DECOMPOSITIONS OF E^3 INTO STRONGLY CELLULAR SETS

BY

ARLO W. SCHURLE

1. Introduction

Bean [3] has asked whether a monotone decomposition of E^3 into points and a null sequence of strongly cellular sets yields E^3 as decomposition space. This has been answered positively by M. Gerlach [9]. We may now ask whether Bean's question has a positive answer when "null sequence" is replaced by "countably many". Armentrout [1] has posed this problem for tame 3-cells. This paper shows that these questions are equivalent.

Bing and Kirkor introduced the notion of strong cellularity in [5] and proved that strongly cellular arcs in E^3 are tame. Other results concerning strong cellularity can be found in [4], [6], and [7]. In particular, in [7] it is shown that a one-dimensional strongly cellular set in E^3 is a tame dendrite. A recent result of Snay [8] is that decompositions of E^3 into points and countably many tame dendrites yields E^3 as decomposition space. Thus among Armentrout's questions 1, 2 and 3 in [1] and the generalization of Bean's in [3], the most crucial one yet open is that concerning tame 3-cells.

2. Preliminaries and statement of results

We first recall the original definition of strong cellularity given in [5] as modified in [6]. I denotes the real interval $[0, 1]$. A homotopy of S in T is a continuous function H from $S \times I$ into T, and H_t denotes the function given by $H_t(x) = H(x, t)$. Also, if C is a cell, then Bd C, Int C denote its combinatorial boundary and interior respectively.

A set Z in E^n is strongly cellular if there is an n-cell C in E^n and a homotopy H of C in C such that, if S = Bd C, then

(1) H_0 is the identity map, and $H_t \mid Z$ is the identity for all t,
(2) $H_t \mid S$ is a homeomorphism and $H_t(S) \cap Z = \emptyset$ for $t < 1$,
(3) $H_t(S) \cap H_u(S) = \emptyset$ for $t \neq u$, and
(4) $H_1(C) = Z$.

The following proposition is obvious.

PROPOSITION 1. *If Z is a strongly cellular subset of E^n and h is a homeomorphism of an open subset of E^n containing Z into E^n, then $h(Z)$ is strongly cellular.*

Using conditions (2) and (3) in the definition of strong cellularity and

Received May 15, 1972.

well-known results from the topology of E^3 we easily obtain the following proposition.

PROPOSITION 2. *Let Z be a strongly cellular subset of E^3 and ε a positive number. Then there is a tame 3-cell C in E^3 and a homotopy H of C into itself such that*

(1) *H_0 is the identity map and $H_t \mid Z$ is the identity for all t,*
(2) *$H \mid \mathrm{Bd}\ C \times [0, 1)$ is a homeomorphism onto $C \backslash Z$*
(3) *H_t is a homeomorphism and $H_t(C)$ is tame for $t < 1$,*
(4) *$H_1(C) = Z$, and*
(5) *$H(p \times [0, 1])$ has diameter less than ε for each p in C.*

Recall that an upper semi-continuous decomposition of a topological space X is a collection G of pair-wise disjoint subsets of X such that $\bigcup G = X$ and whenever U is an open subset of X containing an element g of G, then there is an open subset V of X with $g \subset V \subset U$ and such that V is the union of elements of G. It is well known that G is an upper semi-continuous decomposition of X if and only if there is a closed continuous mapping f of X onto a space Y such that $G = \{f^{-1}(y) : y \in Y\}$. If G is an upper semi-continuous decomposition of X, then H_G will denote the set of nondegenerate elements of G. Also, P_G will denote the projection map from X onto the decomposition space X/G associated with G.

A notion of equivalent decompositions was introduced in [2] for the case in which $\mathrm{Cl}(P_G(\bigcup H_G))$ is compact and zero-dimensional. (Cl denotes closure.) We extend this notion to the general case and in so doing make it more restrictive, in that two decompositions may be equivalent in the sense of [2] but not in our sense.

