CORRECTION TO MY PAPER "A COMBINATORIAL ANALOGUE OF A THEOREM OF MYERS"

BY

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This is a correction of my paper [3]. The proof of Proposition 1 is mistaken (the assertion "$L > |\alpha| + (|\beta_1| - |\alpha|)/3$" does not follow from "$|\partial c| \geq 3$ for any 2-cell").

I use the same notation as in [3]. If K is a 2-manifold, v a vertex, then the curvature $R^*(v)$ is defined to be

$$2 - \sum \{(1 - 2/|\partial c|) \text{ for all 2-cells } c \text{ containing } v\}.$$

PROPOSITION 1. *Let K be a connected cell complex which is a 2-manifold without boundary. Assume:*

(1) *there is a number N such that $|\partial c| \leq N$ for every 2-cell c;*

(2) *there is a number $R > 0$ such that $R^*(v) \geq R$ for every vertex of K. Then K is finite.*

Proof. Metrize K so that each 2-cell is a regular polygon of side-length 1. K has a simplicial subdivision K' which introduces no new vertices. At each vertex $v \in K'$ the piecewise linear curvature (see, for example, [2]) is $\pi R^*(v)$. Moreover each 2-simplex of K' is isometric to one of a finite list of planar triangles by (1). Using the methods of Gluck ([1], quoted in [2]) it is not hard to show that in the intrinsic metric thus defined on $|K'|$, $|K'|$ is complete, and every ball of finite radius contains only finitely many vertices of K'. Since $|K'|$ is complete, Theorem 3 of [2] implies that $|K'|$ is compact. Hence K' has only finitely many vertices; and it follows that K is finite.

The remarks following Proposition 1 are correct, except that in Remark 2 there is no estimate for diam K^*. A corresponding form of Corollary 2 cannot be proved by the method of proof of the present Proposition 1.

The material on variational fields and variations of a path in the dual cell complex to a simplicial manifold need be altered only in the last step: the definition of Ricci curvature. The *Ricci curvature of K^* at b_1 in the direction a_1—a_2* is redefined to be

$$R^*(b, a_1\text{—}a_2) = 8n - \sum_{j=1}^{2n-1} \{|\partial c_j| \text{ for } a_1 < c_j \text{ or } a_2 < c_j\}.$$

In terms of K this is equivalent to redefining

$$R(s, t\text{—}t'') = 8n - \sum_{j=1}^{2n-1} \{N(u_j) \text{ for } u_j < t \textit{ or } < t'' \text{ and } \dim u_j = n - 2\}.$$

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THEOREM 3. *Let K be a connected simplicial n-manifold without boundary, $n \geq 3$. Assume:*

(1) *there is a number N such that $N(u) \leq N$ for all u;*

(2) *there is an $R > 0$ such that $R(s, t\text{—}t'') \geq R$ for all s and $t\text{—}t''$.*

Then K is finite and diam $K \leq (2N - 4)/R + 2$.

Remark. Hypothesis (2) implies that (1) holds with $N = 2n + 6$, since $N(u) \geq 3$ for every u.

Proof. Let K^* be the cell complex dual to K. Let $\alpha = (a_1, \dots, a_r)$ be a path in K^* with $a_i \cap a_{i+1} = b_i$ for $i = 1, \dots, r - 1$; assume α is a shortest path between its endpoints. Let $C_1, \dots, C_n$ be the special variational fields, and $\beta_1, \dots, \beta_n$ the corresponding variations, of α. For each b_i the curvature hypothesis may be formulated as

$$\sum_{c \in C_1 \cup \cdots \cup C_n,\, b_i \in c} (|\partial c| - 2) \leq 4n + 2 - R.$$

Adding these inequalities for $i = 1, \dots, r - 1$ gives

$$(*) \qquad \sum_{i=1}^{r-1} \sum_{c \in C_1 \cup \cdots \cup C_n,\, b_i \in c} (|\partial c| - 2) \leq (4n + 2 - R)(|\alpha| - 1).$$

Let L denote the left-hand side of $(*)$. Then

$$L = \sum_{j=1}^{n} \sum_{c \in C_j} (|\partial c| - 2)(\text{number of } b_i \text{ which are} \in c).$$

Say $C_j = (c_1^j, \dots, c_{s(j)}^j)$. Then for $c \in C_j$, the number of b_i which are in c is $|\partial_\alpha c|$ if $c = c_1^j$ or $c_{s(j)}^j$, and $|\partial_\alpha c| + 1$ otherwise. So

$$\begin{aligned} L &= \sum_{j=1}^{n} \left[\sum_{c \in C_j} (|\partial c| - 2)(|\partial_\alpha c| + 1) - (|\partial c_1^j| - 2) - (|\partial c_{s(j)}^j| - 2) \right] \\ &\geq \sum_{j=1}^{n} \sum_{c \in C_j} (|\partial c| - 2)(|\partial_\alpha c| + 1) - 2n(N - 2), \end{aligned}$$

using hypothesis (1). Now

$$\begin{aligned} |\partial c| &= |\partial_\alpha c| + |\partial_{\beta_j} c| + 1 \quad \text{if } c = c_1^j \text{ or } c_{s(j)}^j, \\ &= |\partial_\alpha c| + |\partial_{\beta_j} c| + 2 \quad \text{otherwise.} \end{aligned}$$

