AN INVARIANCE PROPERTY OF SPECTRAL SYNTHESIS

BY

KAREL DE LEEUW AND CARL HERZ

1. In this paper we study the behavior under group representations of certain spaces of functions that arise in harmonic analysis. We indicate in this section the essential nature of the results we obtain.

Let G_1 and G_2 be locally compact abelian groups, $h : G_1 \to G_2$ a faithful representation, that is, a one-one continuous mapping that is an algebra homomorphism. For example, take G_1 to be the real line, G_2 the torus, and h an injection of G_1 as a dense one-parameter subgroup of G_2; or G_1 arbitrary, G_2 the Bohr compactification of G_1, and h the canonical imbedding of G_1 in G_2.

Let S_1 be a compact subset of G_1 and S_2 its image in G_2 under h.

If $A(S_i)$ is the space of functions in S_i that are restrictions of Fourier transforms in G_i, it is easy to demonstrate the inclusion

$$\{f \circ h : f \in A(S_2)\} \subset A(S_1). \tag{1.1}$$

That equality holds in (1.1) is the main point of Theorem 1 below.

Let $\hat{G}_i$ be the dual group of G_i, $\hat{h} : \hat{G}_2 \to \hat{G}_1$ the adjoint of $h : G_1 \to G_2$. If $B'(S_i)$ is the space of bounded continuous functions on $\hat{G}_i$ having spectrum contained in S_i, it is easy to establish the inclusion

$$\{\phi \circ \hat{h} : \phi \in B'(S_1)\} \subset B'(S_2). \tag{1.2}$$

That equality holds in (1.2) is the main point of Theorem 2 below.

From the fact that equality holds in (1.1) and (1.2) it is possible to conclude that either both S_1 and S_2 are sets of spectral synthesis, or neither is a set of spectral synthesis. A slightly more general result is Theorem 4 below.

A consequence of equality in (1.2) is the following, which we prove as Theorem 5. A bounded function on a discrete abelian group G, that is discontinuous in some locally compact topology on G, cannot have in its spectrum only characters continuous in that topology.

2. In this section we define the spaces of functions with which we shall be concerned.[1]

Let G be a locally compact abelian group, $\hat{G}$ its dual, $C(\hat{G})$ the Banach space of bounded continuous complex-valued functions on $\hat{G}$, $M(\hat{G})$ the Banach algebra of finite measures on $\hat{G}$ under convolution. We denote by $A(G)$ the Banach algebra of functions on G that are Fourier transforms of measures in $M(\hat{G})$. $A(G)$ is isometrically isomorphic to $M(\hat{G})$ under the Fourier transform mapping $\mu \to \hat{\mu}$.

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[1] These spaces all occur in [H]. There, the spaces we denote by $A(S)$, $A'(S)$ and $B'(S)$ are denoted by $V(S)$, $E(S)$ and $E'(S)$ respectively.

Each compact subset S of G has intrinsically associated to it certain quotient algebras $A(S)$ and $B(S)$ of $A(G)$, and certain linear subspaces $A'(S)$ and $B'(S)$ of $C(\hat{G})$, which are defined below. The purpose of this paper is to establish, in a sense to be made precise, that these spaces do not depend on the group G in which S is imbedded.

1°. $A(S)$ is defined to be the quotient algebra

$$A(G)/\{f \in A(G) : f = 0 \text{ on } S\}.$$

$A(S)$ can be identified with the algebra of restrictions to S of functions in $A(G)$. Since S is compact, the same algebra is obtained whether we use all of $M(\hat{G})$ or only the subalgebra $L^1(\hat{G})$ of absolutely continuous measures (see Theorem 4.1(i) of [H]).

