AN EXAMPLE OF NON-LOCALIZATION FOR FOURIER SERIES ON $SU(2)$

BY

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Let $G = SU(2)$ and for each integer $n > 0$ let χ_n be the n-dimensional irreducible character of G. Any function $f \in L^1(G)$ has a Fourier series

$$f \sim \sum_{n=1}^{\infty} P_n f, \qquad P_n f = f * n\chi_n$$

where $*$ denotes convolution. Let N be a subset of G and f a measurable function on G. We will say that f *lives on* N if f vanishes on the complement N' of N.

The Riemann localization theorem says that if x is any point of the circle group $\mathbf{T}$, then any integrable function on $\mathbf{T}$ which vanishes on a neighborhood of x has a convergent Fourier series at x. In [4], Theorem C, it was shown that the analogous theorem for $G = SU(2)$ fails in a strong way: if $y \in G$ and V is any neighborhood of y such that V' has an interior, then there is a function g of bounded variation on G such that g lives on V' and the Fourier series for g diverges at y. In this paper we will show that the function g can be chosen so that its Fourier series diverges at y and $-y$ *and nowhere else.* (It follows from Lemma 1 below that if g vanishes near y and the Fourier series for g diverges at y then the Fourier series for g must also diverge at $-y$.)

THEOREM. *Let $x_0 \in G$ and let N be any non-void open subset of G. Then there exists a bounded function f of bounded variation on G, such that f lives on N, f is infinitely differentiable except on a closed set of measure zero, and the Fourier series for f diverges on $\{x_0\} \cup \{-x_0\}$ and converges to f everywhere else. If f is a function in $L^1(G)$ such that f vanishes near x_0 and the Fourier series for f diverges at x_0, then the Fourier series for f also diverges at $-x_0$. Thus the set $\{x_0\} \cup \{-x_0\}$ in the conclusion of the theorem cannot be replaced by $\{x_0\}$.*

Proof of the theorem. Without loss of generality we assume that $x_0 = e$ is the identity for G. Let

$$\theta(x) = \arccos \tfrac{1}{2}\chi_2(x), \qquad x \in G.$$

Choose $a \in N$ such that $a \neq \pm e$ and $\theta(a) \neq \pi/2$. For $r > 0$ let

$$B_r(a) = \{x \in G : \theta(x^{-1}a) < r\}$$

and let

$$S_a = \{x \in G : \theta(x) = \theta(a)\}.$$

Choose $\varepsilon > 0$ so that $B_\varepsilon(a) \subset N$ and $(B_\varepsilon(a))^- \cap \{e, -e\} = \emptyset$ (where the bar

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denotes closure). By compactness of S_a choose $s_1, \cdots, s_k \in S_a$ so that

$$\bigcup_{n=1}^{k} B_\varepsilon(s_n) \supset S_a .$$

Then $B_\varepsilon(s_1), \cdots, B_\varepsilon(s_k), S_a'$ is an open cover for G. Let $f_1, \cdots, f_{k+1}$ be a C^∞ partition of unity subordinate to this cover, so $\operatorname{supp} f_i \subset B_\varepsilon(s_i)$, $1 \leq i \leq k$, and $\operatorname{supp} f_{k+1} \subset S_a'$. Let λ_a be the function on G defined by

$$\begin{aligned} \lambda_a(x) &= 0 \quad \text{if} \quad \chi_2(x) < \chi_2(a) \\ &= \tfrac{1}{2} \quad \text{if} \quad \chi_2(x) = \chi_2(a) \\ &= 1 \quad \text{if} \quad \chi_2(x) > \chi_2(a). \end{aligned} \tag{1}$$

Then λ_a is infinitely differentiable except on S_a, and in [4], Lemma 3.30, it is shown that the Fourier series for λ_a diverges at $\pm e$ and converges to λ_a everywhere else. Let $g_n = f_n \lambda_a$ $(1 \leq n \leq k+1)$ so that

$$\lambda_a = \sum_{n=1}^{k+1} g_n .$$

Since g_{k+1} is a C^∞ function it has an everywhere convergent Fourier series, and it follows that some g_j $(1 \leq j \leq k)$ has a divergent Fourier series at e. Since $\theta(s_j) = \theta(a)$ we have $s_j = uau^{-1}$ for some $u \in G$. Now define

$$f(x) = g_j(uxu^{-1}) = f_j(uxu^{-1})\lambda_a(x).$$

Then f lives on $B_\varepsilon(a)$ and hence f vanishes near $\pm e$. Also

$$\sum_{k=1}^{n} P_k f(e) = \sum_{k=1}^{n} P_k g_j(e) \qquad \text{for all } n$$

so f has a divergent Fourier series at e. In Section 4 of [4] it is shown that all of the first order derivatives of λ_a are measures, and hence λ_a is a function of bounded variation. Since f is the product of λ_a and a C^∞ function, it follows that f is a function of bounded variation (it was observed in [4] that the functions of bounded variation form a module over the C^∞ functions). Also f is clearly infinitely differentiable off of S_a which is a closed set of measure zero. Hence the theorem will follow if we prove the following two lemmas.

