# ON THE NORM OF THE SUM OF CERTAIN OPERATORS

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In [1] the need arises to calculate the norm of the sum of two operators, one a unitary operator arising from an ergodic point transformation on a measure space, the other a projection associated with a subset of the measure space. The purpose of this note is to show that we can often replace the projection operator by a more general operator, a multiplication by a simple function, and still obtain an explicit expression for the norm of the sum. The interplay between the point transformation and the simple function will determine the norm in question.

Let us fix terminology. Let $(X, \mu)$ be a measure space where $\mu$ is a finite, nonatomic measure normalized so that $\mu(X) = 1$. Let $\alpha$ be an ergodic, measure preserving, invertible transformation of $X$ onto itself. (It is assumed that $\alpha$ and $\alpha^{-1}$ are measurable. *Ergodic* means that any measurable subset of $X$ invariant under $\alpha$ has measure 0 or 1.) $L^\infty(X)$ is the space of all bounded measurable functions on $X$. We associate a unitary operator with $\alpha$ and a bounded linear operator with each function $g$ in $L^\infty(X)$ in the standard way: define $U_\alpha$ by $U_\alpha f = f \circ \alpha^{-1}$, for each $f \in L^2(X)$, and define $L_g$ by $L_g f = gf$, for each $g \in L^\infty(X)$ and each $f \in L^2(X)$. It is $\|U_\alpha + L_g\|$ which we wish to calculate. In the event $g$ is the characteristic function $\chi_E$ of some measurable subset $E$ of $X$, we write $P_E$ in place of $L_{\chi_E}$. The fact that $U_\alpha$ is unitary follows from the assumption that $\alpha$ is measure preserving.

We prove first a simple lemma.

LEMMA. *Let $g \in L^\infty(X)$ and let $E$ be a measurable subset of $X$ such that $g(x) = 0$ if $x \notin E$. Let $F$ be a measurable subset of $E$ satisfying $\alpha(F) \cap E \subseteq F$ and $\alpha^{-1}(F) \cap E \subseteq F$. Then $P_{F \cup \alpha(F)}$ commutes with $L_g U_\alpha^{-1} + I$.*

*Proof.* Of course, it is sufficient to show that $P_{F \cup \alpha(F)}$ commutes with $L_g U_\alpha^{-1}$. To do this we shall show that $L_g U_\alpha^{-1}$ leaves invariant both the subspace $L^2(F \cup \alpha(F))$ and its orthogonal complement in $L^2(X)$.

Assume that $f \in L^2(F \cup \alpha(F))$. We may assume that $f(x) = 0$ whenever $x \notin F \cup \alpha(F)$. For every $x$, $(L_g U_\alpha^{-1} f)(x) = g(x)f(\alpha(x))$. To prove that $L_g U_\alpha^{-1} f$ lies in $L^2(F \cup \alpha(F))$, we must show that $g(x)f(\alpha(x)) = 0$ whenever $x \notin F \cup \alpha(F)$. This is evidently the case whenever $x \notin E$ so we consider only $x \in E$. Since $\alpha^{-1}(F) \cap E \subseteq F$ and $x \notin F$ we must have $x \notin \alpha^{-1}(F)$, and hence $\alpha(x) \notin F$. But $x \notin F$ also implies $\alpha(x) \notin \alpha(F)$, so $\alpha(x) \notin F \cup \alpha(F)$ and we have $f(\alpha(x)) = 0$. This proves the invariance of $L^2(F \cup \alpha(F))$.

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Now suppose $f$ is orthogonal to $L^2(F \cup \alpha(F))$. Hence, we may assume $f(x) = 0$ whenever $x \in F \cup \alpha(F)$. We wish to prove that $g(x)f(\alpha(x)) = 0$, assuming that $x \in F \cup \alpha(F)$. Again, this is evident when $x \notin E$, so we assume that $x \in E$. Then

$$x \in E \cap (F \cup \alpha(F)) \subseteq F;$$

hence $\alpha(x) \in \alpha(F) \subseteq F \cup \alpha(F)$ and, consequently, $f(\alpha(x)) = 0$. Thus $L_g U_\alpha^{-1} f$ is orthogonal to $L^2(F \cup \alpha(F))$ and the lemma is proven.