2°. $B(S)$ is defined to be the quotient of $A(G)$ modulo the closure of the ideal

$$\{f \in A(G) : f = 0 \text{ in some neighborhood of } S\}.$$

3°. $A'(S)$ is defined to be the closed linear subspace of $C(\hat{G})$ consisting of those ϕ satisfying

$$\int_{\hat{G}} \phi \, d\mu = 0$$

for all μ in $M(\hat{G})$ with $\hat{\mu} = 0$ on S. $A'(S)$ consists of those functions in $C(\hat{G})$ that can be "synthesized" from the characters of $\hat{G}$ corresponding to points of S (see [H, p. 186]). $A'(S)$, in the supremum norm, is naturally isomorphic and isometric to the dual space $A(S)^*$ under the pairing $\langle \cdot , \cdot \rangle$ defined by

$$\langle f, \phi \rangle = \int_{\hat{G}} \phi \, d\mu, \qquad f \in A(S), \phi \in A'(S),$$

where μ is chosen to be any measure in $M(\hat{G})$ with $\hat{\mu} = f$ on S (see Theorem 4.1 (i) of [H]).

4°. $B'(S)$ is defined to be the closed linear subspace of $C(\hat{G})$ consisting of those ϕ satisfying

$$\int_{\hat{G}} \phi \, d\mu = 0$$

for all μ in $M(\hat{G})$ with $\hat{\mu} = 0$ on some neighborhood of S. $B'(S)$ consists of those functions in $C(\hat{G})$ having spectrum in S (see [H, p. 186]). $B'(S)$, in the supremum norm, is naturally isomorphic and isometric to the dual space $B(S)^*$ under the pairing $\langle \cdot , \cdot \rangle$ defined by

$$\langle F, \phi \rangle = \int_{G} \phi \, d\mu, \qquad F \in B(S), \phi \in B'(S),$$

where μ is chosen to be any measure in $M(G)$ belonging to coset F (see Theorem 4.2 of [H]).

Obviously $A'(S)$ is a subset of $B'(S)$. S is called a set of *spectral synthesis* if there is equality, $A'(S) = B'(S)$.

3. We next investigate the behavior under group representations of the spaces defined in the preceding section.

Suppose that G_1 and G_2 are locally compact abelian groups and $h : G_1 \to G_2$ a representation, that is, a continuous mapping that is an algebraic homomorphism. Let S_1 be a compact subset of G_1 and $S_2 = h(S_1)$. h induces in a natural manner the mappings

$$
\begin{aligned}
&A(S_1) \xleftarrow{\alpha} A(S_2), \qquad B(S_1) \xleftarrow{\beta} B(S_2),\\
&A'(S_1) \xrightarrow{\alpha'} A'(S_2), \qquad B'(S_1) \xrightarrow{\beta'} B'(S_2).
\end{aligned}
\tag{3.1}
$$

These mappings are defined as follows. We denote by $\hat{h} : \hat{G}_2 \to \hat{G}_1$ the representation adjoint to $h : G_1 \to G_2$. α and β are obtained by taking quotients in the mapping

$$A(G_2) \to A(G_1)$$

which is the Fourier transform of the mapping

$$M(\hat{G}_2) \to M(\hat{G}_1)$$

defined by taking each measure on $\hat{G}_2$ into its image on $\hat{G}_1$ under $\hat{h}$. It is possible to define α directly by

$$\alpha(f) = f \circ h, \qquad f \in A(S_2). \tag{3.2}$$

α' and β' are defined as restrictions of the mapping

$$C(\hat{G}_1) \to C(\hat{G}_2) \tag{3.3}$$

induced by $\hat{h}$. (That (3.3) takes $A'(S_1)$ into $A'(S_2)$ and $B'(S_1)$ into $B'(S_2)$ is shown on p. 216 of [H].)

It is simple to check that the mappings of (3.1) are linear and do not increase norm. Also that α' is dual to α and β' is dual to β, that is

$$
\begin{aligned}
\langle \alpha(f), \phi \rangle &= \langle f, \alpha'(\phi) \rangle, \qquad f \in A(S_2),\ \phi \in A'(S_1),\\
\langle \beta(F), \phi \rangle &= \langle F, \beta'(\phi) \rangle, \qquad F \in B(S_2),\ \phi \in B'(S_1).
\end{aligned}
$$

Finally, α is one-one, because of (3.2) and the fact that h maps S_1 onto S_2.

