}{3}\Phi(0)\left(\frac{1}{Q_1}-\frac{1}{Q_2}\right)\left\{x^3-(x-u)^3\right\} \\
&\quad - \frac{1}{2}\left\{\Phi(0)+\Phi(-1)\right\}\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} \\
&\quad + uG_4(x;Q_1,Q_2) + O(Q_2u) + O\left(\frac{x^2}{\log^A x}\right) \\
&= \frac{1}{3}\Phi(0)\left(\frac{1}{Q_1}-\frac{1}{Q_2}\right)\left\{x^3-(x-u)^3\right\} \\
&\quad + \frac{1}{2}\left(C+\Phi(0)-\Phi(-1)\right)\left\{x^2-(x-u)^2\right\}\log\frac{Q_2}{Q_1} \\
&\quad + uG_4(x,u;Q_1,Q_2) + O(uQ_2) + O\left(\frac{x^2}{\log^A x}\right).
\end{aligned}
$$

17. The asymptotic formula for $G^*(x,u;Q_1,Q_2)$

The evaluation of $G^*(x,u;Q_1,Q_2)$ results from (142) and what has gone before. First, proceeding from $G_3(x,u;Q_1,Q_2)$ to $G_3^*(x,u;Q_1,Q_2)$ by partial summation, let us transform (142) into

$$
\begin{aligned}
(143) \qquad G_3^*(x,u;Q_1,Q_2) &= \frac{1}{3}\Phi(0)\left\{x^3-(x-u)^3\right\}\log\frac{Q_2}{Q_1} \\
&\quad + \frac{1}{2}\left(C+\Phi(0)-\Phi(-1)\right)\left\{x^2-(x-u)^2\right\}(Q_2-Q_1) \\
&\quad + u\int_{Q_1}^{Q_2} t\,dG_4(x;Q_1,t) + O(uQ_2^2) + O\left(\frac{x^2Q_2}{\log^A x}\right) \\
&= \frac{1}{3}\Phi(0)\left\{x^3-(x-u)^3\right\}\log\frac{Q_2}{Q_1} \\
&\quad + \frac{1}{2}\left(C+\Phi(0)-\Phi(-1)\right)\left\{x^2-(x-u)^2\right\}(Q_2-Q_1) \\
&\quad + uG_4^*(x;Q_1,Q_2) + O(uQ_2^2) + O\left(\frac{x^3}{\log^A x}\right), \text{ say,}
\end{aligned}
$$

which furnishes an estimate for the third constituent in the formula (47) for $G^*(x,u;Q_1,Q_2)$. With expressions (52) and (71) for the first and second constituents, this produces the preliminary estimate

(144)

$$\begin{aligned}
&G^*(x, u; Q_1, Q_2) \\
&\quad = \Phi(0)\left[x^2u - x\left\{x^2 - (x-u)^2\right\} + \frac{1}{3}\left\{x^3 - (x-u)^3\right\}\right]\log\frac{Q_2}{Q_1} \\
&\qquad + \left[-\Phi(0)ux + \frac{1}{2}\left\{C + \Phi(0) - \Phi(-1)\right\}\left\{x^2 - (x-u)^2\right\}\right](Q_2 - Q_1) \\
&\qquad + uG_4^*(x; Q_1, Q_2) \\
&\qquad + O\left(x^2 Q_2^{\frac{1}{2}+\epsilon}\right) + O(uQ_2^2) + O\left(x^3 \log^{-A} x\right) \\
&\quad = \frac{1}{3}\Phi(0)u^3 \log\frac{Q_2}{Q_1} + \left\{C - \Phi(-1)\right\} ux(Q_2 - Q_1) \\
&\qquad - \frac{1}{2}\left\{C + \Phi(0) - \Phi(-1)\right\} u^2(Q_2 - Q_1) + uG_4^*(x; Q_1, Q_2) \\
&\qquad + O(uQ_2^2) + O\left(x^3 \log^{-A} x\right) \\
&\quad = \left\{C - \Phi(-1)\right\} u(Q_2 - Q_1)x + uG_4^*(x; Q_1, Q_2) \\
&\qquad + O(uQ_2^2) + O\left(x^3 \log^{-A} x\right)
\end{aligned}$$

provided that we still recall the given conditions (46).