Our goal is to calculate $\|U_\alpha + L_g\|$ whenever $g$ is a simple function whose support is less than the whole space $X$; more precisely, for which $\mu(\{x \mid g(x) = 0\}) > 0$. It is advantageous to consider first a special case and then use the lemma above to reduce the general case to the special one.

Let $E_1, E_2, \ldots, E_n$ be disjoint measurable subsets of $X$ and assume $g$ has the form $g = \sum_{i=1}^n a_i \chi_{E_i}$ where the $a_i$ are arbitrary complex numbers. Assume further that $\alpha(E_i) = E_{i+1}$ for $i = 1, \ldots, n-1$ and that $\alpha(E_n)$ is disjoint from $\bigcup_{i=1}^n E_i$. Define a real $n+1$ by $n+1$ matrix $B = (b_{ij})$ as follows:

$$\begin{aligned} b_{i,i} &= |a_{i-1}|^2 && \text{for } i = 2, \ldots, n+1 \\ b_{i+1,i} &= b_{i,i+1} = |a_i| && \text{for } i = 1, \ldots, n \\ b_{i,j} &= 0 && \text{for all other } i, j. \end{aligned}$$

This matrix has the sequence $0, |a_1|^2, |a_2|^2, \ldots, |a_n|^2$ down the main diagonal and the sequence $|a_1|, |a_2|, \ldots, |a_n|$ immediately above and below the main diagonal.

PROPOSITION. $\|U_\alpha + L_g\| = (1 + \|B\|)^{1/2}$.

*Proof.* Since $\|L_g + U_\alpha\| = \|L_g U_\alpha^{-1} + I\|$, we need only calculate the latter. Let $E = \bigcup_{i=1}^n E_i$. By the lemma, $P_{E \cup \alpha(E)}$ commutes with $L_g U_\alpha^{-1} + I$; therefore

$$\|L_g U_\alpha^{-1} + I\| = \max\left\{\|(L_g U_\alpha^{-1} + I)P_{E \cup \alpha(E)}\|, \|(L_g U_\alpha^{-1} + I)(I - P_{E \cup \alpha(E)})\|\right\}.$$

It is routine to check that $L_g U_\alpha^{-1}(I - P_{E \cup \alpha(E)}) = 0$, and from this we obtain

$$\|(L_g U_\alpha^{-1} + I)(I - P_{E \cup \alpha(E)})\| \leq 1.$$

Therefore, we can prove the proposition by showing that

$$\|(L_g U_\alpha^{-1} + I)P_{E \cup \alpha(E)}\| = (1 + \|B\|)^{1/2}.$$

Suppose $f \in L^2(E \cup \alpha(E))$ and $\|f\| = 1$. Let $f_i = f \mid E_i$, for $i = 1, \ldots, n$, and $f_{n+1} = f \mid \alpha(E_n)$. Note that, for each $i = 1, \ldots, n$, $U_\alpha^{-1} f_{i+1} = f_{i+1} \circ \alpha$ lies in $L^2(E_i)$. Then

$$(L_g U_\alpha^{-1} + I)f = \sum_{i=1}^n (a_i f_{i+1} \circ \alpha + f_i) + f_{n+1}.$$