So

$$\begin{aligned} L &\geq 2 \sum_{j=1}^{n} \sum_{c \in C_j} (|\partial c| - 2) + \sum_{j=1}^{n} \sum_{c \in C_j} (|\partial c| - 2)(|\partial_\alpha c| - 1) - 2n(N - 2) \\ &= 2 \sum_{j=1}^{n} \left[\sum_{c \in C_j} (|\partial_\alpha c| + |\partial_{\beta_j} c|) - 2 \right] + L' - 2n(N - 2), \end{aligned}$$

where L' denotes the second double-sum in the previous inequality. Hence

$$L \geq 2 \sum_{j=1}^{n} (|\alpha| + |\beta_j|) + L' - 2nN$$
$$= 2\left(n|\alpha| + \sum_{j=1}^{n} |\beta_j|\right) + L' - 2nN.$$

To evaluate L', observe that by construction of the C_j, $|\partial_\alpha c| \geq 1$ for all c. Hence

$$L' = \sum \{(|\partial c| - 2)(|\partial_\alpha c| - 1) \quad \text{for } c \in C_1 \cup \cdots \cup C_n \text{ and } |\partial_\alpha c| \geq 2\}$$
$$= \sum_{i=1}^{r-1} \sum \{(|\partial c| - 2) \qquad \text{for } a_i \cup a_{i+1} \subseteq c\},$$

since in this last expression, $(|\partial c| - 2)$ is counted $(|\partial_\alpha c| - 1)$ times. For each i the inner sum reduces to one term. Moreover, for α to be a geodesic—that is, to minimize length locally—$|\partial c|$ must be ≥ 4 whenever the inner sum is nonzero. (Otherwise $a_i \cup a_{i+1}$ could be replaced by the third side of c.) So

$$L' \geq \sum_{i=1}^{r-1} 2 = 2(|\alpha| - 1).$$

Substituting for L and L' in (*) gives

$$2\left(n|\alpha| + \sum_{j=1}^{n} |\beta_j|\right) + 2(|\alpha| - 1) - 2nN \leq (4n + 2 - R)(|\alpha| - 1).$$

Hence $\sum_{j=1}^{n} |\beta_j| \leq n|\alpha| - [|\alpha|R/2 - R/2 - n(N - 2)]$. To assume α is as short as possible is to assume each $|\beta_j| \geq |\alpha|$. Hence

$$|\alpha|R/2 - R/2 - n(N - 2) \leq 0;$$

that is, $|\alpha| \leq 2n(N - 2)/R + 1$.

Thus $\operatorname{diam} K^* \leq 2n(N - 2)/R + 1$. As in [3] it follows that

$$\operatorname{diam} K \leq 2 + (\operatorname{diam} K^* - 1)/n \leq (2N - 4)/R + 2.$$

Theorem 3 is proved.

Remark 1. When $n = 3$, the curvature hypothesis (2) reduces to

$$\sum_{j=1}^{5} N(u_j) \leq 23$$

whenever $u_1, \ldots, u_5$ are edges of a tetrahedron. This occurs, for example, if each tetrahedron has four edges with $N(u) = 4$, one with $N(u) = 5$, and one with $N(u) = 6$.

Remark 2. For general n, the curvature hypothesis (2) requires that "on the average" $N(u)$ be $\leq 4 + (4 - R)/(2n - 1)$. This number times the dihedral angle of a regular n-simplex at an $(n-2)$-face is $< 2\pi$; hence Theorem 3 is in the spirit of Theorem 4 of [2] (quoted in [3]). The Theorem 3 claimed in [3] required that "on the average" $N(u)$ be less than about $4 + (R+2)/(n-1)$. However, here the average is arithmetic; there it was harmonic, which is a weaker requirement on the $N(u_j)$ being averaged.

Remark 3. The main example of [3] is correct (the triangulation of $S^{n-1} \times R$). In this complex $R(s, t\text{—}t'')$ takes values ≥ -2. Thus Theorem 3 cannot be much strengthened. I conjecture that in dimension 3 there is a true theorem with the (weaker) curvature hypothesis of Theorem 3 of [3].

Corollary 4 cannot be proved using the method of proof of Theorem 3.

THEOREM 5. *Theorem* 3 *holds if K is a geometrical n-circuit* (*see* [3]), *with $R(s, t\text{—}t'')$ redefined as*

$$8n - \sum_{j=1}^{2n-1} \{N(u_j; s) \text{ for } u_j < t \text{ or } < t'' \text{ and } \dim u_j = n-2\}.$$

Here $N(u; s)$ denotes the number of 1-*simplexes in that component of* link (u, K) *to which* link (u, s) *belongs.*

REFERENCES

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