It remains to examine the entity $G_4^*(x; Q_1, Q_2)$ that is defined by means of (19), (141) and (143). Recollecting that $I^*(u)$ certainly has a continuous first differential coefficient, we first set

$$\Xi(s_1) = x^2 I^*\left(x/s_1\right)$$

and through the substitution $s_1 = x/s$ evaluate the first entity

$$x\int_{x/t}^{x/Q_1} I^*(s)ds$$

in $G_4(x; Q_1, t)$ as

$$\int_{Q_1}^{t} \frac{\Xi(s_1)ds_1}{s_1^2}$$

when $Q_1 \le t \le Q_2$. Hence

$$G_4(x, Q_1, t) = \int_{Q_1}^{t} \frac{\Xi(s_1)ds_1}{s_1^2} + \frac{\Xi(t)}{t} - \frac{\Xi(Q_1)}{Q_1} = \int_{Q_1}^{t} \frac{\Xi'(s_1)ds_1}{s_1}$$

and we conclude that

$$G_4^*(x:Q_1,Q_2) = \int_{Q_1}^{Q_2} \Xi'(t)dt = \Xi(Q_2) - \Xi(Q_1) \tag{145}$$
$$= x^2 I^*(x/Q_2) - x^2 I^*(x/Q_1).$$

At long last our theorem on $G^*(x,u;Q_1,Q_2)$ is available when in addition to (46) we assume that $Q_2 = o(x)$ in accordance with $(45)_2$ and the succeeding comment, although there are two cases to consider. In the first case where $D_1 = C - \Phi(-1) > 0$, we merely extract from (145) and (22) the order relation

$$G_4^*(x;Q_1,Q_2) = o(Q_2 x)$$

and then draw from (142) the asymptotic formula

$$G^*(x,u;Q_1,Q_2) = D_1 u(Q_2 - Q_1)x + o(uQ_2x) + O\left(x^3 \log^{-A} x\right),$$

which may be stated alternatively as

$$\begin{aligned} G^*(x,u;Q_1,Q_2) &= u\{G(x,Q_2) - G(x,Q_1)\} + o(uQ_2x) + O\left(x^3 \log^{-A} x\right) \\ &= uG(x;Q_1,Q_2) + o(uQ_2x) + O\left(x^3 \log^{-A} x\right) \end{aligned} \tag{146}$$

according to (6) and an obvious notation. In the other case, the first main term in (144) is now zero but, for $Q = Q_1$ or Q_2,

$$x^2 I^*(x/Q) = G(x,Q) + O\left(x^2 \log^{-A} x\right)$$

by (20), for which reasons we then get

$$\begin{aligned} G^*(x,u;Q_1,Q_2) &= u\{G(x,Q_2) - G(x,Q_1)\} + O(uQ_2^2) \\ &\quad + O\left(x^3 \log^{-A} x\right) \\ &= uG(x;Q_1,Q_2) + O(uQ_2^2) + O\left(x^3 \log^{-A} x\right) \end{aligned} \tag{147}$$

through (145).

Of similar appearance, these two formulae when taken together seem to confirm our speculations about the connection between the sums $G^*(x,u;Q_1,Q_2)$ and $G(x;Q_1,Q_2)$ for the larger values of Q_1 and Q_2. Yet some care needs to be taken in their interpretation ere strict assertions are made. In the first case, for instance, it is enough to assume, for any chosen positive constant A_1, that condition (46) is slightly strengthened by insisting that

$$u > x \log^{-A_1} x; \quad u < Q_1 < \frac{1}{2}Q_2; \quad Q_2 = o(x), \tag{148}$$

in which event, taking $A = 2A_1 + 1$, we know from (6) that

$$G(x; Q_1, Q_2) > \frac{1}{3} D_1 Q_2 x + O\left(\frac{x^2}{\log^A x}\right) > \frac{1}{4} D_1 Q_2 x$$

with the result that

$$\begin{aligned} G^*(x, u; Q_1, Q_2) &= uG(x; Q_1, Q_2) + o(uQ_2 x) + O\left(\frac{uQ_2 x}{\log x}\right) \\ &= uG(x; Q_1, Q_2) + o(uQ_2 x) \\ &\sim uG(x; Q_1, Q_2). \end{aligned} \tag{149}$$