No further assertion concerning the mappings (3.1) can be made without some restriction on the kernel of h. So let us assume that h is faithful. Then the image $\hat{h}(\hat{G}_2)$ is dense in $\hat{G}_1$, and as a consequence the mapping α', being a restriction of (3.3), is isometric. So we have a one-one mapping α whose dual α' is isometric. It is a consequence of Banach space theory (see [DS, Section VI. 6]) that both α and α' must then be onto and isometric.

We have established the following:

THEOREM 1. *Let $h : G_1 \to G_2$ be a faithful representation. If S_1 is a compact* *subset of* G_1 *and* $S_2 = h(S_1)$, *the mappings*

$$A(S_1) \xleftarrow{\alpha} A(S_2), \qquad A'(S_1) \xrightarrow{\alpha'} A'(S_2)$$

are isometric Banach space isomorphisms.

That α is onto, which is the essential point of the above, can be restated as follows. For each measure μ in $M(\hat{G}_1)$ one can find a measure μ_2 in $M(\hat{G}_2)$ satisfying $\hat{\mu}_1 = \hat{\mu}_2 \circ h$ on S_1. Our proof involves the use of the Hahn-Banach theorem, and seems to yield no information concerning the problem of finding such a μ_2, given μ_1.

4. We next take up the analogue of Theorem 1 for β and β'. That result would be

THEOREM 2. *Let* $h : G_1 \to G_2$ *be a faithful representation. If* S *is a compact subset of* G_1 *and* $S_2 = h(S_1)$, *the mappings*

$$B(S_1) \xleftarrow{\beta} B(S_2), \qquad B'(S_1) \xrightarrow{\beta'} B'(S_2)$$

are isometric Banach space isomorphisms.

This result appears to be deeper than Theorem 1. We have not been able to find as direct a proof as we have given for that theorem. β' is isometric, as was α', since $\hat{h}(\hat{G}_2)$ is dense in $\hat{G}_1$. It is not obvious, however, that β is one-one, as was the case with α. That α was one-one followed from the fact that α is the mapping on functions dual to the map $h : S_1 \to S_2$, and h takes S_1 onto S_2. On the contrary, β is not in general a map of functions; so one cannot conclude a priori that it is one-one.

Theorem 2 will be a consequence of results that we establish below.

5. If we drop the assumption that h is faithful, but still assume that it is one-one on S_1, the mapping

$$\alpha : A(S_2) \to A(S_1)$$

is one-one and takes $A(S_2)$ onto a separating subalgebra of $A(S_1)$. Still, α need not be onto.

Example. Take $G_1 = T^2$, $G_2 = T^1$ where T denotes the additive group of reals modulo 2π. Let F be a twice differentiable real-valued function defined on T which is not linear. We put

$$S_1 = \{(x, y) \in T^2 : y = F(x)\},$$

$S_2 = T^1$, and $h : T^2 \to T^1$ the projection map $h(x, y) = x$. It is known (see [K], esp. Th. I, a result due to Leibenson) that given F there exists a sequence $\{a_n\}$ such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$ but $\sum_{-\infty}^{\infty} a_n \exp\{inF(x)\}$ does not have an absolutely convergent Fourier series. Let ψ be the element of $A(S_1)$ determined by $\psi(x, y) = \sum_{-\infty}^{\infty} a_n \exp(iny)$ for $(x, y) \in S_1$. An element $\phi \in A(S_2) =$

$A(T)$ is an absolutely convergent Fourier series

$$\phi(x) = \sum_{-\infty}^{\infty} b_n \exp(inx), \qquad \sum_{-\infty}^{\infty} |b_n| < \infty.$$

Were it the case that $\psi = \alpha(\phi)$ we should have

$$\sum a_n \exp\{inF(x)\} = \sum b_n \exp(inx)$$

which contradicts the assumed property of $\{a_n\}$. Hence, the element $\psi \in A(S_1)$ is not in the image of $A(S_2)$ under α.

It is even easier to see that the mapping

$$\beta : B(S_2) \to B(S_1)$$

need not be onto.