LEMMA 1. *Let $f \in L^1(G)$. If f vanishes near $b \in G$ and the Fourier series for f diverges at b then the Fourier series for f also diverges at $-b$.*

LEMMA 2. *Let a be an element of G such that $\theta(a) \neq \pi/2$, let λ_a be as in (1) and let $g \in C^\infty(G)$. Then the Fourier series for $g\lambda_a$ converges to $g\lambda_a$ except possibly at $\pm e$.*

Proof of Lemma 1. If $f \in L^1(G)$, the Riemann Lebesgue set for f is

$$r(f) = \{x \in G : \lim_{n\to\infty} P_n f(x) = 0\}.$$

If f vanishes near b and the Fourier series for f diverges at b, then it follows from Theorem C of [5] that $b \notin r(f)$. Let U_n be an irreducible n dimensional matrix representation of G. Then $U_n(-e) = (-1)^{n+1} I_n$ where I_n is the

$n \times n$ identity matrix, so $U_n(-b) = (-1)^{n+1}U_n(b)$ for all $b \in G$. Since $P_n f$ is a linear combination of the coordinates of U_n it follows that $P_n f(-b) = (-1)^{n+1}P_n f(b)$ for all n. Hence $-b \notin r(f)$ and hence the Fourier series for f diverges at $-b$.

The proof of Lemma 2 will require a number of preliminary lemmas, and before considering these lemmas we give a general outline of the proof.

First we show that if g is in the representative ring of G then $r(g\lambda_a)$ contains all points of G except possibly $\pm e$ (Lemmas 3–6). From this we will conclude that the Fourier series for $g\lambda_a$ converges to $g\lambda_a$ except possibly at $\pm e$ for any such g. Next we show that if $b \neq \pm e$ is an element of G which is not conjugate to $-a$, and h is any function in $C^\infty(G)$ which vanishes at b together with all of its derivatives of order ≤ 6, then the Fourier series for $h\lambda_a$ converges to 0 at b. Since any $h \in C^\infty(G)$ can be written $h = h_1 + h_2$ where h_1 is in the representative ring of G and h_2 vanishes at b together with its derivatives of order ≤ 6 (Lemma 10), we conclude that for any $h \in C^\infty(G)$ the Fourier series for $h\lambda_a$ converges to $h\lambda_a$ except possibly at $\pm e$ and on the set S_{-a} of points conjugate to $-a$. Since $\theta(a) \neq \pi/2$, a and $-a$ are not conjugate, and the Fourier series for $h\lambda_a$ converges on S_a. Using this fact we show that the Fourier series for $h\lambda_a$ must also converge on S_{-a}, and Lemma 2 follows.

The Lie algebra $\mathfrak{g}$ of G is isomorphic to the Lie algebra $\mathfrak{g}'$ of 2×2 skew Hermitian matrices with zero trace under the map $M \rightarrow D_M$ where

$$D_M f(x) = \frac{d}{dt} f(x \exp tM)\,|_{t=0}, \qquad M \in \mathfrak{g}', D_M \in \mathfrak{g}, f \in C^\infty(G). \tag{2}$$

Since χ_2 has a maximum at e, $D\chi_2(e) = 0$ for all $D \in \mathfrak{g}$. It is easy to verify that

$$(D_M)^2\chi_2 = -(\det M)\chi_2, \qquad M \in \mathfrak{g}'. \tag{3}$$

Let M_1, M_2, M_3 be a basis for $\mathfrak{g}'$, and let $D_i = D_{M_i}$ $(1 \leq i \leq 3)$. Let a_0, a_1, a_2, a_3 be complex numbers such that

