SOME GENERALIZATIONS, TO CERTAIN LOCALLY FINITE GROUPS, OF THEOREMS DUE TO CHAMBERS AND ROSE

BY

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1. Introduction

In this paper we examine further the properties of the class $\mathfrak{U}$ introduced in [4] and studied again in [9] and [10]. We refer the reader to these papers for our notation and terminology. The class $\mathfrak{U}$ resembles in many ways the class of finite soluble groups. Indeed a result for finite soluble groups which makes sense in the wider context usually holds for $\mathfrak{U}$-groups. This is certainly the case with Gaschutz' theory of formations which was carried over to arbitrary QS-closed subclasses $\mathfrak{K}$ of $\mathfrak{U}$ by Gardiner, Hartley and Tomkinson [4]. We have also extended [7] our earlier work [5], [6] on $\mathfrak{F}$-reducers and $\mathfrak{F}$-subnormalizers to such classes $\mathfrak{K}$. In addition the results of Alperin [1] on system normalizers, Carter subgroups and the relation between them in finite soluble groups, were extended to $\mathfrak{U}$-groups in [9]. It is our aim to show here that many of the results of Chambers [3] and Rose [15] hold in appropriate subclasses of $\mathfrak{U}$. Generalizing Chambers [3] we show, for example, that if $\mathfrak{K}$ is a QS-closed subclass of $\mathfrak{U}$ and $\mathfrak{F}$ a saturated $\mathfrak{K}$-formation then the $\mathfrak{F}$-normalizers are pronormal in $\mathfrak{K}_A$-groups (i.e. $\mathfrak{K}$-groups with abelian Sylow p-subgroups for each prime p). This yields a partial extension of Alperin's [1, Theorem 1] for $\mathfrak{F}$-normalizers and $\mathfrak{F}$-projectors of $\mathfrak{K}_A$-groups. We shall also show that the $\mathfrak{F}$-normalizers of $\mathfrak{K}_A$-groups are characterized as those subgroups which cover the $\mathfrak{F}$-central and avoid the $\mathfrak{F}$-eccentric chief factors. We shall extend Chambers' work in Section 2 and Rose's in Section 3. In the third and final section we shall consider the class $\mathfrak{D}$ of $\mathfrak{U}$-groups with pronormal basis normalizers. For example, we prove that $\mathfrak{D}$ is a $\mathfrak{U}$-formation (in the sense of [4, §1]) and derive many of its properties from our work [7] on reducers in $\mathfrak{U}$-groups.

2. $\mathfrak{U}$-groups with abelian Sylow subgroups

If $\mathfrak{X}$ is a subclass of $\mathfrak{U}$ we denote by $\mathfrak{X}_A$ the class of $\mathfrak{X}$-groups with abelian Sylow p-groups for each prime p. In this section we study the class $\mathfrak{U}_A$ showing in particular that most of Chambers' results on finite soluble A-groups can be extended to the class $\mathfrak{U}_A$ or appropriate subclasses of it.

It is clear that if $\mathfrak{X}$ is a QS-closed subclass of $\mathfrak{U}$ then so is $\mathfrak{X}_A$ (cf. [4, 2.1]).

LEMMA 2.1. *Every $\mathfrak{U}_A$-group is soluble.*

Proof. If $G \in \mathfrak{U}_A$ then G has a finite normal series with locally nilpotent factors. Since $\mathfrak{U}_A$ is QS-closed and every locally nilpotent $\mathfrak{U}_A$-group is abelian, each of these factors is abelian. Hence G is soluble as claimed.

Received January 26, 1972.

LEMMA 2.2. *Suppose the* $\mathfrak{U}$*-group* G *has abelian Sylow* p*-subgroups for some prime* p. *Then the* p*-length* $\ell_p(G)$ *of* G *is at most one. In particular* $\mathfrak{U}_A$*-groups have* p*-length at most one for all primes* p.

Proof. Let P be a Sylow p-subgroup and H/K a p-chief factor of G. Then $H/K \leq PK/K$ and since P is abelian by hypothesis, P centralizes H/K. Therefore $P \leq O_{p'p}(G)$ by [4, 3.8]. Hence $G/O_{p'p}(G)$ is a p'-group and G has p-length at most one, as required.

Suppose p is a prime and H_p is a Sylow p-subgroup of a subgroup H of a $\mathfrak{U}$-group G. We say H is p-normally embedded in G if H_p is a Sylow p-subgroup of some normal subgroup of G. It is easy to see that H is p-normally embedded in G if and only if H_p is a Sylow p-subgroup of the normal closure of H_p in G.

We now use [7, Theorem 3.22] to establish the following lemma which will later yield the pronormality of $\mathfrak{F}$-normalizers in $\mathfrak{K}_A$-groups.

LEMMA 2.3. *Suppose* $V \leq G \in \mathfrak{U}$ *and* V *is* p*-normally embedded in* G *for each prime* p. *Then* V *is pronormal in* G.

Proof. Suppose $\mathbf{S}$ is a Sylow basis of G and $\mathbf{S}$, $\mathbf{S}^x$ reduce into V for some $x \in G$. Then $\mathbf{S} \cap V$ and $\mathbf{S}^x \cap V$ are Sylow bases of V so there is some element $y \in V$ such that $S_p \cap V = S_p^{xy} \cap V = V_p$ say, for each prime p.

Let p be any prime. Since V is p-normally embedded in G there is a normal subgroup M of G such that V_p is a Sylow p-subgroup of M. Now V_p is contained in both S_p and S_p^{xy} so it follows that $V_p = S_p \cap M = S_p^{xy} \cap M$ and hence that $V_p = V_p^{xy}$.

Since p was an arbitrary prime and $V = \langle V_p \,;\, \text{all } p\rangle$ we therefore have $V = V^{xy}$ and hence $xy \in N_G(V)$. Thus $x \in N_G(V)$ and from [7, 3.22] we deduce that V is pronormal in G.