In the second case more discretion must be exercised because we have demonstrated in XIV that the magnitude of $G(x, Q)$ can then fluctuate fairly violently. To accommodate this point it in fact suffices to fortify (46) by the stipulations

$$u > x \log^{-A_1} x; \quad u < Q_1 < Q_2^2/x; \quad Q_2 = o(x) \tag{150}$$

together with the condition laid down in (45).

With these conditions and the value $2A_1 + 1$ for A, it is seen from the basic Barban-Davenport-Halberstam type formula (5) that

$$G(x, Q_1) = O(Q_1 x) + O\left(x^2 \log^{-A_1} x\right) = O(Q_1 x) = O(Q_2^2)$$

and hence that

$$\{G(x, Q_2) - G(x, Q_1)\} / Q_2^2 \to \infty,$$

from which again (149) follows in the light of the first condition in (150) that asserts that

$$x^3 \log^{-2A_1 - 1} x = o(uQ_2^2) = o\big[u\{G(x, Q_2) - G(x, Q_1)\}\big].$$

In summation, our deductions in this section amount to

Theorem 2. *Defining $E(x; a, k)$ as in the Introduction, let us write*

$$G(x; Q_1, Q_2) = \sum_{Q_1 < k \le Q_2} \sum_{0 < a \le k} E^2(x; a, k)$$

and

$$G^*(x, u; Q_1, Q_2) = \sum_{Q_1 < k \le Q_2} k \sum_{0 < a \le u} E^2(x; a, k);$$

let also A, A_1 be any chosen positive constants. Then, under the condition (46)*, we have either*

$$G^*(x, u; Q_1, Q_2) = uG(x; Q_1, Q_2) + o(uQ_2 x) + O\left(x^3 \log^{-A} x\right)$$

or

$$G^*(x, u; Q_1, Q_2) = uG(x; Q_1, Q_2) + O(uQ_2^2) + O\left(x^3 \log^{-A} x\right)$$

according as the constant D_1 in the asymptotic formula (6) *of X be non-zero or zero.*

The asymptotic equality

$$G^*(x, u; Q_1, Q_2) \sim uG(x; Q_1, Q_2)$$

holds if either

(a) $D_1 > 0$ *and* (148) *above be given (condition* (45) *then holds)*

or

(b) $D_1 = 0$ *and both* (45) *and* (150) *above be given.*

18. Asymptotic formulae for $G_\lambda(x, Q)$ and $G_{\mu,\lambda}(x, Q)$

Theorem 2 is the penultimate milestone on our journey. It has shed light on the situation we have been exploring but is too hemmed in by subsidiary conditions to be altogether satisfactory as a final result. Its main importance is that it provides a portal for a theorem on $G_\lambda(x, Q)$ that is subject to less qualification and that is almost fully complementary to our conclusions in the earlier §5.

The situation in which we shall work is still that described by the restraints on x, Q given in $(45)_2$ so that $G(x, Q)/Q^2 \to \infty$ by $(45)_1$. Then, for conciseness, we largely ignore the case $D_1 > 0$ because this is the easier to treat and because within it the unboundedness of $G(x, Q)/Q^2$ is guaranteed.