$$a_0\chi_2 + a_1 D_1\chi_2 + a_2 D_2\chi_2 + a_3 D_3\chi_2 = 0.$$

By evaluating at e we get $a_0 = 0$ and $D_M\chi_2 = 0$ where

$$M = a_1 M_1 + a_2 M_2 + a_3 M_3.$$

By (3), $\det M = 0$ and hence $M = 0$ since any non-zero element of $\mathfrak{g}'$ has a non-zero determinant. We conclude that $\{x_2, D_1\chi_2, D_2\chi_2, D_3\chi_2\}$ is linearly independent. Let E_n be the two sided ideal in $L_2(G)$ with generating idempotent $n\chi_n$. Since each E_n is invariant under every $D \in \mathfrak{g}$, and $\dim E_2 = 4$, we see that $\{\chi_2, D_1\chi_2, D_2\chi_2, D_3\chi_2\}$ is a basis for E_2. Let $\mathfrak{I}_n$ be the subspace of $C(G)$ consisting of all functions of the form $P(\chi_2, D_1\chi_2, D_2\chi_2, D_3\chi_2)$ where P is a complex polynomial in 4 variables of degree $\leq n$. Then $\mathfrak{I}_n$ is left and right translation invariant, and hence is a two-sided ideal in $L_2(G)$. Since we can write $\chi_n = p(\chi_2)$ where p is a polynomial of degree $n - 1$, $\chi_j \in \mathfrak{J}_n$ for $1 \leq j \leq n + 1$. By the structure theory for ideals in $L^2(G)$ (see [2, page 158]) $\mathfrak{J}_n \supset E_1 \oplus \cdots \oplus E_{n+1}$. The space $\mathfrak{J} = \bigcup_{n=0}^{\infty} \mathfrak{J}_n = \bigcup_{n=1}^{\infty} E_n$ is the representative ring of G. We will call $\mathfrak{J}$ the space of trigonometric polynomials, and $\mathfrak{J}_n$ the space of trigonometric polynomials of degree $\leq n$. If $n > 0$ then every element f of $\mathfrak{J}_n$ can be written in the form

$$f = f_0 \chi_2 + f_1 D_1 \chi_2 + f_2 D_2 \chi_2 + f_3 D_3 \chi_2, \qquad f_j \in \mathfrak{J}_{n-1}, 0 \leq j \leq 3. \tag{4}$$

Also any $f \in \mathfrak{J}$ can be written in the form

$$f = a + b\chi_2^p + f_1 D_1 \chi_2 + f_2 D_2 \chi_2 + f_3 D_3 \chi_2$$

where, $a, b \in \mathbf{C}$, p is a positive integer, $f_1, f_2, f_3 \in \mathfrak{J}$. If $f(e) = 0$ then $a + 2^p b = 0$, and this implies that $a + b\chi_2^p = (2 - \chi_2)f_0$ for some $f_0 \in \mathfrak{J}$. Thus any $f \in \mathfrak{J}$ which vanishes at 0 can be written in the form

$$f = (2 - \chi_2)f_0 + f_1 D_1 \chi_2 + f_2 D_2 \chi_2 + f_3 D_3 \chi_2, \quad f_i \in \mathfrak{J}, 0 \leq i \leq 3. \tag{5}$$

Let $\mathbf{D}$ be the algebra of all left invariant differential operators on G, and for each $n \geq 0$ let $\mathbf{D}^{(n)}$ be the subspace of $\mathbf{D}$ consisting of all operators of degree $\leq n$. Let $\mathbf{D}^{(-1)}$ be the zero subspace of $\mathbf{D}$.

LEMMA 3. *Let $n \geq 0$ and let $X \in \mathbf{D}^{(n)}$. Then there exists an integer $k \geq 0$, a finite subset $\{f_1, \cdots, f_k\}$ of E_2 and a finite subset $\{Y_1, \cdots, Y_k\}$ of $\mathbf{D}^{(n-1)}$ such that*

$$\chi_2 X \chi_m = X\chi_{m-1} + X\chi_{m+1} + \textstyle\sum_{j=1}^{k} f_j Y_j \chi_m \qquad \text{for all } m \geq 1. \tag{6}$$

For each $D \in \mathfrak{g}$ and $X \in \mathbf{D}^{(n)}$ there is an integer $l \geq 0$, a finite subset $\{g_1, \cdots, g_l\}$ of E_2 and a finite subset $\{Z_1, \cdots, Z_l\}$ of $\mathbf{D}^{(n-1)}$ such that

$$D\chi_2 X\chi_m = m^{-1}XD(\chi_{m+1} - \chi_{m-1}) + \textstyle\sum_{j=1}^{l} g_j Z_j \chi_m \text{ for all } m \geq 1. \tag{7}$$