LEMMA 2.4. *Suppose the* $\mathfrak{U}$*-group* G *has abelian Sylow* p*-subgroups for some prime* p, *and* H *is a subgroup of* G *containing a* p*-complement of* G. *Then* H *is* p*-normally embedded in* G.

Proof. Suppose H contains the p-complement S of G. Now G has p-length at most one by 2.2 so $G = O_{p'pp'}(G)$.

Suppose first that $O_{p'}(G) = 1$. Then G has a unique Sylow p-subgroup P which is abelian by hypothesis. Since $G = PS$ it follows that $G = HP$ and hence that $H \cap P$ is a normal subgroup of G. But P contains the unique Sylow p-subgroup of H so $H \cap P \in Syl_p(H)$.

Now suppose that the general case prevails and let H_p be a Sylow p-subgroup of H. Then $H_p O_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $HO_{p'}(G)/O_{p'}(G)$ so by the case just considered $H_p O_{p'}(G)/O_{p'}(G)$ is a normal subgroup of $G/O_{p'}(G)$. Clearly H_p is a Sylow p-subgroup of the normal subgroup $H_p O_{p'}(G)$ of G, so H is p-normally embedded in G, as claimed.

We shall show later that in $\mathfrak{K}_A$-groups the $\mathfrak{F}$-normalizers complement the

$\mathfrak{F}$-residual. To do this we require the following generalization of a result of Taunt [17] (cf. also [14, VI 14.3]).

LEMMA 2.5. *Let p be a prime and suppose the Sylow p-subgroup P of the $\mathfrak{U}$-group G is abelian. Then* $P \cap G' \cap Z(G) = 1$.

Here, as usual, G' denotes the derived group and $Z(G)$ the centre of G.

Proof of 2.5. Suppose that there exists a non-trivial element x in $P \cap G' \cap Z(G)$. Then $x = [y_1, z_1] \cdots [y_n, z_n]$ for some $y_i, z_i \in G$ $(1 \leq i \leq n)$. Let $G_1 = \langle y_i, z_i ; 1 \leq i \leq n \rangle$. Then G_1 is a finite group and if P_1 is a Sylow p-subgroup of G_1 containing $P \cap G_1$ then $P_1 \cap G_1' \cap Z(G_1) = 1$, by the finite case of the lemma [14, VI 14.3]. But this gives a contradiction since x clearly belongs to $P_1 \cap G_1' \cap Z(G_1)$. The result now follows.

COROLLARY 2.6 *If G is a $\mathfrak{U}_A$-group then* $G' \cap Z(G) = 1$.

For the remainder of this paper $\mathfrak{K}$ will denote a QS-closed subclass of $\mathfrak{U}$, $\mathfrak{f}$ an integrated $\mathfrak{K}$-preformation function on a set of primes π, and $\mathfrak{F}$ the saturated $\mathfrak{K}$-formation defined by $\mathfrak{f}$ (cf. [4, §1]). If σ is a set of primes then $\mathfrak{K}_{A,\sigma}$ will denote the class of $\mathfrak{K}$-groups with abelian Sylow p-subgroups for each prime p in σ. In particular if p is a prime then $\mathfrak{K}_{A,p}$ denotes the class of $\mathfrak{K}$-groups with abelian Sylow p-subgroups.

LEMMA 2.7. *Suppose $p \in \pi$ and $G \in \mathfrak{K}_{A,p}$. If S is a Sylow p'-subgroup of G and $C_p(G)$ is the $\mathfrak{f}(p)$-centralizer of G then $N_G(S \cap C_p(G))$ is p-normally embedded in G.*

Proof. Since S certainly normalizes $S \cap C_p(G)$ this result is an immediate consequence of Lemma 2.4.

COROLLARY 2.8. *Suppose $p \in \pi$ and $G \in \mathfrak{K}_{A,p}$. If D is an $\mathfrak{F}$-normalizer of G then D is p-normally embedded in G.*

Proof. Suppose D is the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\mathbf{S}$ of G. Then

$$S_p \cap D = S_p \cap N_G(S_{p'} \cap C_p(G))$$

is a Sylow p-subgroup of both D and $N_G(S_{p'} \cap C_p(G))$ by [4, 2.13(i)]. The result is now immediate from 2.7.

We now establish the following generalization of [3, 3.5].

THEOREM 2.9. *Suppose D is an $\mathfrak{F}$-normalizer of the $\mathfrak{K}_{A,\pi}$-group G. Then D is p-normally embedded in G for all primes p and hence is pronormal in G.*

Proof. If $p \in \pi$ then D is p-normally embedded in G by 2.8. If $p \notin \pi$ then $D_p = 1$ since D is a π-group, so D is trivially p-normally embedded in G in this case. Therefore D is p-normally embedded in G for all p, and so, by 2.3, is pronormal in G.

COROLLARY 2.10. *The $\mathfrak{F}$-normalizers of $\mathfrak{K}_A$-groups are pronormal.*

COROLLARY 2.11. *If D_1 and D_2 are $\mathfrak{F}$-normalizers of the $\mathfrak{K}_{A,\pi}$-group G contained in the same $\mathfrak{F}$-projector E of G then D_1 and D_2 are conjugate in E.*

Proof. There is an element $y \in G$ such that $D_1 = D_2^y$. Now D_2 is pronormal in G by 2.9, so D_2 and D_2^y are conjugate in $\langle D_2, D_2^y\rangle$. The result now follows since $\langle D_2, D_2^y\rangle \leq E$.

Remark. Corollary 2.11 is a partial extension of a well-known result of Alperin [1, Theorem 1], but the conclusion of 2.11 does not hold for all $\mathfrak{K}$-groups. For in [12] Hawkes gives an example of a saturated $\mathfrak{S}^*$-formation $\mathfrak{F}$ and an $\mathfrak{S}^*$-(that is finite soluble) group G with $\mathfrak{F}$-normalizers contained but not conjugate in the same $\mathfrak{F}$-projector of G.

