NORM-CONSTANT ANALYTIC FUNCTIONS AND EQUIVALENT NORMS

BY

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Let X be a complex Banach space, Δ the open unit disc in C and let $f: \Delta \to X$ be an analytic function satisfying $\|f(\zeta)\| \equiv 1$ $(\zeta \in \Delta)$. If X is strictly c-convex [1] then by a result of Thorp and Whitley [7] f is a constant (see also [5]). If X is not strictly c-convex then there are always nonconstant analytic functions from Δ to X having constant norm on Δ. Such functions were studied in [2], [3] and certain necessary and sufficient conditions were obtained for an analytic function to have constant norm.

Suppose that a nonconstant analytic function $f: \Delta \to X$ has constant norm on an open subset of Δ. An easy application of the Hahn-Banach theorem shows that such an f does not have any zeros on Δ. This shows that there are many analytic functions from Δ to X whose norm is not constant on any open subset of Δ and in any norm on X, equivalent to the original one. In the present paper we give a surprisingly simple complete description of such functions.

Throughout, Δ is the open unit disc in C. If X is a complex Banach space we denote by $S(X)$, X', $L(X)$ the unit sphere of X, the dual space of X and the Banach algebra of all bounded linear operators from X to X, respectively. The image of $x \in X$ under $u \in X'$ is denoted by $\langle x \mid u \rangle$. If T is a subset of X we denote by $\overline{\text{sp}}\ T$ the closed linear subspace spanned by the elements of T.

THEOREM. *Let X be a complex Banach space and let*

$$f(\zeta) = a_0 + \zeta a_1 + \zeta^2 a_2 + \cdots$$

be a nonconstant analytic function from Δ to X. Then

$$a_0 \notin \overline{\text{sp}}\ \{a_1, a_2, a_3, \ldots\}$$

if and only if there exist an equivalent norm $|||\ \ |||$ on X and an open subset $U \subset \Delta$ such that $|||f(\zeta)|||$ is constant on U.

LEMMA 1. *Let X be a complex Banach space and let $f: \Delta \to X$ be an analytic function. Suppose that $\|f(\zeta)\| \equiv c > 0$ on some open subset of Δ. Then $f(\Delta) \subset f(\zeta_0) + \operatorname{Ker} u$ where $\zeta_0 \in \Delta$, $u \in X'$ and $f(\zeta_0) \notin \operatorname{Ker} u$.*

Proof. Assume that $\|f(\zeta)\| \equiv c > 0$ $(\zeta \in U)$ where $U \subset \Delta$ is an open set and let $\zeta_0 \in U$. By the Hahn-Banach theorem there exists $u \in S(X')$ satisfying $\langle f(\zeta_0) \mid u \rangle = c$. Since $|\langle f(\zeta) \mid u \rangle| \leq \|f(\zeta)\| \cdot \|u\| = c$ $(\zeta \in U)$ it follows that

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$\langle f(\zeta) \mid u \rangle = c$ ($\zeta \in U$) and $\langle f(\zeta) \mid u \rangle = c$ ($\zeta \in \Delta$) by the identity theorem. Hence $f(\Delta) \subset f(\zeta_0) + \operatorname{Ker} u$. By $\langle f(\zeta_0) \mid u \rangle = c > 0$ it follows that $f(\zeta_0) \notin \operatorname{Ker} u$, Q.E.D.

LEMMA 2. *Let X be a complex Banach space and let $f\colon \Delta \to X$ be a function which is locally bounded on Δ. Suppose that $f(\Delta) \subset H$ where H is a closed hyperplane in X disjoint from* 0. *Then given a compact subset $K \subset \Delta$* there exists an equivalent *norm $\| \ \|_K$ on X such that $\|f(\zeta)\|_K \equiv 1$* ($\zeta \in K$).