Example. Let h be the projection of R^3 onto R^2 and let S_1 be a hemisphere in R^3 projecting one-one on a disk S_2 in R^2. There is a commutative diagram

$$\begin{tikzcd} B(S_2) \arrow[r, "\beta"] \arrow[d, "\pi_2"'] & B(S_1) \arrow[d, "\pi_1"] \\ A(S_2) \arrow[r, "\alpha"] & A(S_1) \end{tikzcd}$$

where π_1 and π_2 are the natural quotient mappings. S_2 is a set of spectral synthesis; so π_2 is one-one. S_1 is not a set of spectral synthesis; so π_1 is not one-one; but α is one-one. Therefore, no non-trivial element of the kernel of π_1 can be in the range of β.

If in addition to h being one-one on S we assume that the kernel of h is discrete, both α and β must be onto. This is our main result.

THEOREM 3. *Let $h : G_1 \to G_2$ be a representation. Suppose that S_1 is a compact subset of G_1 such that h is one-one on a neighborhood of S_1. Let $S_2 = h(S_1)$. Then the mappings*

$$A(S_1) \xleftarrow{\alpha} A(S_2), \qquad B(S_1) \xleftarrow{\beta} B(S_2),$$

$$A'(S_1) \xrightarrow{\alpha'} A'(S_2), \qquad B'(S_1) \xrightarrow{\beta'} B'(S_2)$$

are Banach space isomorphisms.

The above theorem is of interest even in classical analysis. Consider an example. $G_1 = R^1$, the real line, $G_2 = T^1$, and circle $=$ reals modulo 2π, $h : G_1 \to G_2$ the natural projection, and $S_1 = [-\tau, +\tau]$, with $0 \leq \tau < \pi$. It is easy to see that $B'(S_1)$ consists of the entire complex analytic functions of exponential type $\leq \tau$ which are bounded on the real axis. The mapping

$$\beta' : B'(S_1) \to B'(S_2)$$

associates to a function its sequence of values at integer points. Theorem 3 gives the known result that this map is an isomorphism.

Theorem 3 has as a corollary:

THEOREM 4. *Let $h : G_1 \to G_2$ be a representation. Suppose that S_1 is a compact subset of G_1 such that h is one-one on a neighborhood of S_1. Let $S_2 = h(S_1)$. Then either S_1 and S_2 are both sets of spectral synthesis or both not sets of spectral synthesis.*

Proof. Since α' and β' are one-one and onto, $A'(S_1) = B'(S_1)$ if and only if $A'(S_2) = B'(S_2)$.

For faithful representations, that S_2 a set of spectral synthesis implies S_1 a set of spectral synthesis has been obtained independently by I. Glicksberg. For representations that are homomorphisms (that is, have $h(G_1)$ closed in G_2 and $h(U)$ relatively open in $h(G_1)$ for each open U in G_1), Theorem 4 is due to Reiter (see [R] or Theorem 6.1 of [H]).

6. The proof of Theorem 3 is rather long. It consists of a piecemeal attack on the problems.

If $h : G_1 \to G_2$ is a representation, we shall write $T(h)$ to mean that the statement of Theorem 3 is valid for h.

h is called a *homomorphism* if $h(G_1)$ is closed in G_2 and the image of an open set in G_1 is relatively open in $h(G_1)$, a *monomorphism* if it is a one-one homomorphism, and an *epimorphism* if it is an onto homomorphism.

