CONVERGENCE ON FILTERS AND SIMPLE EQUICONTINUITY[1]

BY

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Convergence on filters is conceived as a replacement for simple uniform convergence [5, Dictionnaire]. Besides retaining the notable characteristic of simple uniform convergence, preservation of continuity, the new convergence is intimately related to the linear topological structure of function spaces.

The first section defines the concept of convergence on a filter and shows that it leads to a necessary and sufficient condition for a filter of functions continuous at a point x to converge at x to a function which is also continuous there. The second section develops the associated uniformity and shows that the topology of almost uniform convergence is the special case of convergence on all ultra filters [2], [3]. The last of the four applications given in Section three can be considered as a localization of the method used in obtaining a Stone-Čech compactification.

Section four presents a weakened form of equicontinuity called simple equicontinuity. The interesting properties which it has in common with equicontinuity are displayed. It is used to characterize the relatively compact sets for the topology of pointwise convergence in the space of continuous functions. The result is an analogue of Ascoli's theorem [9]. Combining the present result with Ascoli's theorem leads to another characterization of compact sets.

The last section examines simple equicontinuity in a locally convex linear topological space. It terminates in a strengthened form of the Alaoglu-Bourbaki theorem in which simple equicontinuity replaces equicontinuity [6], [8].

1. Convergence on a filter

Throughout this paper $G(S, E)$ denotes a space of functions whose common domain is a set S and whose ranges are in the Hausdorff uniform space E. The space E^S of all functions from S into E is denoted by $J(S, E)$.

The concept of simple uniform convergence originated with Dini [7]. To observe its relationship to convergence on a filter, consider the filter $\mathfrak{F}$ of Definition 1.2 as the filter of neighborhoods of the point s_0 in Definition 1.1. Besides the obvious replacement of the sequence $\{f_n\}$ by the filter $\mathcal{G}$, note that no pointwise convergence is required in 1.2 and every refinement of the filter $\mathcal{G}$ will also converge on $\mathfrak{F}$.

1.1 Definition. (*Simple Uniform Convergence*) [5, Dictionnaire]. A se-

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quence $\{f_n\}$ of functions defined on a topological space S with values in a uniform space E is *simply uniformly convergent* at a point s_0 of S to a function f_0 if it converges pointwise on a neighborhood of s_0 and for every entourage V of E and positive integer n_0 there exist a neighborhood U of s_0 and an integer $n \geqq n_0$ such that $(f_n(s), f_0(s))$ is in V for all s in U.

1.2 DEFINITION. (*Convergence on a Filter*) *A filter* $\mathcal{G}$ *composed of subsets of* $G(S, E)$ *converges to a function* f_0 *on a filter* $\mathfrak{F}$ *of subsets of* S if for every entourage U of E there is a D in $\mathcal{G}$ such that for each f in D there is an F_f in $\mathfrak{F}$ with the property that $(f(s), f_0(s))$ is in U for all s in F_f.

The basic relationship between convergence on a filter and continuous functions is given in the following theorem. It is interesting to note that pointwise convergence of the filter of functions may occur only at s_0.

1.3 THEOREM. *Let* $\mathcal{G}$ *be a filter in* $G(S, E)$ *where* S *is a topological space and every* g *in* $\mathcal{G}$ *is continuous at* s_0 *in* S. *The filter* $\mathcal{G}$ *converges at the point* s_0 *to a function* f_0 *which is continuous at* s_0 *if and only if it converges to* f_0 *on the filter of neighborhoods of* s_0.

Proof. Assume f_0 is continuous at s_0 and for each entourage U from E there is an F in $\mathcal{G}$ such that $(f(s_0), f_0(s_0))$ is in U for all f in F. Since f and f_0 are both continuous at s_0 there is a neighborhood V_f of s_0 which places $(f(s), f_0(s))$ in U for all s in V_f.