Proof. Write $H = x_0 + \operatorname{Ker} u$ where $x_0 \in X$, $u \in X'$ and $x_0 \notin \operatorname{Ker} u$. Hence $y \in H$ if and only if $\langle y \mid u \rangle = \gamma$ where $\gamma = \langle x_0 \mid u \rangle \neq 0$. Further, let $r = \sup \{\|f(\zeta)\|; \zeta \in K\}$. By the compactness of K and by the local boundedness of f we have $r < \infty$. On the other hand $r > 0$ since $f(\Delta) \subset H$. Define

$$\|x\|_K = \max \left\{ \frac{\|x\|}{r}, \left| \frac{\langle x \mid u \rangle}{\gamma} \right| \right\} \quad (x \in X).$$

Clearly $\| \ \|_K$ is a norm on X. We have

$$\|x\|_K \leq \max \left\{ \frac{\|x\|}{r}, \frac{\|x\| \cdot \|u\|}{|\gamma|} \right\} = \left[\max \left\{ \frac{1}{r}, \frac{\|u\|}{|\gamma|} \right\} \right] \cdot \|x\| \quad (x \in X)$$

and

$$\|x\| \leq \max \left\{ \|x\|, r \cdot \left| \frac{\langle x \mid u \rangle}{\gamma} \right| \right\} = r \cdot \|x\|_K \quad (x \in X)$$

which shows that $\| \ \|_K$ is equivalent to $\| \ \|$. Finally, since $f(\Delta) \subset H$ we have

$$\left| \frac{\langle f(\zeta) \mid u \rangle}{\gamma} \right| = 1 \quad (\zeta \in \Delta)$$

and since

$$\frac{\|f(\zeta)\|}{r} \leq 1 \quad (\zeta \in K)$$

it follows that

$$\|f(\zeta)\|_K = \max \left\{ \frac{\|f(\zeta)\|}{r}, \left| \frac{\langle f(\zeta) \mid u \rangle}{\gamma} \right| \right\} \equiv 1 \quad (\zeta \in K), \qquad \text{Q.E.D.}$$

Proof of the theorem. For each fixed $\zeta \in \Delta$ let $\gamma(\zeta)$ be the closed linear span of all vectors of the form $f(\eta) - f(\zeta)$ where $\eta \in \Delta$, i.e.

$$\gamma(\zeta) = \overline{\operatorname{sp}} \{f(\eta) - f(\zeta); \eta \in \Delta\}.$$

By $f(\eta) - f(\zeta_1) = (f(\eta) - f(\zeta_2)) - (f(\zeta_1) - f(\zeta_2))$ ($\eta, \zeta_1, \zeta_2 \in \Delta$) it follows that $\gamma(\zeta)$ does not depend on $\zeta \in \Delta$. Further, we have

$$\overline{\operatorname{sp}} \{a_1, a_2, a_3, \ldots\} = \gamma(0).$$

To see this, observe that given $u \in X$ we have $\langle a_i \mid u \rangle = 0$ ($i = 1, 2, \ldots$) if and only if $\langle f(\zeta) - f(0) \mid u \rangle = 0$ ($\zeta \in \Delta$) and then apply the Hahn-Banach theorem.

Now, suppose that $a_0 \notin \overline{\text{sp}}\ \{a_1, a_2, a_3, \ldots\}$. By the above discussion it follows that $f(0) \notin \gamma(0)$. By the Hahn-Banach theorem there exists a closed hyperplane H containing $\gamma(0)$ and disjoint from $f(0)$ so $f(\Delta) \subset f(0) + H$, $f(0) \notin H$. Clearly f is continuous and so by Lemma 2 given any open subset $U \subset \Delta$ with closure contained in Δ there exists an equivalent norm $|||\ \ |||$ on X such that $|||f(\zeta)||| \equiv 1$ $(\zeta \in U)$.

To prove the converse suppose that there exist an open subset $U \subset \Delta$ and an equivalent norm $|||\ \ |||$ on X such that $|||f(\zeta)||| \equiv c$ $(\zeta \in U)$. We want to prove that $a_0 \notin \overline{\text{sp}}\ \{a_1, a_2, a_3, \ldots\}$ hence we may assume with no loss of generality that $\|f(\zeta)\| \equiv c(\zeta \in U)$. By the assumption f is not a constant so $c > 0$. By Lemma 1 there exist $\zeta_0 \in \Delta$ and $u \in X'$ such that $f(\Delta) \subset f(\zeta_0) + \text{Ker}\ u$ where $f(\zeta_0) \notin \text{Ker}\ u$. It follows that $\gamma(\zeta_0) \subset \text{Ker}\ u$ hence $f(\zeta_0) \notin \gamma(\zeta_0)$. Since $f(0) - f(\zeta_0) \in \gamma(\zeta_0)$ it follows that

$$a_0 = f(0) = f(\zeta_0) + (f(0) - f(\zeta_0)) \notin \gamma(\zeta_0) = \gamma(0) = \overline{\text{sp}}\ \{a_1, a_2, \ldots\},$$

Q.E.D.