The terms in this sum are all mutually orthogonal, so

$$\begin{aligned}
\|(L_g U_\alpha^{-1} + I)f\|^2 \\
&= \sum_{i=1}^{n} \|a_i f_{i+1} \circ \alpha + f_i\|^2 + \|f_{n+1}\|^2 \\
&= \sum_{i=1}^{n} |a_i|^2 \|f_{i+1} \circ \alpha\|^2 + \sum_{i=1}^{n} 2 \operatorname{Re} \bar{a}_i \langle f_i, f_{i+1} \circ \alpha \rangle + \sum_{i=1}^{n+1} \|f_i\|^2 \qquad (*) \\
&= 1 + \sum_{i=1}^{n} |a_i|^2 \|f_{i+1}\|^2 + \sum_{i=1}^{n} 2 \operatorname{Re} \bar{a}_i \langle f_i, f_{i+1} \circ \alpha \rangle \\
&\le 1 + \sum_{i=1}^{n} |a_i|^2 \|f_{i+1}\|^2 + \sum_{i=1}^{n} 2|a_i| \, \|f_i\| \, \|f_{i+1}\|.
\end{aligned}$$

(Note that in the above we have made use of the facts that $\sum_{i=1}^n \|f_i\|^2 = \|f\|^2 = 1$ and $\|f_i \circ \alpha\| = \|f_i\|$, for each $i$.)

Consider the quadratic form

$$Q(x) = \sum_{i=1}^{n} |a_i|^2 x_{i+1}^2 + \sum_{i=1}^{n} 2|a_i| x_i x_{i+1}$$

on $\mathbf{R}^{n+1}$. We claim that $Q(x) = \langle Bx, x \rangle$, where $B$ is the matrix described in the paragraph preceding the proposition. Indeed, if

$$x = (x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$$

then

$$\begin{aligned}
Bx = (&|a_1| x_2, |a_1| x_1 + |a_1|^2 x_2 + |a_2| x_3, \ldots, |a_{n-1}| x_{n-1} \\
&+ |a_{n-1}|^2 x_n + |a_n| x_{n+1}, |a_n| x_n + |a_n|^2 x_{n+1}),
\end{aligned}$$

whence

$$\begin{aligned}
\langle Bx, x \rangle = {} & |a_1| x_2 x_1 + |a_1| x_1 x_2 + |a_1|^2 x_2^2 + \cdots \\
&+ |a_n| x_{n+1} x_n + |a_n| x_n x_{n+1} + |a_n|^2 x_{n+1}^2 = Q(x).
\end{aligned}$$

Since $B$ is a self-adjoint matrix, $\sup_{\|x\|=1} Q(x) = \|B\|$. It follows from the inequality in (*) that $\|(L_g U_\alpha^{-1} + I)f\|^2 \le 1 + \|B\|$, from which we obtain $\|L_g U_\alpha^{-1} + I\| \le (1 + \|B\|)^{1/2}$.

It remains to show that we actually have equality. Fix

$$x = (x_1, \ldots, x_{n+1})$$

in $\mathbf{R}^{n+1}$ such that $\|x\| = 1$ and $Q(x) = \sup_{\|y\|=1} Q(y)$. Such an $x$ exists since the unit sphere in $\mathbf{R}^{n+1}$ is compact. Since replacing each $x_i$ by $|x_i|$ can only raise the value of $Q$, we may assume each $x_i \ge 0$. Let $f_1$ be any element of $L^2(E_1)$ such that $\|f_1\| = x_1$. We define $f_{i+1}$, $i = 1, \ldots, n$, recursively as follows:

If $x_i \ne 0$, let

$$f_{i+1} = \frac{\bar{a}_i}{|a_i|} \cdot \frac{x_{i+1}}{x_i} f_i \circ \alpha^{-1}.$$

If $x_i = 0$, let $f_{i+1}$ be any element of $L^2(E_{i+1})$ with $\|f_{i+1}\| = x_{i+1}$.