For the converse consider an arbitrary entourage U from E. There is a G in $\mathcal{G}$, an f in G, a neighborhood V of s_0, and an entourage W from E such that $W \circ W \circ W \subset U$, and $(f(s), f_0(s))$ and $(f(s), f(s_0))$ are in W for all s in V. Thus $(f_0(s), f_0(s_0))$ is in U for all s in V and f_0 is continuous at s_0.

The above theorem can be considered as a localization of Arzela's theorem on quasi-uniform convergence [1], [2]. In the same manner many results through out this paper are localizations of items found in references [2], [4].

2. The topology of convergence on a filter

2.1 THEOREM. *Let* $\mathfrak{F}$ *be a fixed filter in* S *and denote by* $\mathfrak{U}$ *all sets of the form* $U(V, \mathfrak{F}) = \{(f, g) :$ *there is an* F_{fg} *in* $\mathfrak{F}$ *such that* $(f(s), g(s))$ *is in* V *for all* s *in* $F_{fg}\}$, V *ranging over a base for the entourages of* E. *Then in* $G(S, E)$ *the class of all filters which converge on the filter* $\mathfrak{F}$ *has an associated topology obtained from the uniformity having* $\mathfrak{U}$ *as a base. It is called the topology of convergence on the filter* $\mathfrak{F}$. *The topology is Hausdorff if and only if for each pair,* f *and* g, *of functions in* $G(S, E)$ *there exists an entourage* V *from* E *such that for every* F *in* $\mathfrak{F}$ *there is an* s *in* F *such that* $(f(s), g(s))$ *is not in* V.

Proof. The fact that the sets $U(V, \mathfrak{F})$ form a base for the entourages of a uniformity can be seen by observing that if V, V' and W are entourages from E and $V' \circ V' \subset V$, then

$$U(V', \mathfrak{F}) \circ U(V', \mathfrak{F}) \subset U(V, \mathfrak{F}) \quad \text{and} \quad U(V, \mathfrak{F}) \cap U(W, \mathfrak{F}) = U(V \cap W, \mathfrak{F}).$$

Assuming that $\mathcal{G}$ is a filter in $G(S, E)$ converging to f_0 on the filter $\mathfrak{F}$, there exists for each entourage V from E a G in $\mathcal{G}$ such that for each g in G there is an F_g in $\mathfrak{F}$ with the property that $(g(s), f_0(s))$ is in V for all s in F_g. In other words, $\mathcal{G}$ converges to f_0 for the topology associated with the uniformity. The converse is also immediate.

The necessary and sufficient condition for the topology to be Hausdorff is verified by observing that it is equivalent to saying that the intersection of all sets of the form $U(V, \mathfrak{F})$ is the diagonal of $G(S, E) \times G(S, E)$.

2.2 THEOREM. *If E is a locally convex linear topological space and $G(S, E)$ is a linear space of functions, then the topology of convergence on a filter $\mathfrak{F}$ makes $G(S, E)$ a locally convex topological group. The topology is linear if and only if for every f in $G(S, E)$ and every neighborhood V of the zero element of E, there exist an F in $\mathfrak{F}$ and a positive integer n such that $f(F) \subset nV$. (Note this does not in general imply that f is bounded on some member of $\mathfrak{F}$.)*

Proof. Let $W(V, \mathfrak{F})$ be a neighborhood of the zero function as determined by the entourages in Theorem 2.1. In other words, for each f in $W(V, \mathfrak{F})$ there is a F_f in $\mathfrak{F}$ such that $f(s)$ is in V for all s in F_f. When V is a closed convex circled neighborhood of the zero of E, $cW(V, \mathfrak{F}) = W(cV, \mathfrak{F})$ for all scalars c, and $dW(V, \mathfrak{F}) \subset W(V, \mathfrak{F})$ for all scalars d, $|d| \leqq 1$. It also follows that $W(V, \mathfrak{F})$ is convex because if f and g are in $W(V, \mathfrak{F})$, and c and d are non-negative numbers such that $c + d = 1$, there exists an F in $\mathfrak{F}$ such that $f(s)$ and $g(s)$ are in V for all s in F. Thus $cf(s) + dg(s)$ is in V for all s in F.