Proof. We will prove (7) by induction on the order of X. (The proof of (6) is similar.) Since

$$D\chi_2 \cdot \chi_m = m^{-1}(D\chi_{m+1} - D\chi_{m-1}) \qquad \text{for all } D \in \mathfrak{g} \tag{8}$$

by [4, Lemma 3.3], (7) holds for $X \in \mathbf{D}^{(0)}$. Assume that (7) holds for all $X \in \mathbf{D}^{(n)}$, and let $Y \in \mathbf{D}^{(n)}$, $D' \in \mathfrak{g}$. Then

$$D\chi_2(D'Y)\chi_m = D'(D\chi_2 \cdot Y\chi_m) - D'D\chi_2 \cdot Y\chi_m$$

since D' is a derivation. Express $D\chi_2 \cdot Y\chi_m$ by (7) and then use the fact that D' is a derivation and the fact that any operator in $\mathbf{D}$ maps E_2 into itself to conclude that (7) holds for all operators in $\mathbf{D}^{(n+1)}$ of the form $D'Y$, $D' \in \mathfrak{g}$, $Y \in \mathbf{D}^{(n)}$. Thus (7) holds for all $X \in \mathbf{D}^{(n+1)}$ since $\mathbf{D}^{(n+1)}$ is generated by $\mathbf{D}^{(n)}$ and elements of the form $D'Y$.

LEMMA 4. *For any $x, y \in G$ let J_{xy} be the linear functional on $C(G)$ defined by*

$$(9) \qquad J_{xy}(f) = \int_G f(x^{-1}uyu^{-1})\, du.$$

If x, y are both distinct from $\pm e$ then

$$(10) \qquad \lim_{m\to\infty} m^{-n} J_{xy}(fX\chi_m) = 0$$

for all trigonometric polynomials f, and all $X \in \mathbf{D}^{(n)}$, $0 \leq n < \infty$.

Proof. First observe that for any $f \in E_m$ we have

$$(11) \qquad J_{xy}(f) = m^{-1} f(x^{-1})\chi_m(y).$$

This is easily verified if f is a coordinate function of an irreducible m dimensional representation of G, (cf. [6, page 87]) and these coordinate functions form a basis for E_n. Since any $X \in \mathbf{D}$ maps each ideal E_n into itself we have by (11)

$$(12) \qquad m^{-n} J_{xy}(X\chi_m) = m^{-n-1} X\chi_m(x^{-1})\chi_m(y).$$

Using the relations

$$\chi_m(x) = \sin m\theta(x)/\sin \theta(x)$$

and

$$(13) \qquad D\chi_m = ((m+1)\chi_{m-1} - (m-1)\chi_{m+1})(3 - \chi_3)^{-1} D\chi_2$$

(see [4, Lemma 3.3]), together with the fact that $3 - \chi_3$ vanishes only at $\pm e$, one can easily prove by induction on n ($=$ order X) that the set $\{m^{-n} X\chi_m(x^{-1}) : 1 \leq m < \infty\}$ is bounded for each $X \in \mathbf{D}^{(n)}$, $0 \leq n < \infty$, $x \neq \pm e$. Hence it follows from (12) that if x and y are both distinct from $\pm e$ then

$$\lim_{m\to\infty} m^{-n} J_{xy}(X\chi_m) = 0, \qquad X \in \mathbf{D}^{(n)},\ 0 \leq n < \infty.$$

Thus (10) holds for $f = 1$ for all $X \in \mathbf{D}$. We will now prove (10) by induction on the degree of f. Assume the result for all trigonometric polynomials of degree $\leq p$ and all $X \in \mathbf{D}$. By (4) we see that (10) holds for all trigonometric polynomials of degree $\leq p + 1$ if and only if

$$(14) \quad \lim_{m\to\infty} m^{-n} J_{xy}(f\chi_2 X\chi_m) = 0, \qquad \lim_{m\to\infty} m^{-n} J_{xy}(fD\chi_2 X\chi_m) = 0$$

for all $f \in \mathfrak{Z}_p$, $D \in \mathfrak{g}$, $0 \leq n < \infty$, $X \in \mathbf{D}^{(n)}$. We will prove (14) (for any $f \in \mathfrak{Z}_p$, $D \in \mathfrak{g}$) by induction on n. For $n = 0$ we have

$$\lim_{m\to\infty} J_{xy}(f\chi_2 \cdot \chi_m) = \lim_{m\to\infty} [J_{xy}(f \cdot \chi_{m+1}) + J_{xy}(f \cdot \chi_{m-1})] = 0$$

for $f \in \mathfrak{Z}_p$, and by (8)

$$\lim_{m\to\infty} J_{xy}(fD\chi_2 \cdot \chi_m) = \lim_{m\to\infty} [m^{-1} J_{xy}(f \cdot D\chi_{m+1}) - m^{-1} J_{xy}(f \cdot D\chi_{m-1})] = 0.$$