Let F and G be upper semi-continuous decompositions of a space X. Then we say that F and G are equivalent if there is a homeomorphism h of X/G onto X/F such that $h \mid P_G(H_G)$ is a homeomorphism onto $P_F(H_F)$. We may now state our main theorem as follows.

THEOREM. *Let G be an upper semi-continuous decomposition of E^3 such that H_G consists of countably many strongly cellular sets. Then G is equivalent to an upper semi-continuous decomposition F of E^3 such that H_F consists of tame 3-cells.*

Since tame cells in E^3 are strongly cellular, this result shows that the generalized question of Bean [3] and Armentrout [1, p. 5, Question 1] are equivalent. It also shows that if all decompositions into points and tame 3-cells yield E^3 [1, p. 5, Question 1] then so do all decompositions into points and countably many tame disks [1, p. 5, Question 2]. Further, an application of Theorem 9 of [2] yields the following result.

COROLLARY 1. *Let G be an upper semi-continuous decomposition of E^3 such that H_G consists of countably many strongly cellular sets. Then G is equiva-*

lent to an upper semi-continuous decomposition F of E^3 such that H_F consists of polyhedral 3-cells.

Since a point is a strongly cellular set, we can apply Theorem 9 of [2] and the procedures used in the proof of our main theorem to obtain the following.

COROLLARY 2. *Let G be an upper semi-continuous decomposition of E^3 such that H_G is countable and E^3/G is topologically E^3. Then G is equivalent to an upper semi-continuous decomposition F of E^3 such that H_F is a null sequence of polyhedral 3-cells.*

3. Proof of the theorem

Let G be an upper semi-continuous decomposition of E^3 such that H_G consists of the strongly cellular sets $g_1, g_2, \cdots$. Let V be an open subset of E^3 such that $\cup H_G \subset V$ and each component of V is bounded.

We inductively construct maps $\phi_0, \phi_1, \cdots, \phi_r, f_0, f_1, \cdots, f_r$, tame 3-cells $C_1, C_2, \cdots, C_r$ and for each i a homotopy H^i of C_i into itself satisfying the eleven conditions to follow. We let ϕ_0 and f_0 be the identity map of E^3, and for notational convenience we let

$$D_i = H^i(C_i \times \{\tfrac{1}{2}\}), \quad F_i = f_1 \circ f_2 \circ \cdots \circ f_i \quad \text{and} \quad \Phi_i = \phi_i \circ \phi_{i-1} \circ \cdots \circ \phi_0 .$$

Φ_i is defined only on the set of points on which the composition is defined. The first step of the induction is omitted since it follows the inductive step almost exactly. The conditions to be satisfied are as follows. (d denotes the usual metric on E^3.)

(1) C_i and H^i satisfy the conditions of Proposition 2 for $Z = \Phi_{i-1}(g_i)$ and an ε to be chosen in the proof.

(2) C_i and D_j are disjoint for $i \neq j$.

(3) C_i is a subset of V.

(4) ϕ_i is a homeomorphism of $E^3 \setminus \Phi_{i-1}(g_i)$ onto $E^3 \setminus D_i$.

(5) $d(\phi_i(p), p) < (\tfrac{1}{2})^i$ for all $p \in E^3$.

(6) ϕ_i is the identity on $E^3 \setminus C_i$.

(7) f_i is a uniformly continuous closed mapping of E^3 onto itself.

(8) $d(f_i(p), p) < (\tfrac{1}{2})^i$ for all $p \in E^3$.

(9) $f_i(D_i) = \Phi_{i-1}(g_i)$.

(10) $f_i = \phi_i^{-1}$ on $E^3 \setminus D_i$.

(11) $d(F_i(p), F_{i-1}(p)) < (\tfrac{1}{2})^i$ for all $p \in E^3$.