The above information about the sets of the type $W(V, \mathfrak{F})$ makes it straightforward to verify that $G(S, E)$ is a topological group for the resulting topology.

In order to see when the topology is linear consider an arbitrary f in $G(S, E)$ and $W(V, \mathfrak{F})$ as above. If there exist an F in $\mathfrak{F}$ and a positive integer n such that $f(F) \subset nV$ then f is in $nW(V, \mathfrak{F})$ and the topology is linear. Reverse the argument for the converse.

In the above theorems the uniform structure and topology on $G(S, E)$ were obtained by considering convergence on a single filter. There is no problem in extending these results to convergence on a family of filters. The results and their proofs are omitted due to their close analogy to the same situation in the theory of uniform convergence [9].

The final theorem of this section gives the relationship between the topology of almost uniform convergence [2] and the topology of convergence on a specific family of filters.

2.3 THEOREM. *On $G(S, E)$ the uniformity obtained from convergence on all ultra filters in S is the same as the uniformity of almost uniform convergence on S.*

2.4 COROLLARY. *On $G(S, E)$ the topology of convergence on all ultra filters in S is the same as the topology of almost uniform convergence.*

Proof of Theorem 2.3. Let V be an entourage from E and consider a subset $Q(V)$ of $G(S, E) \times G(S, E)$ having the following two properties. (i) For every finite set $(f_1, g_1), (f_2, g_2), \cdots, (f_k, g_k)$ from the complement of $Q(V)$ there is an s in S such that $(f_i(s), g_i(s))$ is not in V for $i = 1, 2, \cdots, k$. (ii) There is no proper subset of $Q(V)$ having property (i). The family of sets of the form $Q(V)$, V ranging over the entourages from E, is a base for the uniformity on $G(S, E)$ which is associated with the almost uniform convergence on S.

Consider an arbitrary ultra filter $\mathfrak{F}$ in S and the entourage $U(V, \mathfrak{F})$ from the uniformity of convergence on $\mathfrak{F}$. Let $\{(f_1, g_1), \cdots, (f_k, g_k)\}$ be a finite set from the complement of $U(V, \mathfrak{F})$. For each F in $\mathfrak{F}$, $(f_i(F), g_i(F))$ is not a subset of V for $i = 1, 2, \cdots, k$. Since $\mathfrak{F}$ is an ultra filter there is an F_0 in $\mathfrak{F}$ such that the intersection of $(f_i(F_0), g_i(F_0))$ and V is empty for $i = 1, 2, \cdots, k$. Thus $U(V, \mathfrak{F})$ contains an entourage from the uniformity of almost uniform convergence on S.

For the converse consider a set $Q(V)$ from the base for the uniformity of almost uniform convergence on S as described at the beginning of this proof. For each finite set $\{(f_1, g_1), \cdots, (f_k, g_k)\}$ from the complement of $Q(V)$ let

$$F = \{s : s \text{ in } S, (f_i(s), g_i(s)) \text{ not in } V \text{ for } i = 1, 2, \cdots, k\}.$$

Denote by $\mathfrak{F}_0$ an ultra filter in S containing all such sets. Thus $U(V, \mathfrak{F}_0)$ is a subset of $Q(V)$ and the proof is completed.

3. Applications

The first theorem of this section utilizes convergence on filters to characterize continuous vector-valued functions as the limit of finite linear combinations of scalar-valued functions with the coefficients being taken from the vector space. Convergence on a filter can be used to preserve only partial continuity as is seen in Theorem 3.2 for the case of upper semi-continuous functions. Theorem 3.3 is an example where convergence on a filter preserves discontinuities. The last theorem of the section is concerned with continuous extensions of functions.