LEMMA 2.12. *Suppose the $\mathfrak{K}$-group G has pronormal $\mathfrak{F}$-normalizers and D is an $\mathfrak{F}$-normalizer of G contained in the subgroup H of G. Then D is contained in some $\mathfrak{F}$-normalizer of H.*

Proof. Suppose D is the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\mathbf{S}$ of G and let $\mathbf{S}^x$ be a Sylow basis of G which reduces into both H and D. Now $\mathbf{S}$ reduces into D by [4, 2.13(ii)] so $x \in R_G(D)$, the reducer of D in G (cf. [7, §3]). Since D is by hypothesis pronormal in G we have $x \in N_G(D)$ by [7, 3.22]. Thus $D = D^x$ is the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\mathbf{S}^x$ and $\mathbf{S}^x$ reduces into H. The result now follows from [4, 4.10].

COROLLARY 2.13. *If D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}_{A,\pi}$-group G contained in a subgroup H of G then D is contained in some $\mathfrak{F}$-normalizer of H.*

Remark. Shamash [16, 4.3(2)] gives an example of a finite soluble group G with a subgroup H containing a basis normalizer D of G such that D normalizes no Sylow basis of H. In general therefore we cannot hope to improve much upon 2.12.

We now discuss Chambers' characterization of $\mathfrak{F}$-normalizers (of finite soluble A-groups) by the covering/avoiding property. We shall prove that a similar characterization holds for the $\mathfrak{F}$-normalizers of $\mathfrak{K}_{A,\pi}$-groups.

LEMMA 2.14. *Suppose $p \in \pi$ and $G \in \mathfrak{K}_{A,p}$. Let H be a p-subgroup of G which avoids every $\mathfrak{F}$-eccentric p-chief factor of G. Then*

$$H \leq N_G(S \cap C_p(G))$$

for every p-complement S of G.

Proof. Let S be a p-complement of G and $N = N_G(S \cap C_p(G))$; S is clearly contained in N. Let $P = O_{p'p}(G)$. Then G/P is a p' group by 2.2 so $G = PS$ and hence $G = PN$. Now $O_{p'}(G) \leq P \cap N \leq P$ and $P/O_{p'}(G)$ is abelian by hypothesis, so $P \cap N$ is a normal subgroup of $PN = G$. Let $C = P \cap N$.

By [4, 3.1] and the definition of $C_p(G)$, N covers the $\mathfrak{F}$-central and avoids the $\mathfrak{F}$-eccentric p-chief factors of G. Now N avoids every chief factor of G between C and P so such factors are $\mathfrak{F}$-eccentric. By hypothesis therefore H avoids every chief factor of G between C and P. It follows that $H \leq C$. For let $x \to \bar{x}$ be the natural epimorphism of G onto $\bar{G} = G/O_{p'}(G)$. Then $\bar{H} \leq \bar{P}$ since H is a p-subgroup of G. Suppose $\bar{H}$ is not contained in $\bar{C}$ and let $\bar{x}$ be an element of $\bar{H} - \bar{C}$. Then $\bar{x} \in \bar{P}\text{-}\bar{C}$ so there is a chief factor $\bar{U}/\bar{V}$ of $\bar{G}$ such that $\bar{C} \leq \bar{V} < \bar{U} \leq \bar{P}$ and $\bar{x} \in \bar{U}\text{-}\bar{V}$. Now H avoids U/V so $\bar{H}$ avoids $\bar{U}/\bar{V}$. Thus $\bar{H} \cap \bar{U} = \bar{H} \cap \bar{V}$ and we have a contradiction since $\bar{x} \in \bar{H} \cap \bar{U} - \bar{H} \cap \bar{V}$. In view of this contradiction we have $\bar{H} \leq \bar{C}$ and hence $H \leq C$. Since $C \leq N$ the proof is complete.

LEMMA 2.15. *Suppose H is a subgroup of a $\mathfrak{K}_{A,\pi}$-group G and H avoids every $\mathfrak{F}$-eccentric chief factor of G. Then H is contained in some $\mathfrak{F}$-normalizer of G.*

Proof. We show first that H is a π-group. If this is not the case then we find an element x of order q in H for some prime $q \notin \pi$. Let $(\Lambda_\sigma, V_\sigma ; \sigma \in \Omega)$ be a chief series of G. Then x lies in some layer Λ_σ/V_σ $(\sigma \in \Omega)$. Since

$$x \in (H \cap \Lambda_\sigma) - (H \cap V_\sigma),$$

H does not avoid Λ_σ/V_σ which by our hypothesis must then be $\mathfrak{F}$-central. Thus if Λ_σ/V_σ is a p-factor then $p \in \pi$. But now xV_σ is a non-trivial element of order q in Λ_σ/V_σ so $q = p \in \pi$, a contradiction. Therefore H is a π-group as claimed above.

Let $\mathbf{S}$ be a Sylow basis of G which reduces into H. If $p \in \pi$ then $H_p = H \cap S_p$ is a p-subgroup of G which avoids every $\mathfrak{F}$-eccentric p-chief factor of G, so, by 2.14,

$$H_p \leq N_G(S_{p'} \cap C_p(G)).$$

Since $H_{p'} = H \cap S_{p'} \leq S_{p'} \leq N_G(S_{p'} \cap C_p(G))$ it follows that $H = H_p H_{p'}$ normalizes $S_{p'} \cap C_p(G)$. But H is a π-group so $H = H \cap S_\pi$ and hence

$$H \leq D = S_\pi \cap \textstyle\bigcap_{p \in \pi} N_G(S_{p'} \cap C_p(G)),$$

the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\mathbf{S}$ of G. This establishes the lemma.