The order of the steps in the proof is

PROPOSITION 1. *If h is a monomorphism then $T(h)$.*

PROPOSITION 2. *If h is an epimorphism then $T(h)$.*

PROPOSITION 3. *If h is the imbedding of a group into its Bohr compactification then $T(h)$.*

PROPOSITION 4. *If h is a faithful representation of R^n into a compact group then $T(h)$.*

PROPOSITION 5. *Let $G_1 = R^n \otimes K$ where K is a compact group and suppose $h : G_1 \to G_2$ is a faithful representation into a compact group. Then, if $S = Q \otimes K$ where Q is a cube in R^n, the set hS is a set of spectral synthesis.*

PROPOSITION 6. *Let $h : G_1 \to G_2$ be faithful. Then, if S is a compact subset of G_1, there exists a compact neighborhood U of S such that hU is a set of spectral synthesis in G_2.*

PROPOSITION 7. *Proposition 6 implies Theorem 2.*

Theorem 3 follows easily from Theorems 1 and 2 via Proposition 2. For, denoting the kernel of h by D, $h : G_1 \to G_2$ can be factored as

$$G_1 \xrightarrow{\quad j \quad} G_1/D \xrightarrow{\quad k \quad} G_2,$$

with k faithful, j an epimorphism.

Proposition 1 is due to Reiter (see [R]). Proposition 2 is also known (see Theorem 5.9 of [H]).

7. Propositions 3 and 4 have similar proofs.

Proof of Proposition 3. Let G^* be the Bohr compactification of G and $h : G \to G^*$ the canonical imbedding. Let us assume provisionally that G is compactly generated. It then follows that given S compact in G, there exists a closed discrete subgroup D of G such that G/D is compact and the vector difference $S - S$ meets D only in the identity. Let D^* be the closure of hD in G^*. We have a sequence of maps

$$G \xrightarrow{\ e\ } G \otimes D^* \xrightarrow{\ f\ } G \otimes D^*/D' \xrightarrow{\ g\ } G^*$$

where e is the injection of G into the direct product, D' is the subgroup of $G \times D^*$ consisting of elements of the form (x, hx) with $x \in D$, f is the projection map, and g is defined by $g(x, y) = hx - y$ for $x \in G$, $y \in D^*$ so that g is defined on $G \otimes D^*/D'$. Let us observe that D' is a closed discrete subgroup of $G \otimes D^*$, for if U is a neighborhood of the identity in G meeting D only in the identity then $U \otimes D^*$ is a neighborhood of the identity in $G \otimes D^*$ meeting D' only in the identity. Thus f is an epimorphism. It is clear that f is one-to-one on a neighborhood of eS because of the way D was chosen.

If M is a compact subset of G with $M + D = G$, then

$$(M \otimes D^*) + D' = G \otimes D^*,$$

so $G \otimes D^*/D'$ is compact. Therefore g is a homomorphism (any representation of a compact group is a homomorphism). Finally, g is one-one. For let $D^\perp$ be the annihilator in $\hat{G}$ of D. G^* is the Bohr compactification of G, so $\hat{h}$ maps $G^{*\wedge}$ onto $\hat{G}$. Thus $\hat{h}^{-1}D^\perp$ is the annihilator of D^* in $G^{*\wedge}$. So if $x \in G$ and $hx \in D^*$, x is annihilated by every character in $D^\perp$, and thus must be in D. This shows that hG and D^* meet only in points of hD; equivalently, g is one-one.

It is obvious that $h = gfe$. We have $T(e)$ and $T(g)$ by Proposition 1. $T(f)$ is valid by Proposition 2 since f is one-one on a neighborhood of $e(S)$. This completes the proof of Proposition 3 in the case that G is compactly generated.

If G is not compactly generated, let H be the smallest subgroup of G containing a given compact neighborhood of the set S in question. H is compactly generated. It is open in G and hence closed. We have the commutagive diagram.

$$\begin{array}{ccc} G & \xrightarrow{\ h\ } & G^* \\ \uparrow & & \uparrow \\ H & \xrightarrow{\ \bar{h}\ } & H^*. \end{array}$$

The vertical arrows are monomorphisms and we have just proved $T(\bar{h})$. Thus $T(h)$ follows from Proposition 1.