3.1 THEOREM. *Consider a mapping T from a normal space S into a linear topological space E with real or complex scalars. Let $C(S, [0, 1])$ denote all continuous functions with domain S and range the closed unit interval. T is continuous if and only if there is a set A composed of finite linear combinations of functions from $C(S, [0, 1])$ with coefficients from E such that T is a cluster point of A for the topology of convergence on all convergent filters of neighborhoods in S.*

Proof. Let $\{\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_k\}$ be a finite collection of filters of neighborhoods converging to the distinct points $s_1, s_2, \cdots, s_k$ respectively, and let V denote a neighborhood of the zero element of E. Assuming the continuity of T along with the normality of S it is known that there exists a finite collection $\{F_1, F_2, \cdots, F_k\}$ of subsets of S such that their closures are pairwise disjoint, F_i is in $\mathfrak{F}_i$ and $T(F_i) - T(s_i)$ is a subset of V for $i = 1, 2, \cdots, k$. It also follows that there is a finite set $\{f_1, f_2, \cdots, f_k\}$ of functions from $C(S, [0, 1])$ such that $f_i(s) = 1$ for all s in F_i and $f_i(s) = 0$ for all s in F_j, $j \neq i$.

Let $K(s) = \sum_{i=1}^{k} T(s_i)f_i(s)$ and observe that $K(s) - T(s)$ is in V for s in the union of the F_i's, $i = 1, 2, \cdots, k$. In this manner the required set is constructed.

The converse follows from Theorem 1.3.

3.2 THEOREM. *Let $\mathcal{G}$ be a filter of real-valued functions which are upper semi-continuous at a point s_0 of their domain. If the filter $\mathcal{G}$ converges to f_0 on the filter $\mathfrak{F}$ of neighborhoods of s_0, then f_0 is upper semi-continuous at s_0.*

Proof. Given an arbitrary positive number ε there exist G in $\mathcal{G}$, g in G, and F in $\mathfrak{F}$ such that

$$| g(s) - f_0(s) | < \varepsilon/3 \quad \text{and} \quad g(s) - g(s_0) < \varepsilon/3$$

for all s in F. Putting this information together one has the following:

$$f_0(s) - f_0(s_0) \leqq f_0(s) - g(s) + g(s) - g(s_0) + g(s_0) - f_0(s_0) < \varepsilon$$

for all s in F. In other words, f_0 is upper semi-continuous at s_0.

3.3 THEOREM. *Let f_0 be a meromorphic function on a domain D and let $\mathcal{G}$ be a filter of meromorphic functions with the same domain. Consider a point z_0 in D and a filter $\mathfrak{F}$ composed of the neighborhoods of z_0 with z_0 deleted. The filter $\mathcal{G}$ converges to f_0 on the filter $\mathfrak{F}$ if and only if either* (a) *z_0 is not a pole of f_0, there is a G in $\mathcal{G}$ such that z_0 is not a pole of g for all g in G and the filter $\mathcal{G}$ converges to $f_0(z_0)$ at z_0, or* (b) *f_0 has a pole at z_0 with principal part P, there is a G in $\mathcal{G}$ such that the principal part of g at z_0 is P for all g in G, and the filter base having sets of the form $\{g(z_0) - P(z_0) : g \text{ in } G\}$ for all G in $\mathcal{G}$ converges to $f_0(z_0) - P(z_0)$.*

Proof. Assume convergence on the filter. Given a positive number ε there is a G in $\mathcal{G}$ such that for each g in G there exists an F_g in $\mathfrak{F}$ with the property that $| g(z) - f_0(z) | < \varepsilon$ for all z in F_g. Thus either g and f_0 are both continuous or both have the same principal part at z_0 for all g in G. The continuity of either g or $g - P$ at z_0 leads to the pointwise convergence at z_0. The converse follows from Theorem 1.3.

To place the following theorem in perspective one should view the Stone-Čech compactification of a completely regular space X as the embedding of X in a topological space such that all ultra filters converge and all continuous real-valued functions on X with bounded ranges can be continuously extended over the closure of X. The present theorem gives a necessary and sufficient condition for a localization in which only one filter in X is considered and a less restrictive class of functions is utilized.