Assume that (14) holds for all $f \in \mathfrak{Z}_p$ and $X \in \mathbf{D}^{(n)}$. Then (10) holds for all

$f \in \mathfrak{Z}_{p+1}$ and $X \in \mathbf{D}^{(n)}$. Let $X_0 \in \mathbf{D}^{(n+1)}$, and express $\chi_2 X_0 \chi_m$ and $D\chi_2 X_0 \chi_m$ by (6) and (7). We then conclude that (14) holds with $X = X_0$ from the fact that (10) holds for all $f \in \mathfrak{Z}_p$, $X \in \mathbf{D}$, and all $f \in \mathfrak{Z}_{p+1}$, $X \in \mathbf{D}^{(n)}$.

LEMMA 5. *Let $D \in \mathfrak{g}$. Then the set of numbers $\{n^{-1} \| D\chi_n \|_2 : n > 0\}$ is bounded.*

Proof. We assume without loss of generality that D has norm 1 with respect to the Killing form on $\mathfrak{g}$. Write $D = D_1$ and choose D_2, D_3 so that $\{D_1, D_2, D_3\}$ is an orthonormal basis for $\mathfrak{g}$ with respect to the Killing form. Then $\Delta = D_1^2 + D_2^2 + D_3^2$ is the Laplace operator for G and there exists a constant Λ such that

$$\Delta\chi_n = \Lambda(n^2 - 1)\chi_n \qquad \text{for } n \geq 1. \tag{15}$$

Thus $\| D\chi_n \|_2^2 \leq \sum_{i=1}^3 (D_i \chi_n, D_i \chi_n) = -(\Delta\chi_n, \chi_n) = -\Lambda(n^2 - 1) \| \chi_n \|_2^2$, and the lemma follows from this.

If f is any function on G and $x \in G$, let $L(x)f$ be the function on G defined by $L(x)f(y) = f(x^{-1}y)$. Note that if f is a trigonometric polynomial so is $L(x)f$.

LEMMA 6. *Let $a, x \in G$, $a \neq \pm e$. Let λ_a be as in* (1) *and let f be a function in $C^1(G)$ such that*

$$J_{xa}(D\chi_n L(x^{-1})(fD\chi_2)) = o(n)$$

for all $D \in \mathfrak{g}$. Then x is in the Riemann Lebesgue set of $f\lambda_a$. In particular, if f is a trigonometric polynomial then the Riemann Lebesgue set of $f\lambda_a$ contains all points of G except possibly $\pm e$ (*see Lemma* 4).

Proof. Let D_1, D_2, D_3 be a basis for $\mathfrak{g}$ which is orthonormal with respect to the Killing form. Then for any $f \in C^1(G)$ we have for all $n > 1$ (cf. 15)

$$\begin{aligned} P_n(f\lambda_a)(x) &= (f\lambda_a) * n\chi_n(x) \\ &= \sum_{i=1}^3 \frac{n}{\Lambda(n^2 - 1)} (D_i(fL(x)D_i \chi_n), \lambda_a) \\ &\qquad - \sum_{i=1}^3 \frac{n}{\Lambda(n^2 - 1)} (D_i \chi_n, L(x^{-1})(\lambda_a D_i \bar{f})). \end{aligned} \tag{16}$$

Since $\lambda_a D_i \bar{f} \in L_2(G)$ it follows from Lemma 5 and the fact that $\{D_i \chi_n : 1 \leq n < \infty\}$ is an orthogonal set in $L_2(G)$ that the second sum in (16) tends to zero as $n \to \infty$. Thus $x \in r(f\lambda_a)$ provided that

$$(D(fL(x)D\chi_n), \lambda_a) = o(n)$$

for all $D \in \mathfrak{g}$. In [4] (4.10) it was shown that

$$(Df, \lambda_a) = -\pi^{-1} \sin \theta(a) \int_G f(uau^{-1})D\chi_2(uau^{-1})\, du \tag{17}$$

for any $D \in \mathfrak{g}$ and $f \in C^{\infty}(G)$ (and hence any $f \in C^1(G)$). Thus

$$(D(fL(x)D\chi_n), \lambda_a) = -\pi^{-1} \sin\theta(a) J_{xa}(D\chi_n \cdot L(x^{-1})(fD\chi_2)),$$

and the lemma follows.