We assume that we have ϕ_i, f_i, C_i, and H^i for $i = 1, 2, \cdots, r$, and we now produce $\phi_{r+1}, f_{r+1}, C_{r+1}$, and H^{r+1}. Conditions (3), (4), and (6) imply that Φ_r is a homeomorphism of $E^3 - \cup_{i=1}^r g_i$ onto $E^3 - \cup_{i=1}^r D_i$ which is the identity on $E^3 - V$. Proposition 1 then states that $\Phi_r(g_{r+1})$ is a strongly cellular subset of V. By condition (7), F_r is uniformly continuous, so that there is a positive number δ such that $d(p, q) < \delta$ implies $d(F_r(p), F_r(q)) < (\tfrac{1}{2})^{r+1}$.

An application of Proposition 2 with $Z = \Phi_r(g_{r+1})$ and

$$\varepsilon = \min\{\delta, (\tfrac{1}{2})^{r+1}, d(\Phi_r(g_{r+1}), E^3\backslash(V \cup \bigcup_{i=1}^{r} D_i))\}$$

delivers a tame 3-cell C_{r+1} and a homotopy H^{r+1} satisfying conditions (1), (2) and (3).

Define a function ϕ_{r+1} by

$$\begin{aligned}\phi_{r+1}(p) &= p && \text{if} \quad p \notin C^{r+1}\\ &= H^{r+1}(p', t/2) && \text{if} \quad p = H^{r+1}(p', t) \text{ with } p' \in \operatorname{Bd} C^{r+1} \text{ and } t < 1.\end{aligned}$$

It is easy to check that ϕ_{r+1} satisfies conditions (4), (5), and (6).

Now define a function f_{r+1} by

$$\begin{aligned}f_{r+1}(p) &= p && \text{if} \quad p \notin \operatorname{Int} C_{r+1} \quad \text{or} \quad p \in \Phi_r(g_{r+1})\\ &= H^{r+1}(p', \min\{2t, 1\}) && \text{if} \quad p = H^{r+1}(p', t) \quad \text{for} \quad p' \in \operatorname{Bd} C_{r+1}.\end{aligned}$$

The conditions on H^{r+1} imply that f_{r+1} is a well-defined continuous function. Since f_{r+1} is the identity outside the compact set C_{r+1}, condition (7) is satisfied. Using our choice of ε and the fact that

$$\operatorname{diam} H^{r+1}(p' \times [0, 1]) < \varepsilon \quad \text{for} \quad p \in C_{r+1},$$

we can easily show that conditions (8) and (11) are satisfied. Now

$$H^{r+1}(C_{r+1} \times \{\tfrac{1}{2}\})$$

is the same as $H^{r+1}(\operatorname{Bd} C_{r+1} \times [\tfrac{1}{2}, 1]) \cup \Phi_r(g_{r+1})$, so that

$$f_{r+1}(D_{r+1}) = f_{r+1}(H^{r+1}(C_{r+1} \times \{\tfrac{1}{2}\})) = \Phi_r(g_{r+1})$$

by definition of f_{r+1}. Finally, it is easy to check that f_{r+1} and Φ_{r+1} satisfy condition (10).

Our induction is complete, so that we have sequences of maps, three-cells, etc. satisfying the stated eleven conditions. Conditions (7) and (11) imply that $F_1, F_2, \cdots$ is a uniformly convergent sequence of closed continuous mappings of E^3 onto itself. Hence the limit $F = \lim_{i\to\infty} F_i$ is a closed continuous mapping of E^3 onto itself. We will be done when we show that $F^{-1}(g)$ is a tame 3-cell for $g \in H_G$ and $F^{-1}(p)$ is a point for $p \in E^3\backslash \bigcup H_G$, since it is easy to check that the map taking $F^{-1}(p)$ to the element of G containing p is the desired homeomorphism between E^3/G and the decomposition space induced by F.

Conditions (2), (6) and (10) imply that $F = f_1 \circ f_2 \circ \cdots \circ f_j$ on D_j. Thus by conditions (1), (9), and (10) we have that $F(D_j) = g_j$ for all positive integers j, so that $F^{-1}(g_j) \supset D_j$.