3.4 THEOREM. *Consider a linear space $G(S, R)$ of real-valued functions and* *a filter $\mathfrak{F}$ of subsets of S. The topology of convergence on $\mathfrak{F}$ is linear if and only if S is a dense subset of a topological space X, every ultra filter in X which contains $\mathfrak{F}$ converges to a point of X, and every function in $G(S, R)$ can be continuously extended over X.*

Proof. Assuming the topology is not linear, there is an f in $G(S, R)$ such that for every positive integer m and every F in $\mathfrak{F}$ there can be found an s in F for which $|f(s)| > m$. Let

$$B_{F,m} = \{s : s \text{ in } F, |f(s)| > m\}.$$

Consider an ultra filter in S which contains these sets. Regardless of how S is embedded f can not be continuously extended to the limit of that ultra filter.

For the converse let X be the union of the set S and a set which is in one to one correspondence with the family of all ultra filters in S which contain $\mathfrak{F}$. Consider a point x in X. If it is in the subset S, define its neighborhoods to be all subsets of X containing it. If it is not in S, define its neighborhoods to be every set containing it and a set from the ultra filter which corresponds to it. Let $\mathfrak{F}_0$ be an ultra filter in X containing $\mathfrak{F}$. Then $\mathfrak{F}_0$ also contains an ultra filter in S which corresponds to a point x_0 in X. Because of the linearity there exists for each function f an F_f in $\mathfrak{F}$ such that $f(F_f)$ is contained in a compact set. Thus $\mathfrak{F}_0$ converges to x_0 and f can be continuously extended to x_0.

4. Simple equicontinuity

The classical definition for equicontinuity of a family H of functions at a point s is equivalent to saying that the filter of neighborhoods of s converge uniformly on H. Simple equicontinuity is obtained when uniform convergence is replaced by convergence on all ultra filters of H.

4.1 DEFINITION. A set H of functions from $J(S, E)$ has simple equicontinuity at s_0 relative to a given topology on S if the filter of neighborhoods of s_0 converges to s_0 on every ultra filter in H. A set H is said to have simple equicontinuity if it has simple equicontinuity at every point of S.

4.2 THEOREM. *Let S be a topological space and let H be a subset of $J(S, E)$. The following statements are equivalent.*

(i) *H has simple equicontinuity at s_0 in S.*

(ii) *The filter of neighborhoods of s_0 converges almost uniformly on H.*

(iii) *Every f in H is continuous at s_0 and for every subset B of S having s_0 as an accumulation point and every entourage U from E, there is a finite subset M of B such that for each f in H, $(f(s), f(s_0))$ is in U for at least one s in M.*

Proof. The equivalence of (i) and (ii) follows from Theorem 2.3.

Statement (ii) implies that the filter of neighborhoods of s_0 converges pointwise on H [2]. Thus every member of H is continuous at s_0. To obtain the remainder of statement (iii), assume that B is a subset of S having s_0 as an accumulation point and that V is an entourage for E. Let $\mathfrak{F}$ be a refinement of the filter of neighborhoods such that $\mathfrak{F}$ is eventually in B. Since $\mathfrak{F}$ converges almost uniformly to s_0 on H there is a finite subset $\{s_i : i = 1, 2, \cdots, k\}$ of B such that for every f in H, $(f(s_i), f(s_0))$ is in V for at least one s_i (see [2]).

Reversing the argument gives the converse.

Simple examples show that many sets having simple equicontinuity are not equicontinuous, yet some properties of equicontinuous sets are present in sets having simple equicontinuity (see Theorems 4.3 and 4.6).