Proof of Proposition 4. The crucial point of the argument of Proposition 3 is this: we have $h : G \to K$ a faithful representation of a compactly generated group G into a compact group K; thus $\hat{h}$ maps $\hat{K}$ onto a dense subgroup of $\hat{G}$, but we had to know that $D^{\perp} \subset \hat{h}\hat{K}$ for suitable discrete subgroups $D^{\perp}$ of $\hat{G}$.Unfortunately it is not clear every time we have a representation $\hat{h} : \hat{K} \to \hat{G}$ of a discrete group onto a dense subset that $\hat{h}\hat{K}$ contains a closed discrete subgroup $D^{\perp}$ of $\hat{G}$ such that $\hat{G}/D^{\perp}$ is compact. In case $G = R^n$, however, there is no problem. Let the compact set S be given and choose a number s so that S is contained in the interior of the coordinate cube of side $2s$. Since $h\hat{K}$ is dense in $\hat{G} = R^n$, we can find $\xi_1, \cdots \xi_n \in \hat{h}\hat{K}$ such that ξ_1 is within $\pi/3s$ of the point $(2\pi/3s, 0, 0, \cdots, 0)$, ξ_2 is within $\pi/3s$ of the point $(0, 2\pi/3s, 0, 0, \cdots, 0)$, etc. The points $\xi_1, \cdots \xi_n$ generate a closed discrete subgroup $D^{\perp}$ which is the annihilator in $\hat{G}$ of a subgroup D of G having all the required properties. We have

$$G \xrightarrow{e} G \otimes D^* \xrightarrow{f} G \otimes D^*/D' \xrightarrow{g} K$$

where D^* is the closure of hD in K. Again e and g are monomorphisms (hG and D^* meet only in hD) and f is one-to-one on a neighborhood of S.

8. Since we don't see how to prove the analogue of Proposition 4 directly in case G is not of the form R^n, we have to take an awkward detour. Reiter [R] has established

LEMMA 1. *Let $h : G_1 \to G_2$ be an epimorphism and T a closed subset of G_2. Then T and $h^{-1}T$ are sets of spectral synthesis or not simultaneously.*

Proof of Proposition 5. We have $h : R^n \otimes K \to G_2$ is a faithful representation where G_2 is compact. Thus there is a natural faithful representation $g : R^n \to G_2/hK$. (hK is a closed subgroup of G_2 since K is compact.) Let Q be a compact cube in R^n. Q is a set of spectral synthesis, and therefore, by Proposition 4, gQ is a set of spectral synthesis in G_2/hK. Let $\pi : G_2 \to G_2/hK$ be the projection. We have $h(Q \otimes K) = \pi^{-1}(gQ)$. Lemma 1 applies to π so that $h(Q \otimes K)$ is a set of spectral synthesis in G_2.

Proof of Proposition 6. Here we have $h : G_1 \to G_2$ a faithful representation where G_1 and G_2 are arbitrary locally compact abelian groups. We are given a compact set S in G_1. By structure theory, the closed subgroup of G_1 generated by S is of the form $R^n \otimes K \otimes D$ where K is compact and D is discrete. Thus there are a finite number of compact cubes $Q_1, \cdots Q_m \subset R^n$ and points $x_1, \cdots x_m \in D$ such that

$$S \subset \text{Int} \bigcup_{i=1}^{m}(x_i + Q_i \otimes K).$$

Let U be this union. U is a compact neighborhood of S, and we wish to show that hU is a set of spectral synthesis in G_2. It suffices to show that $h(Q \otimes K)$ is a set of spectral synthesis whenever Q is a compact cube in R^n, for spectral synthesis is preserved by translation and disjoint union of compact sets. We have a sequence of maps

$$R^n \otimes K \to G_1 \xrightarrow{h} G_2 \to G_2^*$$

where G_2^* is the Bohr compactification of G_2. According to Proposition 5, the image of $Q \otimes K$ is a set of spectral synthesis in G_2^*. Therefore, by Proposition 3, the image of $Q \otimes K$ in G_2 is a set of spectral synthesis.

9. We have used Propositions 3, 4, and 5 to get to Proposition 6. There may be a simple direct proof of Proposition 6 which would eliminate these steps. At any rate, we can pass from Proposition 6 to Theorem 2 directly.