LEMMA 2.16. *Suppose D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}_{A,\pi}$-group G.*

(1) *Every chief factor of G below the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is $\mathfrak{F}$-eccentric.*

(2) *D complements $G^{\mathfrak{F}}$ in G.*

Proof. (1) Suppose that the result is false and let H/K be an $\mathfrak{F}$-central chief factor of G below $G^{\mathfrak{F}}$. Then H/K is an $\mathfrak{F}$-central minimal normal subgroup of the $\mathfrak{K}_{A,\pi}$-group G/K contained in $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$, so in obtaining our contradiction we may assume without loss of generality that H is an $\mathfrak{F}$-central minimal normal subgroup of G contained in $G^{\mathfrak{F}}$. Now $G^{\mathfrak{F}} \leq C_G(H)$

since $\mathfrak{f}$ is integrated and H is $\mathfrak{F}$-central, so $H \leq Z(G^{\mathfrak{F}})$. If H is a p-group then $p \in \pi$ since H is $\mathfrak{F}$-central, and $G^{\mathfrak{F}}$ has abelian Sylow p-subgroups by hypothesis. Therefore $H \cap (G^{\mathfrak{F}})' = 1$ by 2.5.

Let $L = G^{\mathfrak{F}}$. Then the $\mathfrak{F}$-residual L/L' of G/L' is abelian so, by [4, 4.12], the $\mathfrak{F}$-normalizers of G/L' complement L/L' in G/L'. Thus, by [4, 4.6], every chief factor of G/L' below L/L' is $\mathfrak{F}$-eccentric. But $H \cap L' = 1$ so H is G-isomorphic to HL'/L'. Thus HL'/L' is an $\mathfrak{F}$-central chief factor of G/L' below L/L', which gives the desired contradiction and establishes (1)

(2) Suppose there is a non-trivial element $x \in D \cap G^{\mathfrak{F}}$. Then taking a chief series of G passing through G^{F} we obtain a chief factor X/Y of G below $G^{\mathfrak{F}}$ such that $x \in X\text{-}Y$. Since

$$x \in (D \cap X) = (D \cap Y)$$

the chief factor X/Y is $\mathfrak{F}$-central [4, 4.6] which contradicts (1). Therefore $G^{\mathfrak{F}} \cap D = 1$. Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$ we also have $G = DG^{\mathfrak{F}}$ by [4, 4.6] and this establishes (2).

We can now prove our generalization of [3, 3.6].

THEOREM 2.17. *Suppose* $H \leq G \in \mathfrak{K}_{A,\pi}$. *Then* H *is an* $\mathfrak{F}$*-normalizer of* G *if and only if* H *covers every* $\mathfrak{F}$*-central chief factor of* G *and avoids every* $\mathfrak{F}$*-eccentric chief factor of* G.

Proof. $\mathfrak{F}$-normalizers certainly have the required property by [4, 4.6] so we need only show that a subgroup H with the given "covering/avoiding" property is an $\mathfrak{F}$-normalizer of G.

Suppose then that H covers every $\mathfrak{F}$-central chief factor of G and avoids every $\mathfrak{F}$-eccentric chief factor of G. Now $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ so in particular $G/G^{\mathfrak{F}}$ is a π-group. Thus, by hypothesis, $G/G^{\mathfrak{F}}$ has abelian Sylow p-subgroups for each prime p, and it follows, as in the proof of 2.1, that $G/G^{\mathfrak{F}}$ is soluble. Now $HG^{\mathfrak{F}}/G^{\mathfrak{F}}$ covers every chief factor of $G/G^{\mathfrak{F}}$ since all such factors are $\mathfrak{F}$-central. Therefore $G = HG^{\mathfrak{F}}$ by [11, Lemma 1]. Now H is contained in some $\mathfrak{F}$-normalizer D of G by 2.15, and D complements $G^{\mathfrak{F}}$ in G by 2.16(2). Therefore

$$D = D \cap HG^{\mathfrak{F}} = H(D \cap G^{\mathfrak{F}}) = H,$$

and the proof is complete.

Remark. Basis normalizers, and hence $\mathfrak{F}$-normalizers, are not usually characterized by their covering/avoiding property as an example of Hawkes [13] demonstrates.

As a corollary to 2.16 we have:

COROLLARY 2.18. *Suppose* $G \in \mathfrak{K}_{A,\pi}$ *and* D *is an* $\mathfrak{F}$*-normalizer of* G. *Then*

$$N_G(D) = D \times (G^{\mathfrak{F}} \cap C_G(D)).$$

Proof. Let $N = N_G(D)$. Now D complements $G^{\mathfrak{F}}$ in G by 2.16 (2), so

$N = D(N \cap G^{\mathfrak{F}})$. Since $[D, N \cap G^{\mathfrak{F}}] \leq D \cap G^{\mathfrak{F}} = 1$, it is clear that

$$N \cap G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap C_G(D).$$

But $D \cap G^{\mathfrak{F}} = 1$ so we finally have $N = D \times (G^{\mathfrak{F}} \cap C_G(D))$, as claimed.

If D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}_A$-group then $R_G(D) = N_G(D)$ by 2.10 and [7, 3.22]. It might be hoped that D satisfies the stronger condition

$$R_G(D; \mathfrak{F}) = N_G(D)$$

when $\mathfrak{F}$ satisfies (1.1), (1.2) and (1.3) of [8]. However this is not the case as the following example shows.

Example 2.19. We take $\mathfrak{K}$ to be the class $\mathfrak{S}^*$ of finite soluble groups and $\mathfrak{F}$ the saturated $\mathfrak{S}^*$-formation of finite supersoluble groups. By [2, 6.1] $\mathfrak{F}$ is defined by the integrated $\mathfrak{S}^*$-formation function $\mathfrak{f}$ which assigns to each prime p the class $\mathfrak{f}(p)$ of finite abelian groups of exponent dividing p-1. Clearly $\mathfrak{f}$ satisfies (1.1), (1.2) and (1.3) of [8].