Lemma 7. *Let g be a trigonometric polynomial, $a \in G$, $a \neq \pm e$, and let λ_a be as defined in* (1). *Then the Fourier series for $g\lambda_a$ converges to $g\lambda_a$ except possibly at $\pm e$.*

Proof. For any $x \in G$, $n \geq 1$, $f \in L^1(G)$ put

$$S_n f(x) = \textstyle\sum_{k=1}^{n} P_k f(x). \tag{18}$$

Then

$$S_n(g\lambda_a)(x) = g(x)S_n \lambda_a(x) + S_n((g - g(x))\lambda_a)(x).$$

Since $S_n \lambda_a(x) \to \lambda_a(x)$ except for $x = \pm e$ the lemma will follow if we show that

$$\lim_{n\to\infty} S_n(h\lambda_a)(x) = 0 \tag{19}$$

for all $h \in \mathfrak{Z}$ such that $h(x) = 0$, $(x \neq \pm e)$. Now

$$S_N(h\lambda_a)(x) = (L(x^{-1})h \cdot L(x^{-1})\lambda_a ,\ \textstyle\sum_{k=1}^{N} k\chi_k) \tag{20}$$

and $L(x^{-1})h$ is a trigonometric polynomial which vanishes at e. Thus if we show that

$$\lim_{N\to\infty} ((2 - \chi_2)fL(x^{-1})\lambda_a ,\ \textstyle\sum_{k=1}^{N} k\chi_k) = 0 \tag{21}$$

$$\lim_{N\to\infty} ((D\chi_2)fL(x^{-1})\lambda_a ,\ \textstyle\sum_{k=1}^{N} k\chi_k) = 0 \tag{22}$$

for all $f \in \mathfrak{Z}$, $D \in \mathfrak{g}$, $x \neq \pm e$, then (19) will follow because of (5). Using the relations

$$(2 - \chi_2) \textstyle\sum_{k=1}^{N} k\chi_k = (N + 1)\chi_N - N\chi_{N+1} \tag{23}$$

$$D\chi_2 \textstyle\sum_{k=1}^{N} k\chi_k = D(\chi_N + \chi_{N+1}) \tag{24}$$

(see [3] (5.12) and [4] (3.5)) we can rewrite (21) and (22) as

$$\lim_{N\to\infty} \frac{N+1}{N} P_N(\lambda_a L(x)f)(x) - \frac{N}{N+1} P_{N+1}(\lambda_a L(x)f)(x) = 0 \tag{21$'$}$$

$$\begin{aligned}\lim_{N\to\infty} (D(L(x)(f(\chi_N + \chi_{N+1}))), \lambda_a) - \lim_{N\to\infty} (L(x^{-1})\lambda_a \\ \cdot Df,\ \chi_N + \chi_{N+1}) = 0.\end{aligned} \tag{22$'$}$$

Now (21′) is a consequence of Lemma 6, and the second limit in (22′) is 0 because $\{\chi_n\}$ $(1 \leq n < \infty)$ is an orthonormal set in $L_2(G)$. To evaluate the first limit in (22′) we use (17) to get

$$\begin{aligned}&\lim_{N\to\infty} (D(L(x)(f(\chi_N + \chi_{N+1}))), \lambda_a) \\ &\qquad = \lim_{N\to\infty} -\frac{\sin\theta(a)}{\pi} J_{xa}(f(\chi_N + \chi_{N+1})L(x^{-1})D\chi_2),\end{aligned}$$

and this limit is zero by Lemma 4. This completes proof of Lemma 7.