Therefore, if $x_i \neq 0$ we have

$$f_{i+1} \circ \alpha = \frac{\bar{a}_i}{|a_i|} \frac{x_{i+1}}{x_i} f_i$$

and hence

$$2 \operatorname{Re} \bar{a}_i \langle f_i, f_{i+1} \circ \alpha \rangle = 2 \operatorname{Re} |a_i| \frac{x_{i+1}}{x_i} \langle f_i, f_i \rangle = 2|a_i| x_i x_{i+1}.$$

If, on the other hand, $x_i = 0$ then $f_i = 0$ also, and again

$$2 \operatorname{Re} \bar{a}_i \langle f_i, f_{i+1} \circ \alpha \rangle = 0 = 2|a_i| x_i x_{i+1}.$$

If we now let $f = \sum_{i=1}^n f_i$, then $f \in L^2(E \cup \alpha(E))$, $\|f\| = 1$, and the equalities in (*) show that

$$\|(L_g U_\alpha^{-1} + I) f\|^2 = 1 + Q(x) = 1 + \|B\|.$$

Thus $\|L_g U_\alpha^{-1} + I\| = (1 + \|B\|)^{1/2}$.

We turn now to the general case for a simple function $g$ with $\mu(\{x \mid g(x) = 0\}) > 0$. We may write $g = \sum_{i=1}^n a_i \chi_{E_i}$, where the $a_i$ are distinct, the $E_i$ are disjoint and $\mu(\bigcup_{i=1}^n E_i) < 1$. Let $E = \bigcup_{i=1}^n E_i$. To each finite sequence $p = (a_{i_1}, \ldots, a_{i_k})$ with values in $\{a_i \mid i = 1, \ldots, n\}$ we associate a set

$$F_p = \{x \in E \mid \alpha^{-1}(x) \notin E, x \in E_{i_1}, \alpha(x) \in E_{i_2}, \ldots, \alpha^{k-1}(x) \in E_{i_k}, \alpha^k(x) \notin E\}.$$

Let $S = \{p \mid \mu(F_p) > 0\}$. The set $S$ may be finite or infinite. Examples where $S$ is infinite, even in the case where $g$ is the characteristic function of a set, may be constructed easily using the "stacking" methods of [2]. Further, since $\alpha$ is ergodic, for almost every $x$ in $E$ there is some positive integer $k$ such that $\alpha^k(x) \notin E$ and some negative integer $j$ such that $\alpha^j(x) \notin E$.

Let $E_p = F_p \cup \alpha(F_p) \cup \cdots \cup \alpha^{k-1}(F_p)$, where $p$ is a sequence of length $k$. Then the comments above say that, up to a set of measure zero, $E = \bigcup_{p \in S} E_p$. Further $E_p$ satisfies

$$\alpha(E_p) \cap E \subseteq E_p \quad \text{and} \quad \alpha^{-1}(E_p) \cap E \subseteq E_p.$$

Therefore $P_{E_p \cup \alpha(E_p)}$ commutes with $L_g U_{\alpha^{-1}} + I$, and so we obtain

$$\|L_g + U_\alpha\| = \|L_g U_{\alpha^{-1}} + I\| = \sup_{p \in S} \|(L_g U_{\alpha^{-1}} + I) P_{E_p \cup \alpha(E_p)}\|.$$

(It will be evident soon that $\|(L_g U_\alpha^{-1} + I)(I - P_{E \cup \alpha(E)})\|$, which should properly be among the set of numbers over which we take the sup, is actually smaller than any of the other numbers and so may be omitted.) For each sequence $p = (a_{i_1}, \ldots, a_{i_k})$ define a $k + 1$ by $k + 1$ matrix $B(p)$ just as in the paragraph preceding the proposition, i.e., $B(p)$ has the sequence $0, |a_{i_1}|^2, \ldots, |a_{i_k}|^2$ on the diagonal and the sequence $|a_{i_1}|, \ldots, |a_{i_k}|$ immediately above and below the diagonal. Then the proposition says that for $p \in S$,