Let A be an alternating group of degree 4, and let G be the wreath product of a cyclic group C_7 of order 7 by A, the wreath product being taken with respect to the natural permutation representation of A on 4 symbols. $|G| = 7^4 \cdot 3 \cdot 2^2$. G is certainly an $\mathfrak{S}^*_A$- (i.e. finite soluble A-) group and is the semidirect product of an elementary abelian group N of order 7^4 with A. The centre Z of G is the "diagonal" of the base group N of G and has order 7. Let V denote the Sylow 2- and U a Sylow 3-subgroup of A. Then $1 < Z < N < NV < NVU = G$ is a chief series of G. The supersoluble central chief factors are just the cyclic chief factors [2, 6.3] so it follows, for example from 2.17, that $D = Z \times U$ is a supersoluble normalizer of G. U is a supersoluble projector of A so UN/N is a supersoluble projector of G/N. Now UN is supersoluble so by Gaschutz' Lemma [4, 5.3] UN is a supersoluble projector of G. Since U is a characteristic subgroup of D it follows that D is not normal in UN. However $UN \leq R_G(D; \mathfrak{F})$ by [7, 3.11], so D is not normal in $R_G(D; \mathfrak{F})$, giving the desired example.

We conclude this section with a brief discussion of the basis normalizers of $\mathfrak{U}_A$-groups. We have the following special cases of our previous results, the first of which generalizes a result of Rose [15, 2.4].

THEOREM 2.20. *The basis normalizers of $\mathfrak{U}_A$-groups are pronormal.*

THEOREM 2.21. *If $H \leq G \in \mathfrak{U}_A$ then H is a basis normalizer of G if and only if H covers every central chief factor of G and avoids every eccentric chief factor of G.*

COROLLARY 2.22. *Let D be a basis normalizer of a $\mathfrak{U}_A$-group G.*

(1) *D complements the derived group of G in G,*

(2) *$N_G(D) = D \times (G' \cap C_G(D))$*

(3) *$N_G(D) = C_G(D)$.*

Proof. Since every locally nilpotent $\mathfrak{U}_A$-group is abelian it follows that the $\mathrm{L}\mathfrak{N}$-residual of G is in fact the derived group of G. Therefore (1) is immediate from 2.16(2), and (2) follows from 2.18. The third statement follows from the second since D, being a locally nilpotent $\mathfrak{U}_A$-group, is abelian.

LEMMA 2.23. (1) *Suppose D is a basis normalizer and N a normal subgroup of the $\mathfrak{U}$-group G. Then $N = (N \cap G')(N \cap D)$.*

(2) *If N is an abelian normal subgroup of the $\mathfrak{U}_A$-group G then*

$$N = (N \cap G') \times (N \cap Z(G)).$$

Proof. (1) Since N is a normal subgroup of G, $[N, G] \leq N \cap G'$. Thus every chief factor of G lying between $N \cap G'$ and N is central and so covered by D, [4, 4.6]. Furthermore

$$N/N \cap G' \leq Z(G/N \cap G')$$

so that $(D \cap N)(N \cap G')$ is a normal subgroup of G. Therefore

$$N = (D \cap N)(N \cap G');$$

for otherwise we can take a chief series of G passing through $(D \cap N)(N \cap G')$ and N, and D will cover no chief factor in this series between $(D \cap N)(N \cap G')$ and N. This establishes (1).

(2) If D is a basis normalizer of G then

$$N = (N \cap G')(N \cap D)$$

by (1) and $D \cap G' = 1$ by 2.22. Thus to prove (2) it suffices to show that $N \cap D = N \cap Z(G)$. Now $Z(G)$ normalizes every Sylow basis of G so we certainly have $N \cap Z(G) \leq N \cap D$. It therefore remains to show that $N \cap D \leq Z(G)$.

Let D_p, N_p denote the unique Sylow p-subgroup of D, N respectively. Then N_p is a characteristic subgroup of N and hence a normal subgroup of G. If D is the normalizer of the Sylow basis $\mathbf{S}$ of G then

$$[N_p \cap D_p, S_{p'}] \leq N_p \cap S_{p'} = 1,$$

so $N_p \cap D_p$ centralizes $S_{p'}$. But $N_p \cap D_p \leq S_p$ and G is a $\mathfrak{U}_A$-group, so S_p is abelian and $N_p \cap D_p$ centralizes S_p. Since $G = S_{p'} S_p$ it follows that $N_p \cap D_p$ is contained in the centre of G. But $N_p \cap D_p$ is the unique Sylow p-subgroup of $N \cap D$, so we finally have $N \cap D \leq Z(G)$ which, as above, completes the proof.

As we have already seen (2.1) every $\mathfrak{U}_A$-group is soluble. This fact gives us the following description of the Hirsch-Plotkin radical of a $\mathfrak{U}_A$-group.

THEOREM 2.24. *Suppose G is a $\mathfrak{U}_A$-group of derived length n. Then the Hirsch-Plotkin radical*

$$\rho(G) = Z(G) \times Z(G') \times \cdots \times Z(G^{(n-1)})$$

where $G^{(i)}$ denotes the i^{th} term in the derived series of G.

Proof. We argue by induction on the derived length n of G, the result being clear when G is abelian. We may therefore suppose that $n > 1$ and $\rho(G') = Z(G') \times \cdots \times Z(G^{(n-1)})$ by induction. Now $\rho(G)$ is a locally nilpotent subgroup of the $\mathfrak{U}_A$-group G so is in fact abelian. Therefore

$$\rho(G) = (\rho(G) \cap G') \times (\rho(G) \cap Z(G))$$

by 2.23(2). But $Z(G)$ is certainly contained in $\rho(G)$ and $\rho(G) \cap G' = \rho(G')$. Thus

$$\rho(G) = Z(G) \times \rho(G') = Z(G) \times Z(G') \times \cdots \times Z(G^{(n-1)})$$

and the proof is complete.

Lemma 2.23 and Theorem 2.24 are generalizations of well-known results of Taunt [17].

We close this section with a characterization of the normalizers of basis normalizers of $\mathfrak{U}_A$-groups.