LEMMA 8. *Let f be a C^∞ function on G which vanishes at e together with all of its derivatives of order ≤ 6. Then f can be written $f = (2 - \chi_2)^2 g$ where $g \in C^1(G)$.*

Proof. The function $g = (2 - \chi_2)^{-2} f$ is clearly of class C^1 except possibly at e. Define $\Phi : \mathbf{R}^3 \to G$ by

$$\Phi(x, y, z) = \exp \begin{pmatrix} iz & x + iy \\ -x + iy & -iz \end{pmatrix}.$$

Φ maps a neighborhood of the origin diffeomorphically onto a neighborhood of e in G. Let r be the function on $\mathbf{R}^3$ defined by $r(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$. A routine calculation shows that $(2 - \chi_2) \circ \Phi = r^2 h$ where h is analytic on $\mathbf{R}^3$ and $h(0, 0, 0) = 1$. Hence the lemma will follow if we show that any function F in $C^\infty(\mathbf{R}^3)$ which vanishes at the origin together with all of its derivatives of order ≤ 6 can be written $F = r^4 G$ where $G \in C^1(\mathbf{R}^3)$. This follows by a straightforward argument using Taylor's theorem.

LEMMA 9. *Let a, x be elements of G such that $a \neq \pm e$, $x \neq \pm e$, and suppose that a and $-x$ are not conjugate in G. Let f be a function in $C^\infty(G)$ which vanishes at x together with all of its derivatives of order ≤ 6. Then the Fourier series for $f\lambda_a$ converges to 0 at x.*

Proof. Using (20) and (23) we get

$$S_N(f\lambda_a)(x) = \frac{N+1}{N} P_N((L(x)(2 - \chi_2)^{-1})f\lambda_a)(x) - \frac{N}{N+1} P_{N+1}((L(x)(2 - \chi_2)^{-1})f\lambda_a)(x),$$

so the lemma will follow if we show that

$$\lim_{N\to\infty} P_N((L(x)(2 - \chi_2)^{-1})f\lambda_a)(x) = 0.$$

By Lemma 8 we have $L(x^{-1})f/(2 - \chi_2) = g(2 - \chi_2)$ where $g \in C^1(G)$. Using this in (16) we get

$$\begin{aligned} &P_n((L(x)(2 - \chi_2)^{-1})f\lambda_a)(x) \\ &\quad = P_n(\lambda_a L(x)(g(2 - \chi_2)))(x) \\ &\quad = \sum_{i=1}^{3} \frac{n}{\Lambda(n^2 - 1)} (D_i(L(x)(g(2 - \chi_2)D_i\chi_n)), \lambda_a) \\ &\qquad - \sum_{i=1}^{3} \frac{n}{\Lambda(n^2 - 1)} (D_i\chi_n, L(x^{-1})(\lambda_a)D_i(\bar{g}(2 - \chi_2)))). \end{aligned} \tag{25}$$

The second sum on the right in (25) tends to zero as $n \to \infty$ by an argument given in Lemma 6. Hence Lemma 9 will follow if we show that

$$(D(L(x)(g(2 - \chi_2)D\chi_n)), \lambda_a) = o(n) \tag{26}$$

for all $g \in C^1(G)$, $D \in \mathfrak{g}$. By (17), (26) is equivalent to

$$J_{xa}(g(2 - \chi_2)D\chi_n L(x^{-1})D\chi_2) = o(n) \tag{27}$$

so the Lemma will certaintly follow if we show that

$$J_{xa}(g(2 - \chi_2)D\chi_n) = o(n) \tag{28}$$

for all $g \in C(G)$. In Lemma 4 we showed that (28) holds if g is a trigonometric polynomial, and since the trigonometric polynomials are dense in $C(G)$, (28) will hold for all $g \in C(G)$ provided that the set of functionals

$$F_{xan} : g \to n^{-1} J_{xa}(g(2 - \chi_2)D\chi_n) \qquad (n = 1, 2, \cdots)$$

is bounded in the dual space of $C(G)$. Now

$$\| F_{xan} \| \leq \sup_{u \in G} n^{-1} \mid D\chi_n(x^{-1} uau^{-1})(2 - \chi_2)(x^{-1} uau^{-1}) \mid.$$

By (13) and the identity $(2 - \chi_2)(2 + \chi_2) = 3 - \chi_3$ we have

$$n^{-1} D\chi_n \cdot (2 - \chi_2) = [(\chi_{n-1} - \chi_{n+1}) + n^{-1}(\chi_{n-1} + \chi_{n+1})]D\chi_2/(2 + \chi_2).$$

Since $\| (\chi_{n-1} - \chi_{n+1}) + n^{-1}(\chi_{n-1} + \chi_{n+1}) \|_\infty \leq 4$, we see that

$$\{ \| F_{xan} \| : n = 1, 2, \cdots \}$$

will be bounded provided that the compact set $\{x^{-1} uau^{-1} : u \in G\}$ does not contain $-e$, i.e. provided that a and $-x$ are not conjugate. Since this is true by hypothesis, the lemma follows.