Proof of Proposition 7. Since the mapping

$$\beta' : B'(S_1) \to B'(S_2)$$

is an isometry, to establish Theorem 2 it is sufficient to show that β' is onto. Let U_1 be a compact neighborhood of S_1 such that $U_2 = h(U_1)$ is a set of spectral synthesis. Then $A'(U_2) = B'(U_2)$, which contains $B'(S_2)$. Since U_1 is a neighborhood of S_1, $A'(U_1)$ contains $B'(S_1)$ (see Theorem 1.3 of [H]). Let ϕ be some function in $B'(S_2)$. We must find a function ψ in $B'(S_1)$ with $\beta'(\psi) = \phi$. Since $\phi \in B'(S_2) \subset A'(U_2)$, it is a consequence of Theorem 1 that there is a function ψ in $A'(U_1)$ with $\phi = \psi \circ \hat{h}$. Since $\phi = \psi \circ \hat{h}$,

$$\text{spectrum}\, \psi = h^{-1}\, (\text{spectrum}\, \phi) \subset h^{-1}(S_2) = S_1 .$$

(See Lemma 5.6 of [H].) Thus $\psi \in B'(S_1)$ and $\phi = \beta'(\psi)$ is simply a restatement of $\phi = \psi \circ \hat{h}$.

10. It is tempting to try to arrange matters so that Theorem 4 can be proved directly. Indeed the heart of the matter is to prove that if $h : G_1 \to G_2$ is a faithful representation and S is a compact set of spectral synthesis in G_1, then hS is a set of spectral synthesis in G_2. Now we always have a commutative diagram

$$\begin{tikzcd} G_1^* \arrow[r, "h^*"] & G_2^* \\ G_1 \arrow[u] \arrow[r, "h"] & G_2 \arrow[u] \end{tikzcd}$$

where $*$ denotes Bohr compactification. Let S^* be the image of S in G_1^*. We know by Proposition 4 that S^* is a set of spectral synthesis. Unfortunately, the representation h^* need not be one-to-one. The kernel is a subgroup K of G_1^* no coset of which contains two elements of S^*. It follows easily from Lemma 1 and Proposition 1 that h^*S^* is a set of spectral synthesis if and only if $S^* \oplus K$ is. The following seems plausible.

CONJECTURE. *The direct sum of a compact set of spectral synthesis and a closed subgroup is a set of spectral synthesis.*

What we do know is that if S is a compact set of spectral synthesis in G, K is a closed subgroup of G, and the group generated by S meets K only in the identity, then $S \oplus K$ is a set of spectral synthesis. This follows from Theorem 4 by considering the map $H \otimes K \rightarrow G$ where H is the group generated by S.

11. We establish here the result concerning the spectrum of discontinuous functions mentioned at the end of the first section.

THEOREM 5. *Let G be a locally compact abelian group, G^d the same group in the discrete topology. Let ϕ be a bounded function on G^d. If the only characters in the spectrum of ϕ are continuous on G, then ϕ must itself be continuous on G.*

Proof. Let $\hat{G}$ be the dual of G, $\hat{G}^*$ its Bohr compactification, with the canonical imbedding

$$\hat{G} \xrightarrow{h} \hat{G}^*.$$

Dual to this is the natural mapping

$$G \xleftarrow{\hat{h}} G^d.$$

Suppose now that ϕ in $C(G^d)$ has spectrum T satisfying the hypothesis of the theorem, that is, $T \subset h(\hat{G})$. Let $S = h^{-1}(T)$. T is a compact subset of $\hat{G}^* \cap h(\hat{G})$, so by Theorem 1.2 of [G], S is compact in $\hat{G}$. Thus we are in the situation of Theorem 2, and the mapping

$$\beta' : B'(S) \rightarrow B'(T)$$

will be onto. Since $\phi \in B'(T)$, there is a function ψ in $B'(S)$ with $\beta'(\psi) = \phi$, that is, $\phi = \psi \circ \hat{h}$. In other words, ϕ is continuous in the topology of G.

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STANFORD UNIVERSITY
PALO ALTO, CALIFORNIA
CORNELL UNIVERSITY
ITHACA, NEW YORK