THEOREM 2.25. *Let D be a basis normalizer of a $\mathfrak{U}_A$-group G, and let $N = N_G(D)$. Suppose H is a subgroup of G containing D. Then $H = N$ if and only if H covers every D-central and avoids every D-eccentric D-composition factor of G.*

Proof. N is the reducer of D in G by 2.20 and [7, 3.22], so certainly N covers every D-central D-composition factor of G by [7, 3.21]. Suppose A/B is a D-composition factor of G covered by N. Then $A = (N \cap A)B$ and hence $[A, D] \leq B$ since N centralizes D by 2.22. Therefore A/B, and hence every D-composition factor covered by N, is D-central. Since N certainly covers or avoids every D-composition factor of G it follows that N covers every D-central and avoids every D-eccentric D-composition factor of G.

Suppose conversely that H covers every D-central and avoids every D-eccentric D-composition factor of G. Then $N \leq H$ by [7, 3.21]. By 2.13, D normalizes some Sylow basis $\mathbf{T}$ of H. Since $\mathbf{T}$ reduces into $N_H(\mathbf{T})$ we certainly have $D_p \leq T_p$ for each prime p. But T_p is abelian since G is a $\mathfrak{U}_A$-group, so D_p centralizes T_p for each prime p.

Let $(\Lambda_\sigma, V_\sigma; \sigma \in \Omega)$ be a D-composition series of G. Then

$$(\Lambda_\sigma \cap H, \quad V_\sigma \cap H; \quad \sigma \in \Omega)$$

is a D-series of H in which every non-trivial factor is D-central. Intersecting this series further with T_p we obtain a series of T_p in which every non-trivial factor is centralized by $D_{p'}$, so, by [4, 4.11], $D_{p'}$ centralizes T_p. Hence $D = D_p D_{p'}$ centralizes T_p for each prime p. Since H is generated by the subgroups T_p it follows that $H \leq C_G(D)$. In particular therefore $H \leq N$ and this, together with our previous inequality, completes the proof.

3. $\mathfrak{U}$-Groups with pronormal basis normalizers

In this section we extend Rose's results [15] on finite soluble groups with pronormal system normalizers. In many cases our results can be proved using Rose's techniques but we have tried where possible to give different proofs using our work on reducers [7].

Using the language of [7, §4] we have the following restatement of [7, 3.22].

THEOREM 3.1. *Suppose* $H \leq G \in \mathfrak{U}$. *Then the following three conditions are equivalent:*

(1) *H is pronormal in G,*

(2) $R_G(H) = N_G(H)$,

(3) *$N_G(H)$ is the strong serializer of H in G.*

We denote by $\mathfrak{D}(\mathfrak{K}, \mathfrak{F})$ the class of all $\mathfrak{K}$-groups with pronormal $\mathfrak{F}$-normalizers. If $\mathfrak{F} \geq \mathfrak{K} \cap \mathrm{L}\mathfrak{N}$ then, by [4, 5.1 and 5.7],

$$\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F} \leq \mathfrak{D}(\mathfrak{K}, \mathfrak{F}).$$

In any case we have $\mathfrak{K}_{A,\pi} \leq \mathfrak{D}(\mathfrak{K}, \mathfrak{F})$ by 2.9.

THEOREM 3.2. *$\mathfrak{D}(\mathfrak{K}, \mathfrak{F})$ is a $\mathfrak{K}$-formation.*

Proof. It is clear that $\mathfrak{D}(\mathfrak{K}, \mathfrak{F})$ is Q-closed, so we need only show that $\mathfrak{K} \cap \mathrm{R}\mathfrak{D}(\mathfrak{K}, \mathfrak{F}) = \mathfrak{D}(\mathfrak{K}, \mathfrak{F})$.

Suppose $G \in \mathfrak{K} \cap \mathrm{R}\mathfrak{D}(\mathfrak{K}, \mathfrak{F})$. Then there exist normal subgroups N_λ of $G (\lambda \in \Lambda)$ such that $G/N_\lambda \in \mathfrak{D}(\mathfrak{K}, \mathfrak{F})$ for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. Let D be an $\mathfrak{F}$-normalizer of G and $\lambda \in \Lambda$. Then DN_λ/N_λ is an $\mathfrak{F}$-normalizer of G/N_λ so, by 3.1 and [7, 3.5],

$$R_G(D)N_\lambda/N_\lambda = N_{G/N_\lambda}(DN_\lambda/N_\lambda).$$

Therefore $DN_\lambda \lhd RN_\lambda$ where $R = R_G(D)$, and hence $[D, R] \leq DN_\lambda$. Since λ was an arbitrary member of Λ we have, using [7, 5.1],

$$[D, R] \leq \bigcap_{\lambda \in \Lambda}(DN_\lambda) = D.$$

This shows that D is a normal subgroup of R and it follows that $R = N_G(D)$. Therefore $G \in \mathfrak{D}(\mathfrak{K}, \mathfrak{F})$ by 3.1, and the proof is complete.

Remark. In general the $\mathfrak{K}$-formations $\mathfrak{D}(\mathfrak{K}, \mathfrak{F})$ are neither subgroup-closed nor saturated. For Rose [15, 3.5 and 5.6] shows that the class $\mathfrak{D}(\mathfrak{S}^*, \mathfrak{N}^*)$ has neither of these properties; $\mathfrak{D}(\mathfrak{S}^*, \mathfrak{N}^*)$ is of course the class of finite soluble groups with pronormal system normalizers.

For the remainder of this section we shall consider the class $\mathfrak{D} = \mathfrak{D}(\mathfrak{U}, \mathrm{L}\mathfrak{N}^*)$ which naturally extends $\mathfrak{D}(\mathfrak{S}^*; \mathfrak{N}^*)$; $\mathfrak{D}$ is the class of $\mathfrak{U}$-groups with pronormal basis normalizers. From 3.2 and 2.20 we have the following generalization of [15, 2.4 and 3.4].

THEOREM 3.3. *$\mathfrak{D}$ is a $\mathfrak{U}$ formation containing the class $\mathfrak{U}_A$.*

LEMMA 3.4. *Suppose D is a basis normalizer of the $\mathfrak{U}$-group G. Then D is pronormal in G if and only if $N_G(D)$ is the serializer of D in G.*

Proof. This result is an immediate consequence of 3.1 and [7, 5.11].