4.3 THEOREM. *On a set H in $J(S, E)$ having simple equicontinuity the topologies of pointwise convergence and pointwise convergence on a dense subset are identical.*

Proof. Consider an arbitrary g_0 in H, s_0 in S, and entourage V from E. Let D be a dense subset of S and let $\mathfrak{F}$ be a filter base composed of the intersections of the neighborhoods of s_0 with D. Take an entourage U of E such that $U \circ U \circ U \subset V$. There is a set F in $\mathfrak{F}$ such that $(g_0(s), g_0(s_0))$ is in U for all s in F. Since the filter base converges almost uniformly on H, there is a finite subset $\{s_i : i = 1, 2, \cdots, k\}$ of $F \subset D$ such that for each f in H $(f(s_i), f(s_0))$ is in U for at least one s_i. It is now possible to conclude that

$$W = \{f \text{ in } H : (f(s_i),\ g_0(s_i)) \text{ in } U \text{ for } i = 1, \cdots, k\}$$

in a subset of $\{f \text{ in } H : (f(s_0), g_0(s_0)) \text{ in } V\}$ because for each f in W there is an s_i such that $(f(s_0), f(s_i))$, $(f(s_i), g_0(s_i))$, and $(g_0(s_i), g_0(s_0))$ are all in U.

The converse is immediate.

4.4 COROLLARY. *The topology of pointwise convergence is metrizable on a subset H of $J(S, E)$ if H has simple equicontinuity, S is separable, and E is metrizable.*

Before going further it is necessary to establish a duality. For this purpose replace the function space $G(S, E)$ by a single function φ having as its domain the cartesian product $G \times S$ of the two sets G and S. The range is in the uniform Hausdorff space E. The terminology of previous sections will be utilized by considering either the elements of G or S as functions on the other.

4.5 THEOREM. *Consider filters $\mathcal{G}$ and $\mathfrak{F}$ of subsets of G and S respectively with the property that $\varphi(\mathcal{G}, s)$ converges to (g_0, s) for all s in at least one F in $\mathfrak{F}$ and $\varphi(g, \mathfrak{F})$ converges to $\varphi(g, s_0)$ for all g in at least one D in $\mathcal{G}$. Then the filter $\mathcal{G}$ converges to g_0 on $\mathfrak{F}$ if and only if $\mathfrak{F}$ converges to s_0 on $\mathcal{G}$. (Note that g_0 and s_0 need not be in any member of $\mathcal{G}$ and $\mathfrak{F}$ respectively.)*

Proof. Assuming that the filter $\mathcal{G}$ converges to g_0 on $\mathfrak{F}$, it will be shown first that $\varphi(\mathcal{G}, s_0)$ is a Cauchy filter.

Given an entourage V from E, there is another entourage V' such that $V' \circ V' \circ V' \circ V' \subset V$. There exists a D in $\mathcal{G}$ such that for each g in D there is an F_g in $\mathfrak{F}$ with the result that $(\varphi(g, s), \varphi(g_0, s))$ is in V' for all s in F_g. Taking arbitrary g' and g'' in D there is an F in $\mathfrak{F}$ such that, $F \subset F_{g'}$, $F \subset F_{g''}$, and, $(\varphi(g', s), \varphi(g', s_0))$ and $(\varphi(g'', s), \varphi(g'', s_0))$ are in V' for all s in F. These facts combine to say that $(\varphi(g', s_0), \varphi(g'', s_0))$ is in V and thus the desired result.

For convenience let K be the limit of $\varphi(\mathcal{G}, s_0)$ in the completion of E.

Proceed next to show that the filter $\varphi(g_0, \mathfrak{F})$ also converges to K. To do this let V be an arbitrary entourage from E and V' another entourage such that $V' \circ V' \circ V' \subset V$. There is a D in $\mathcal{G}$ such that for each g in D there exists an F_g in $\mathfrak{F}$ such that $(\varphi(g, s_0), K)$ and $(\varphi(g, s), \varphi(g_0, s))$ are in V' for all g in D and all s in F_g. Each set F_g has a subset F'_g, F'_g in $\mathfrak{F}$, with the property that $(\varphi(g, s), \varphi(g, s_0))$ is in V' for all s in F'_g. These facts combine to say that $(\varphi(g_0, s), K)$ is in V for all s in F'_g. In other words, $\varphi(g_0, \mathfrak{F})$ converges to K.