LEMMA 10. *Let $f \in C^\infty(G)$, $x \in G$, and let n be an integer ≥ 0. Then there exists a trigonometric polynomial t_n such that $f - t_n$ vanishes at x together with all of its derivatives of order $\leq n$.*

Proof. We assume without loss of generality that $x = e$. Let

$$M_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and let $D_i = D_{M_i}$, $1 \leq i \leq 3$. Then it is easy to verify that

$$D_i^2 \chi_2 = -\chi_2, \qquad i = 1, 2, 3 \tag{29}$$

$$D_i D_j \chi_2 = \operatorname{sgn}(i, j, k) D_k \chi_2, \qquad i \neq j \tag{30}$$

where sgn (i, j, k) is the sign of the permutation (i, j, k). For any 4-tuple (n_0, n_1, n_2, n_3) of non negative integers and any $j = 1, 2, 3$ we have

$$\begin{aligned} &D_j[\chi_2^{n_0} \textstyle\prod_{k=1}^3 (D_k \chi_2)^{n_k}] \\ &\qquad = -n_j(\chi_2)^{n_0+1}(D_j \chi_2)^{n_j-1} \textstyle\prod_{k=1, k\neq j}^3 (D_k \chi_2)^{n_k} + R_j \end{aligned} \tag{31}$$

where R_j vanishes at e together with all of its derivatives of order $\leq n_1 + n_2 + n_3 - 1$. Let p, q, r, a, b, c, be non negative integers with $p + q + r = a + b + c = m$. Then by m applications of (31) we get

$$(D_1^a D_2^b D_3^c)((D_1 \chi_2)^p (D_2 \chi_2)^q (D_3 \chi_2)^r)(e) = \delta_{ap} \delta_{bq} \delta_{cr} (-2)^m p!q!r!.$$

If $f \in C^\infty$ and m is an integer ≥ 0 put

$$T_f^m = (-2)^{-m} \sum_{p+q+r=m} (D_1^p D_2^q D_3^r f)(e) \cdot \frac{(D_1 \chi_2)^p (D_2 \chi_2)^q (D_3 \chi_2)^r}{p!\, q!\, r!}.$$

Then $XT_f^m(e) = 0$ for all $X \in \mathbf{D}^{(m-1)}$ and

$$(D_1^p D_2^q D_3^r) T_f^m(e) = (D_1^p D_2^q D_3^r f)(e)$$

if $p + q + r = m$. Recall that any $Y \in \mathbf{D}^{(m)}$ can be written in the form

$$Y = \textstyle\sum_{0 \leq p+q+r \leq m} A_{pqr} D_1^p D_2^q D_3^r, \qquad A_{pqr} \in \mathbf{C}$$

(see [1, page 98]). The trigonometric polynomials t_n can now be constructed inductively. Take $t_0 = f(e)$, and if t_n is constructed choose $t_{n+1} = t_n + T_{f-t_n}^{n+1}$.

Proof of Lemma 2. Let $g \in C^\infty(G)$ and let $x \in G$ be an element such that $x \neq \pm e$ and x is not conjugate to $-a$. By Lemma 10 we can write $g = g_1 + g_2$ where g_1 is a trigonometric polynomial, and g_2 vanishes at x together with its derivatives of order ≤ 6. Thus

$$\lim_{N \to \infty} S_N(g\lambda_a)(x) = g\lambda_a(x)$$

by Lemmas 7 and 9. Thus the Fourier series for $g\lambda_a$ converges except possibly at $\pm e$ and at points conjugate to $-a$. Now suppose $x_0 \in G$ is conjugate to $-a$. Then $-x_0$ is not conjugate to $-a$ (since $\theta(a) \neq \pi/2$) and hence the Fourier series for $g\lambda_a$ converges at $-x_0$, and $-x_0 \in r(g\lambda_a)$. Thus $x_0 \in r(g\lambda_a)$ since we saw in the proof of Lemma 1 that $r(f) = -r(f)$ for any $f \in L^1(G)$. Also $g\lambda_a$ is infinitely differentiable at x_0 (since x_0 is not conjugate to a). Theorems A and C of [5] imply that the Fourier series of an L_1 function on G converges at any point of the Riemann Lebesgue set of the function at which the function is C^1. Thus the Fourier series for $g\lambda_a$ converges at points conjugate to $-a$, and the proof is complete.

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