From this lemma we deduce the following generalization of [15, 6.1].

THEOREM 3.5. *If D is a basis normalizer of the $\mathfrak{U}$-group G then $G \in \mathfrak{D}$ if and only if $N_G(D)$ is the serializer of D in G.*

COROLLARY 3.6. *Let D be a basis normalizer of the $\mathfrak{U} \cap (\mathrm{L}\mathfrak{N})^3$-group G and C the unique Carter subgroup of G containing D. Then $G \in \mathfrak{D}$ if and only if $D \lhd C$.*

Proof. This result is immediate from 3.5 and [7, 5.5].

Remark. Corollary 3.6 extends a similar result [5, 6.5] of Rose. He also shows that the condition that G is a $\mathfrak{U} \cap (\mathrm{L}\mathfrak{N})^3$-group is necessary in that there exists a finite soluble group G of nilpotent length 4 with basis normalizer D and Carter subgroup C such that $D \lhd C$ but $G \notin \mathfrak{D}^* = \mathfrak{D}(\mathfrak{S}^*, \mathfrak{N}^*)$.

From 2.12 we have the following extension of [15, 4.1].

LEMMA 3.7. *Suppose D is a basis normalizer of the $\mathfrak{D}$-group G contained in a subgroup H of G. Then D is contained in some basis normalizer of H.*

We also obtain an extension of [15, 4.2].

LEMMA 3.8. *If G is a $\mathfrak{D}$-group then each Carter subgroup of G contains a unique basis normalizer of G.*

Proof. Let C be a Carter subgroup of G and suppose D, D^x are basis normalizers of G contained in C $(x \in G)$. Since C is locally nilpotent we have $C \leq R_G(D) \cap R_G(D)^x$ by [7, 3.11]. Now C is abnormal in G by [4, 5.6] so $x \in R_G(D)$. From 3.1 we now obtain $D = D^x$, which completes the proof.

LEMMA 3.9. *Let D be the normalizer of the Sylow basis $\mathfrak{S}$ of the $\mathfrak{U}$-group G. Then* $R_G(D) = \langle C_{S_p}(D_{p'})\,;\ \text{all primes } p\rangle$.

Proof. $\mathfrak{S}$ reduces into D by [4, 2.13] so

$$D_p = D \cap S_p \leq C_{S_p}(D_{p'})$$

for each prime p. Thus $D_{p'} \times C_{S_p}(D_{p'})$ is a locally nilpotent subgroup of G containing D and so, by [7, 5.10], lies in $R_G(D)$. Hence $\langle C_{S_p}(D_{p'})\,;\ \text{all } p\rangle \leq R_G(D)$. We complete the proof by showing that if A/B is a D-central D-composition factor of G then $D(A/B) = N_{DA}(\mathrm{S} \cap DA)$ is contained in $\langle C_{S_p}(D_{p'})\,;\ \text{all } p\rangle$; the result then follows from [7, 5.10].

Let $H = D(A/B) = N_{DA}(\mathrm{S} \cap DA)$. S reduces into the locally nilpotent subgroup H which contains D so $H_p = H \cap S_p$ centralizes $H_{p'}$ and hence $D_{p'}$. Thus $H_p \leq C_{S_p}(D_{p'})$ and hence

$$H \leq \langle C_{S_p}(D_{p'})\,;\quad \text{all } p\rangle,$$

as required.

We use Lemma 3.9 to establish the following extension of [15, 5.2].

THEOREM 3.10. *Suppose the $\mathfrak{U}$-group G has a normal Sylow p-subgroup P such that $G/P \in \mathfrak{D}$, for some prime p. Let D_p, $D_{p'}$ denote the unique Sylow*

p- and Sylow p'-subgroup of the basis normalizer D of G respectively. Then $G \in \mathfrak{D}$ if and only if $D_p \lhd C_P(D_{p'})$.

Proof. Suppose firstly that $G \in \mathfrak{D}$. Then $C_P(D_{p'}) \leq N_G(D)$ by 3.1 and 3.9. Now D_p is a characteristic subgroup of D contained in $C_P(D_{p'})$ so it follows that $D_p \lhd C_P(D_{p'})$.

Conversely suppose $D_p \lhd C_P(D_{p'})$. Let D be the normalizer of the Sylow basis $\mathbf{S}$ of G. Then $\mathbf{S}$ reduces into D by [4, 2.13], so $D_p = D \cap S_p = D \cap P$, $D_{p'} = D \cap S_{p'}$. If $R = R_G(D)$ then $\mathbf{S}$ also reduces into R by [7, 3.3]. Let $R_{p'} = R \cap S_{p'}$. Now DP/P is a basis normalizer of the $\mathfrak{D}$-group G/P so, by [7, 3.5] and 3.1, $DP/P \lhd RP/P$. Therefore

$$[D_{p'}, R_{p'}] \leq S_{p'} \cap DP = S_{p'} \cap D_{p'}P = D_{p'}(S_{p'} \cap P) = D_{p'}.$$

Thus $D_{p'} \lhd R_{p'}$. If $q \neq p$ then D_q is a characteristic subgroup of $D_{p'}$ and $C_{S_q}(D_{q'}) \leq R_{p'}$ by 3.9, so that $C_{S_p}(D_{q'})$ normalizes D_q and hence D. Thus $C_{S_q}(D_{q'}) \leq N_G(D)$ for each prime $q \neq p$. Now $D_p \lhd C_P(D_{p'})$ and $S_p = P$ so $C_{S_p}(D_{p'}) \leq N_G(D)$. Therefore $R_G(D) \leq N_G(D)$ by 3.9. Hence $R_G(D) = N_G(D)$ and $G \in \mathfrak{D}$ by 3.1.