An Application. The above theorem was proved when trying to answer the following question. Let a be an element of a complex Banach algebra with unit e. Can $\|(\lambda e - a)^{-1}\|$ be constant on an open subset of the resolvent set $\rho(a)$ of a? Below we give a partial answer to this question.

PROPOSITION 1. *Let a be an element of a complex Banach algebra with unit e. Then $\|(\lambda e - a)^{-1}\|$ can not be constant on any open subset of the unbounded component of $\rho(a)$.*

Proof. Assume that f is a nonconstant analytic function from an open connected set $D \subset C$ into a complex Banach space X. Suppose that $\|f(\zeta)\| \equiv c$ on an open subset of D. As in the proof of Lemma 1 there is $u \in X'$ such that $\langle f(\zeta) \mid u\rangle \equiv c$ $(\zeta \in D)$. Consequently f is bounded below on D by a positive constant. Now Proposition 1 follows by observing that for R sufficiently large we have

$$\inf \{\|(\lambda e - a)^{-1}\| : |\lambda| > R\} = 0, \qquad \text{Q.E.D.}$$

Note that Proposition 1 holds even if we replace the norm on the algebra by any equivalent norm which is not necessarily an algebra norm.

The situation on other components of $\rho(a)$ is not clear. If $A \in L(H)$ is a bilateral shift on the space H of all bilateral square-summable sequences [4, p. 41] then it is easy to see that

$$A^{-1} \notin \text{sp}\ \{A^{-2}, A^{-3}, \ldots\}$$

hence by the theorem there exists an equivalent norm $|||\ \ |||$ on $L(H)$ making $|||(\lambda I - A)^{-1}|||$ constant in a neighborhood of 0. However, in the original norm, $\|(\lambda I - A)^{-1}\|$ can not be constant on any open subset of $\rho(A)$ by the following proposition.

PROPOSITION 2. *Let X be a uniformly c-convex complex Banach space and let $A \in L(X)$. Then $\|(\lambda I - A)^{-1}\|$ can not be constant on any open subset of $\rho(A)$.*

Proof. Assume the contrary. With no loss of generality we may then assume that $\Delta \subset \rho(A)$ and that $\|(\lambda I - A)^{-1}\| = \|A^{-1}\|$ $(\lambda \in \Delta)$. Since

$$A^{-1} = A^{-1}(\lambda I - A)(\lambda I - A)^{-1} = \lambda A^{-1}(\lambda I - A)^{-1} - (\lambda I - A)^{-1}$$

we have

$$(\lambda I - A)^{-1} = -A^{-1} + \lambda A^{-1}(\lambda I - A)^{-1}.$$

It follows that

$$\| - A^{-1}x + \lambda A^{-1}(\lambda I - A)^{-1}x\| \leq \|A^{-1}\| \quad (x \in S(X), \lambda \in \Delta).$$

Now a sequence $\{x_n\} \subset S(X)$ exists with $\lim \|A^{-1}x_n\| = \|A^{-1}\|$ and since X is uniformly c-convex it follows by Theorem 2 of [1] that

$$\lim A^{-1}(\lambda I - A)^{-1}x_n = 0 \quad (\lambda \in \Delta, \lambda \neq 0)$$

which is clearly not possible.

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REFERENCES

1. J. GLOBEVNIK, *On complex strict and uniform convexity*, Proc. Amer. Math. Soc., vol. 47 (1975), pp. 175–178.
2. ———, *On vector-valued analytic functions with constant norm*, Studia Math., vol. 53 (1975), pp. 29–37.
3. J. GLOBEVNIK AND I. VIDAV, *On operator-valued analytic functions with constant norm*, J. Funct. Anal., vol. 15 (1974), pp. 394–403.
4. P. HALMOS, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967.
5. L. A. HARRIS, *Schwarz's lemma in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A., vol. 62 (1969), pp. 1014–1017.
6. G. KÖTHE, *Topological vector spaces I*, Springer Verlag, New York, 1969.
7. E. THORP AND R. WHITLEY, *The strong maximum modulus theorem for analytic functions into a Banach space*, Proc. Amer. Math. Soc., vol. 18 (1967), pp. 640–646.

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