For the final phase of the proof let V be an arbitrary entourage from E and V' another entourage such that $V' \circ V' \circ V' \subset V$. There are sets D and F from $\mathcal{G}$ and $\mathfrak{F}$ respectively such that $(\varphi(g, s_0), K)$ and $(\varphi(g_0, s), K)$ are in V' for all g in D and s in F. Taking an arbitrary s from F there exists a subset D_s of D such that $(\varphi(g, s), \varphi(g_0, s))$ is in V' for all g in D_s. Putting these results together it is concluded that for every s in F there is a D_s in $\mathcal{G}$ such that $(\varphi(g, s), \varphi(g, s_0))$ is in V for all g in D_s. In other words, the filter $\mathfrak{F}$ converges to s_0 on the filter $\mathcal{G}$.

With the above duality it is now possible to establish a basic theorem.

4.6 THEOREM. *If H is a subset of the set $J(S, E)$ of all functions from S into E, and H has simple equicontinuity at s_0, then the closure of H for the topology of pointwise convergence on S also has simple equicontinuity at s_0.*

Proof. Consider an arbitrary f in the closure of H, but not in H. There is an ultra filter $\mathcal{G}$ in H which converges to f for the topology of pointwise convergence. Theorem 4.2 says that the filter $\mathfrak{F}$ of neighborhoods of s_0 converges almost uniformly on H. Making use of Corollary 2.4 it is observed that the filter $\mathfrak{F}$ converges to s_0 on the ultra filter $\mathcal{G}$. Now apply the duality Theorem 4.5. Thus the filter $\mathcal{G}$ converges to f on the filter $\mathfrak{F}$. The function f is continuous at s_0 (see Theorem 1.3). In other words all the functions in the closure of H are continuous at s_0.

The remainder of the proof is obtained by viewing S as a collection of continuous functions defined on $J(S, E)$. From Theorem 4.1 of [2] it is deduced that the filter $\mathfrak{F}$, which converges almost uniformly on H, converges almost uniformly on the closure of H. Thus the closure of H has simple equicontinuity at s_0.

The way has now been prepared for the analogue of Ascoli's theorem for the topology of pointwise convergence [9].

4.7 THEOREM. *A set H in the space of all continuous functions defined on a topological space S with range in a uniform Hausdorff space E is relatively com-*

pact for the topology of pointwise convergence if and only if $H(s)$ is relatively compact for every s in S and H has simple equicontinuity.

Proof. Tychonoff's theorem gives the compactness of the closure of H in $J(S, E)$, while the preceding theorem guarantees the continuity of all functions in the closure.

4.8 LEMMA. *If a set H of functions has simple equicontinuity and $H(s)$ is totally bounded for all s in a dense subset of S then $H(s)$ is totally bounded for all s in S.*

Proof. Let V be an arbitrary entourage from E and V' another entourage such that $V' \circ V' \circ V' \subset V$. Consider an arbitrary s_0 in S. Since H has simple equicontinuity it follows from Theorem 4.2 that there is a finite set $(s_1, s_2, \cdots, s_k)$ such that

$$H = \bigcup_{i=1}^{k} H_i,$$

$$H_i = \{g : g \text{ in } H, (g(s_i), g(s_0)) \text{ in } V'\} \qquad \text{for } i = 1, 2, \cdots k,$$

where $\{s_1, s_2, \cdots, s_k\}$ is a subset of the dense subset of S for whose elements $H(s)$ is totally bounded. Thus $H_i(s_i)$ is totally bounded and there is a finite subset M_i of H_i such that for each g in H_i there is an f in M_i for which $(g(s_i), f(s_i))$ is in V', $i = 1, 2, \cdots, k$.