COROLLARY 3.11. *Suppose the $\mathfrak{U}$-group G has a normal abelian Sylow p-subgroup P such that $G/P \in \mathfrak{D}$, for some prime p. Then $G \in \mathfrak{D}$.*

If $G \in \mathfrak{U}$ and $p_1, p_2, \cdots, p_r$ are distinct primes we say G has a *Sylow tower of complexion* $p_1, p_2, \cdots, p_r$ if G has a normal series $1 = G_0 \leq G_1 \leq \cdots \leq G_r = G$ such that

$$G_i/G_{i-1} \in \mathrm{Syl}_{p_i}(G/G_{i-1}) \quad \text{for each } i = 1, \cdots, r.$$

Now $\mathfrak{U} \cap (\mathrm{L}\mathfrak{N})^2 \leq \mathfrak{D}$ by [4, 5.1 and 5.6]. Thus by repeated application of 3.11 we obtain:

COROLLARY 3.12. *Suppose the $\mathfrak{U}$-group G has a Sylow tower of complexion $p_1, p_2, \cdots, p_r$ where $r \geq 3$ and abelian Sylow p_i-subgroups for $1 \leq i \leq r-2$. Then $G \in \mathfrak{D}$.*

Remark. Corollaries 3.11 and 3.12 generalize similar results [15, 5.3 and 5.4] of Rose. He also shows [5, 5.6] that in general $r-2$ may not be replaced by $r-3$ in 3.12.

Our final result is a generalization of [15, 5.5]. To prove it we require:

LEMMA 3.13. *Suppose the $\mathfrak{U}$-group G has a normal Sylow p-subgroup P such that $G/P \in \mathfrak{D}$, for some prime p. Suppose further that for each locally nilpotent ascendabnormal subgroup H of G there is a basis normalizer D of G such that $D \lhd H$. Then $G \in \mathfrak{D}$.*

Proof. Let D be a basis normalizer of G. Then

$$D_p = D \cap P \leq C_P(D_{p'}) = P_0.$$

Now $D^* = D_{p'} \times P_0$ is a locally nilpotent subgroup of DP containing D so, by [7, 3.11], is contained in $\bar{R} = R_{DP}(D)$. By [7, 4.23], $\bar{R}$ is a Carter subgroup of DP so $\bar{R}$ is abnormal in DP by [4, 5.6]. Also DP/P is a basis normalizer of G/P so, by [10, 4.6], is subabnormal in G/P. Therefore DP, and hence $\bar{R}$, is subabnormal in G. Since $\bar{R}$ is locally nilpotent there is, by hypothesis, a basis normalizer D^x of G such that $D^x \lhd \bar{R}$. Thus

$$P_0^{x^{-1}} \leq \bar{R}^{x^{-1}} \leq N_G(D) \leq N_G(D_{p'}).$$

Now $P \lhd G$ so therefore

$$P_0^{x^{-1}} \leq P \cap N_G(D_{p'}) = C_P(D_{p'}) = P_0 .$$

Hence $P_0^{x^{-1}} = P_0$ since a subgroup of a locally finite group cannot be conjugate to a proper subgroup of itself (cf. the proof of [9, 3.1]). Thus $P_0 \leq N_G(D) \leq N_G(D_p)$. Since $D_p \leq P_0$ we therefore have $D_p \lhd P_0$, which by 3.10 is enough to prove the lemma.

THEOREM 3.14. *Suppose the* $\mathfrak{U}$*-group* G *has* p*-length at most one for each prime* p*. Then* $G \in \mathfrak{D}$ *if and only if for each locally nilpotent subgroup* H *which is ascendabnormal in* G *there is a basis normalizer* D *of* G *such that* $D \lhd H$.

Proof. Suppose firstly that $G \in \mathfrak{D}$ and let H be a locally nilpotent ascendabnormal subgroup of G. Then H contains some basis normalizer D of G by [10, 4.6], and since $G \in \mathfrak{D}$, $R_G(D) = N_G(D)$. But $H \leq R_G(D)$ by [7, 3.11], so $D \lhd H$ giving the necessity of our condition.

Suppose conversely that for each locally nilpotent ascendabnormal subgroup H of G there is a basis normalizer D of G such that $D \lhd H$. Let N be a normal subgroup of G and H/N a locally nilpotent ascendabnormal subgroup of G/N. Then H is ascendabnormal in G and if X is a Carter subgroup of H then $H/N = XN/N$ and X is abnormal in H. Thus X is a locally nilpotent ascendabnormal subgroup of G, so by hypothesis there is a basis normalizer D of G such that $D \lhd X$. Therefore DN/N is a basis normalizer of G/N such that $DN/N \lhd XN/N = H/N$. Since G/N certainly has p-length at most one for each prime p, the hypothesis on G carry over to G/N and hence to every factor group of G. We are now in a position to prove the result by induction on the $\mathrm{L}\mathfrak{N}$-length of G.

If G has $\mathrm{L}\mathfrak{N}$-length at most two then there is nothing to prove since the class $\mathfrak{U} \cap (\mathrm{L}\mathfrak{N})^2 \leq \mathfrak{D}$. Suppose then that $\ell(G) > 2$ and let $R = \rho(G)$. Then, as shown above, G/R satisfies the same hypotheses as G so by induction $G/R \in \mathfrak{D}$. Let p be any prime. Then $R \leq O_{p'p}(G)$ so $G/O_{p'p}(G) \in \mathfrak{D}$ since this class is Q-closed by 3.3. Now G has p-length at most one so $G/O_{p'}(G)$ has a normal Sylow p-subgroup, namely $O_{p'p}(G)/O_{p'}(G)$. Since $G/O_{p'}(G)$ satisfies the same hypotheses as G and $G/O_{p'p}(G) \in \mathfrak{D}$ we have $G/O_{p'}(G) \in \mathfrak{D}$ by 3.13. But $\mathfrak{D}$ is a $\mathfrak{U}$-formation (3.3) and $\bigcap_p O_{p'}(G) = 1$, so we finally have $G \in \mathfrak{D}$, which completes the proof.

The material in this paper forms part of a thesis submitted to the University of Warwick in 1971 for the degree of Doctor of Philosophy. I would like to thank my supervisor Dr. B. Hartley for all his help and encouragement.

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