With this proviso, we set

$$\gamma(x, Q) = G(x, Q)/Q^2 \text{ and } \eta = \frac{1}{\sqrt{\gamma(x, Q)}}, \tag{151}$$

where, in particular, we have

$$\eta > (A'Q/x)^{\frac{1}{2}} > \log^{-A_1} x \tag{152}$$

by the basic theorem (5). Next, by an obvious extension of earlier notation, let us write

$$G_\lambda(x; Q_1, Q_2) = \sum_{Q_1 < k \le Q_2} \sum_{0 < a \le \lambda k} E^2(x; a, k) = G_\lambda(x, Q_2) - G_\lambda(x, Q_1) \tag{153}$$

for $\lambda < 1$, at which point we slightly weaken earlier conventions regarding the connections between Q_1, Q_2, and Q. Also, again by the basic theorem, we have

$$G_\lambda(x, Q') \le G(x, Q') = O(Q'x) + O\left(x^2 \log^{-3A_1} x\right)$$

so that

$$G_\lambda(x, Q') = \lambda G(x, Q') + O\left(x^2 \log^{-3A_1} x\right) \tag{154}$$

whenever $Q' \le x \log^{-3A_1} x$. Then, to form some kind of appropriate complement of (154) for larger values of Q', we assume in the first place that

$$\lambda \le 1 - \frac{1}{\sqrt{\gamma(x,Q)}} = 1 - \eta = \rho, \text{ say,} \tag{155}$$

and then let R be the least positive integer such that

$$(1+\eta)^{-R} Q \le x \log^{-3A_1} x, \tag{156}$$

denoting the left-side of the inequality by Q_1 and then unrolling the sequence

$$P_0 = Q_1, P_1 = (1+\eta)Q_1, \ldots, P_R = (1+\eta)^R Q_1 = Q.$$

Thus prepared, we form the inequalities

$$\begin{aligned}\frac{1}{P_{r+1}} \sum_{P_r < k \le P_{r+1}} k \sum_{0<a\le \lambda P_r} E^2(x;a,k) &\le G_\lambda(x;P_r,P_{r+1}) \\ &\le \frac{1}{P_r} \sum_{P_r<k\le P_{r+1}} k \sum_{0<a\le\lambda P_{r+1}} E^2(x;a,k),\end{aligned}$$

in the last member of which $\lambda P_{r+1} = \lambda(1+\eta)P_r \le (1-\eta^2)P_r < P_r$. Hence, in the language of (153), we have

$$\frac{1}{P_{r+1}} G^*(x,\lambda P_r;P_r,P_{r+1}) \le G_\lambda(x;P_r,P_{r+1}) \le \frac{1}{P_r} G^*(x,\lambda P_{r+1};P_r,P_{r+1}),$$

which through (147) yields both

$$\begin{aligned}G_\lambda(x;P_r,P_{r+1}) &\le \frac{1}{P_r}\left\{\lambda P_{r+1} G(x;P_r,P_{r+1}) + O(\lambda P_{r+1}^3) + O\left(\frac{x^3}{\log^{7A_1} x}\right)\right\} \\ &= (1-\eta)\lambda G(x;P_r,P_{r+1}) + O(\lambda P_r^2) + O\left(\frac{x^3}{P_0(1+\eta)^r \log^{7A_1}\eta}\right)\end{aligned}$$

and

$$\begin{aligned}G_\lambda(x;P_r,P_{r+1}) &\ge \frac{1}{P_{r+1}}\left\{\lambda P_r G(x;P_r,P_{r+1}) + O(\lambda P_r P_{r+1}^2) + O\left(\frac{x^3}{\log^{7A_1} x}\right)\right\} \\ &> (1+\eta)\lambda G(x;P_r,P_{r+1}) + O(\lambda P_r^2) + O\left(\frac{x^3}{P_0(1+\eta)^r \log^{7A_1}\eta}\right).\end{aligned}$$