We now show that if $F(q)$ is a point of g_j, then q is a point of D_j. An analysis like that in the preceding paragraph shows that if there is an integer m such that $q \notin C_i$ for $i \geq m$, then q must be in D_j. So we suppose that there are integers $r_1 < r_2 < \cdots$ such that $q \in C_{r_i}$, $i = 1, 2, \cdots$. Since a single component of V must contain all the C_{r_i}'s and components of V are bounded, we may assume that the corresponding g_{r_i}'s converge to an element g of G. Since $F(D_{r_i}) = g_{r_i}$ and

$$q \in C_{r_i} \subset B(D_{r_i}, (\tfrac{1}{2})^i),$$

$F(q)$ is an element of g, so that $g = g_j$. We may now suppose that $j < r_1 < r_2 < \cdots$. Then $\Phi_j(g_{r_i})$ converges to D_j as $i \to \infty$. By condition (5), the C_{r_i}'s converge to D_j as $i \to \infty$, so that q must be an element of D_j. Hence $F^{-1}(g_j) = D_j$, a tame 3-cell, for all positive integers j.

Finally, suppose that $F(q) = x \notin \bigcup H_G$. Then $\Phi_i(x)$ is defined for all positive integers i, and by condition (5), $\Phi_i(x)$ converges to a point y as $i \to \infty$. Let ε be a positive number. There is an integer N such that

$$\textstyle\sum_{i=N}^{\infty} (\tfrac{1}{2})^i < \varepsilon \quad \text{and} \quad d(\Phi_i(x), y) < \varepsilon \quad \text{for} \quad i \geq N.$$

By (10), $F_N(\Phi_N(x)) = x$, and since $x \notin \bigcup H_G$, there is a neighborhood U of x such that the restriction of F_N is a homeomorphism between $F_N^{-1}(U)$ and U. Since $\Phi_N(x)$ is in $B(y, \varepsilon)$, we may suppose that $F_N^{-1}(U)$ is a subset of $B(y, \varepsilon)$.

Now $F_i(q)$ converges to x as $i \to \infty$, so that

$$F_N(f_{N+1} \circ f_{N+2} \circ \cdots \circ f_{N+r}(q))$$

is eventually in $F_N^{-1}(U) \subset B(y, \varepsilon)$. But

$$d(q, f_{N+1} \circ f_{N+2} \circ \cdots \circ f_{N+r}(q)) < \textstyle\sum_{i=N}^{\infty} (\tfrac{1}{2})^i < \varepsilon$$

by condition (8) and our choice of N, so that $d(q, y) < 2\varepsilon$. Since ε is an arbitrary positive number, we must have $q = y$. Thus $F^{-1}(x)$ is a single point for each x in $E^3 \setminus \bigcup H_G$. This completes the proof.

References

1. S. Armentrout, *Monotone decompositions of E^3*, Topology Seminar (Wisconsin, 1965), Ann. of Math. Studies, No. 60, Princeton Univ. Press, Princeton, New Jersey, 1966, pp. 1–25.
2. S. Armentrout, L. L. Lininger, and D. V. Meyer, *Equivalent decompositions of R^3*, Pacific J. Math., vol. 24 (1968), pp. 205–227.
3. Ralph J. Bean, *Decompositions of E^3 with a null sequence of starlike equivalent nondegenerate elements are E^3*, Illinois J. Math., vol. 11 (1967), pp. 21–23.
4. ———, *Extending monotone decompositions of 2-spheres to trivial decompositions of E^3*, Duke Math. J., vol. 38 (1971), pp. 539–544.
5. R. H. Bing and A. Kirkor, *An arc is tame in 3-space if and only if it is strongly cellular*, Fund. Math., vol. 55 (1964), pp. 175–180.
6. H. C. Griffith and L. R. Howell, Jr., *Strongly cellular cells in E^3 are tame*, Fund. Math., vol. 65 (1969), pp. 23–32.
7. Arlo W. Schurle, *Strongly cellular subsets of E^3*, Fund. Math., to appear.
8. Richard Snay, *Decompositions of Euclidean 3-space into points and countably many dendrites*, Doctoral dissertation, Indiana University, 1972.
9. Mary Anne Gerlach, Doctoral dissertation, University of Wisconsin.

The University of North Carolina
Charlotte, North Carolina