Consider an arbitrary g in H. At least one H_i contains g. Thus there is an f from the union of the M_i's such that $(g(s_i), f(s_i))$ and $(f(s_i), f(s_0))$ are in V'. Since $(g(s_i), g(s_0))$ is also in V', it follows that $(g(s_0), f(s_0))$ is in V. It is concluded that $H(s_0)$ is totally bounded.

When E is a linear topological space the words "totally bounded" in the above theorem may be replaced by the word "bounded". It should be noted that the above lemma does not follow from Theorem 4.3.

Lemma 4.8 and Theorem 4.7 are combined to give the following theorem.

4.9 THEOREM. *If E is a complete uniform Hausdorff space and H is a subset of $C(S, E)$, then H is relatively compact for the topology of pointwise convergence if and only if H has simple equicontinuity and $H(s)$ is totally bounded for all s in a dense subset of S.*

Combining the above results with Ascoli's theorem leads to further characterizations of compact sets, the following theorem being an example.

4.10 THEOREM. *Let $C(S, E)$ be the space of all continuous functions from the topological space S into the uniform space E. A subset H of $C(S, E)$ is relatively compact for the topology of compact convergence if and only if H has simple equicontinuity, the restriction of H to each compact subset of S is equicontinuous, and $H(s)$ is relatively compact for every s in S.*

5. Applications to linear spaces

Replace the sets S and E of the preceding sections by locally convex linear topological Hausdorff spaces F and E, respectively. The functions under consideration will now be linear and the space of all such functions from F into E is denoted by $L(F, E)$. The following lemma is a consequence of linearity.

5.1. LEMMA. *A subset H of $L(F, E)$ has simple equicontinuity if and only if it has simple equicontinuity at the zero element of F.*

5.2 LEMMA. *If a subset H of $L(F, E)$ has simple equicontinuity, then H is bounded for the topology of pointwise convergence.*

Proof. Assume that there is an x_0 in F such that $H(x_0)$ is not bounded. Thus there is an ultra filter $\mathcal{G}$ in H such that $\mathcal{G}(x_0)$ is a base for an unbounded ultra filter in E. In other words there is a neighborhood W of the origin in E such that for every non-negative real number k there is a D in $\mathcal{G}$ such that $D(x_0)$ is in the complement of kW.

Consider an arbitrary neighborhood V of x_0. There exists a real number $r > 1$ such that rx_0 is in V, and a D in $\mathcal{G}$ such that $D(x_0)$ is in the complement of $\frac{1}{r-1} W$. For arbitrary g in D, $g(x_0)$ is not in $\frac{1}{r-1} W$, and $g(rx_0) - g(x_0)$ is not in W. Thus the filter of neighborhoods of x_0 does not converge on the ultra filter $\mathcal{G}$ giving the desired conclusion.

Theorems 4.7 and 4.9 can be used in $L(F, E)$ because it is a closed subspace of the space $J(F, E)$ for the topology of pointwise convergence. Applications to the weak, weak*, weak operator and strong operator topologies are immediate. Henceforth E' denotes the set of all continuous scalar-valued functions defined on E.

5.3 LEMMA. *A subset H of E' has simple equicontinuity if and only if H has simple equicontinuity relative to every topology on E finer than the weak topology ($\sigma(E, E')$ topology) and coarser than the Mackey topology.*

The validity of the lemma is seen by observing that Lemma 5.2 makes it possible to utilize Theorem 4.7 which is applicable for every topology on E which makes E' the space of all continuous linear scalar-valued functions on E.

The above results now combine to give a strengthened version of the Alaoglu-Bourbaki theorem [6], [8].

5.4 THEOREM. *A subset H of E' is relatively weak* compact (relatively $\sigma(E', E)$ compact) if and only if H has simple equicontinuity.*

By considering the dual situation it is meaningful to speak of a subset of E as having simple equicontinuity. The adjoint space is given the weak* topology ($\sigma(E', E)$ topology) and E is all continuous linear scalar-valued functions on E'.

5.5 THEOREM. *A subset K of a locally convex linear topological space E is relatively compact for the weak topology if and only if K has simple equicontinuity.*

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