Summing these inequalities for all indices r from 0 to $R-1$, we deduce that

$$\begin{aligned}G_\lambda(x;Q_1,Q) &= \lambda G(x;Q_1,Q) + O\left\{\lambda\eta G(x;Q_1,Q)\right\}\\ &\quad + O\left(\lambda Q^2\sum_{r=0}^{\infty}\frac{1}{(1+\eta)^{2r}}\right) + O\left(\frac{x^2}{\log^{4A_1}x}\sum_{r=0}^{\infty}\frac{1}{(1+\eta)^r}\right)\\ &= \lambda G(x;Q_1,Q) + O\left\{\lambda\eta G(x,Q)\right\}\\ &\quad + O\left(\frac{\lambda Q^2}{\eta}\right) + O\left(\frac{x^2}{\eta\log^{4A_1}x}\right)\\ &= \lambda G(x;Q_1,Q) + O\left\{\lambda Q G^{\frac{1}{2}}(x,Q)\right\} + O\left(x^2\log^{-3A_1}x\right)\end{aligned}$$

in view of (156), (151), and (152). If now we add to this the equation (154) formed with the value $Q' = Q_1$, we then conclude that

$$G_\lambda(x,Q) = \lambda G(x,Q) + O\left\{\lambda Q G^{\frac{1}{2}}(x,Q)\right\} + O\left(x^2\log^{-3A_1}x\right). \tag{157}$$

This has been established under the supposition $\lambda \le \rho$ stated in (155). But, in the other case where $1-\eta = \rho < \lambda < 1$ we have

$$G_\rho(x,Q) \le G_\lambda(x,Q) \le G(x,Q),$$

while also

$$\begin{aligned}G_\rho(x,Q) &= \rho G(x,Q) + O\left\{\rho Q G^{\frac{1}{2}}(x,Q)\right\} + O\left(x^2\log^{-3A_1}x\right)\\ &= \lambda G(x,Q) + O\left\{\eta G(x,Q)\right\} + O\left\{\lambda Q G^{\frac{1}{2}}(x,Q)\right\} + O\left(x^2\log^{-3A_1}x\right)\\ &= \lambda G(x,Q) + O\left\{\lambda Q G^{\frac{1}{2}}(x,Q)\right\} + O\left(x^2\log^{-3A_1}x\right)\end{aligned}$$

and similarly

$$G(x,Q) = \lambda G(x,Q) + O\left\{\eta G(x,Q)\right\} = \lambda G(x,Q) + O\left\{\lambda Q G^{\frac{1}{2}}(x,Q)\right\}.$$

Thus (157) subsists in the formerly excluded range of λ and is therefore true when $D_1 = 0$ and $(45)_2$ is given.

A somewhat similar method can manage the case $D_1 > 0$ with the formula

$$\begin{aligned}G_\lambda(x,Q) &= \lambda G(x,Q) + o(\lambda Q x) + O\left(x^2\log^{-2A_1}x\right)\\ &= \lambda D_1 Q x + o(\lambda Q x) + O\left(x^2\log^{-2A_1}x\right)\end{aligned}$$

as its outcome. Therefore, since the final remainder terms in both this and (157) are $o\{\lambda G(x,Q)\}$ when $\lambda > \log^{-A_1}x$, we obtain the result we sought in the form of

THEOREM 3. *For any sequence satisfying Criterion V, let*

$$G_\lambda(x,Q) = \sum_{k\le Q}\sum_{0<a\le\lambda k} E^2(x;a,k).$$

Then, if the sequence conform to condition (45)*, we have*

$$G_\lambda(x,Q) \sim \lambda G(x,Q)$$

whenever $x \log^{-A_1} x < \lambda \leq 1$*,* $Q > x \log^{-A_1} x$*, and* $Q = o(x)$*.*

In interpreting this result, the reader may care to be reminded that a discussion of condition (45) and its opposite is given in §4.

To be complete, our work should be extended to cover the sums $G_{\mu,\lambda}(x,Q)$ that were defined in equation (10) of the Introduction. This can be done by a moderately transparent generalization of our analysis, which was restricted to the special case $\mu = 0$ to avoid further complications and to expose more clearly the principle behind the method. Suffice it then to now state

THEOREM 4. *Under the conditions of Theorem* 3*, we have*

$$G_{\lambda,\mu}(x,Q) \sim \lambda G(x,Q).$$

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