$$\|(L_g U_\alpha^{-1} + I) P_{E_p \cup \alpha(E_p)}\| = (1 + \|B(p)\|)^{1/2}.$$

Thus we have proven that, with the notation above,

$$\|L_g + U_\alpha\| = \sup_{p \in S} (1 + \|B(p)\|)^{1/2}.$$

There are various questions which arise if one wishes to drop one or another of the assumptions made above. For example, suppose we let $g$ be any $L^\infty$-function with $\mu(\{x \mid g(x) = 0\}) > 0$. Then we may approximate $L_g$ uniformly by operators of the form $L_h$ where $h$ is a simple function with the same support as $g$. Hence the norm $\|L_g + U_\alpha\|$ is approximated by $\|L_h + U_\alpha\|$. This procedure is not fully satisfactory as it does not yield a tractable expression for $\|L_g + U_\alpha\|$.

Two other interesting questions are: what happens if we drop the requirement that $\mu(\{x \mid g(x) = 0\}) > 0$ and, what happens if we drop the assumption of ergodicity. In either instance we encounter a similar difficulty: points in the support of $g$ may always remain there under the action of $\alpha$. This means that we do not obtain a decomposition of the support of $g$ into the easily manageable sets $E_p$ associated with finite sequences as above. We may, to each point $x$ associate an infinite sequence of numbers $a_{i_n}$ determined by the action of $\alpha$ on $x$, but uncountably many such sequences might well appear and each may be associated with a set of measure zero. So an approach analogous to what is done above is impossible.

However, if we go to a far extreme from ergodicity, we can once again calculate the norm for a sum $L_g + U_\alpha$ (and with no assumption about the support of the simple function $g$). This can be done if $\alpha$ has no aperiodic part; more precisely, if, for almost all $x$, $\alpha^n(x) = x$ for some $n$ ($n$ may depend on $x$). If

$$E_n = \{x \mid \alpha^n(x) = x \quad \text{and} \quad \alpha^j(x) \neq x, \quad \text{for } j = 1, \dots, n-1\}$$

then $X = \bigcup_{n=1}^\infty E_n$ (up to a null set). Each $E_n$ is invariant under $\alpha$; hence $P_{E_n}$ commutes with $L_g + U_\alpha$ and

$$\|L_g + U_\alpha\| = \sup_n \|(L_g + U_\alpha)P_{E_n}\|.$$

This reduces the problem of calculating $\|L_g + U_\alpha\|$ to the special case where $\alpha^n(x) = x$ for all $x$ and some fixed $n$.

A further reduction is possible. To each $x$ associate the finite sequence $p(x) = (g(\alpha(x)), \dots, g(\alpha^n(x)))$. Define an equivalence relation $\sim$ on $X$ by saying $x \sim y$ if $p(x)$ is a cyclic permutation of $p(y)$. The set $E_x = \{y \mid y \sim x\}$ is measurable and invariant under $\alpha$; hence $P_{E_x}$ commutes with $L_g + U_\alpha$. There are only finitely many distinct $E_x$ and

$$\|L_g + U_\alpha\| = \max \{\|(L_g + U_\alpha)P_{E_x}\|\}.$$

Thus, the problem reduces to the special case where $g = \sum_{i=1}^n a_i E_i$, the $E_i$ are disjoint, $E_k = \alpha^{k-1}(E_1)$ for $k = 1, \dots, n$, and $E_1 = \alpha(E_n)$. (The $a_i$ are not necessarily distinct, but the assumption $\alpha^n(x) = x$, for all $x$, remains in force.) Let $B = (b_{ij})$ be the matrix given by

$$\begin{aligned} b_{i,i} &= |a_i|^2 \\ b_{i,i+1} &= b_{i+1,i} = |a_{i+1}|, \quad i = 1, \ldots, n-1 \\ b_{1,n} &= b_{n,1} = |a_1| \\ b_{i,j} &= 0 \quad \text{otherwise.} \end{aligned}$$

Then using the same techniques as before, one can show

$$\|L_g + U_\alpha\| = (1 + \|B\|)^{1/2}.$$

## Bibliography

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