ON THE SCHUR MULTIPLIER OF A WREATH PRODUCT

BY

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Introduction

This work is a generalization of a paper by N. Blackburn [1] on the Schur multiplier of the wreath product of two finite groups G and H. This wreath product is the group called the *complete* or *unrestricted* wreath product by H. Neumann [5], and the *regular* wreath product by Huppert [3]. We will consider wreath products as defined by Kerber [4] and Huppert, and the notation $G \wr H$ will always be taken to mean a group defined in this way. The regular wreath product of G and H will be denoted by $G \wr_r H$.

Our proofs sometimes follow closely along the lines of those of [1], and where the argument is almost identical, we have omitted the details. To show that our work is in fact a true generalization of Blackburn's work, we note that $G \wr_r H \cong G \wr H^+$, where H^+ is a permutation group on the elements of H which is itself isomorphic to H; indeed, we are able to recover Blackburn's result as a corollary to our main theorem (Theorem 3). We also apply our results to determine the multipliers of the groups $C_l \wr S_n$, $C_l \wr A_n$, $S_l \wr S_n$, $S_l \wr A_n$, $A_l \wr S_n$, $A_l \wr A_n$, where C_l is the cyclic group of order l, and S_l and A_l are respectively the symmetric and alternating groups on l symbols.

Section 1

Let G be a finite group, H a permutation group on the set $X = \{1, \dots, n\}$. We define $G \wr H$ to be the set $\{(f, h) \mid f\colon X \to G,\ h \in H\}$, together with the product $(f, h)(f', h') = (ff'_h, hh')$, where $f'_h(i) = f'(h^{-1}(i))$ for all $i \in X$. This makes $G \wr H$ into a group with identity $(e, 1_H)$, called the wreath product of G with H, where $e(i) = 1_G$ for all $i \in X$. (See [4, p. 24].) Let $G^* = \{(f, 1_H) \mid f\colon X \to G\}$. Then

$$G^* = \mathop{\times}_{i=1}^{n} G_i \lhd G \wr H \quad \text{where } G_i = \{(f, 1_H) \mid f(j) = 1_G \text{ for all } j \neq i\} \cong G.$$

If $H^* = \{(e, h) \mid h \in H\} \cong H$, then $G^* \cap H^* = \{(e, 1_H)\}$, and $G \wr H$ is the semidirect product of G^* and H^*. Thus $|G \wr H| = |G|^n |H|$. Henceforth, we will identify H^* with H.

Let $\{X_i \mid i = 1, \dots, m\}$ be the orbits of H on X, and for simplicity of notation, we assume that $i \in X_i$, $i = 1, \dots, m$. We define $W_i(H) = \{h \in H \mid h(i) =$

Received April 1, 1975.

$i\}$, $i = 1, \ldots, m$. (We will merely write W_i when no confusion arises over the group H in question.) Then for all $i = 1, \ldots, m$, there exist $\{w_j \mid j \in X_i\}$ such that $w_j(j) = i$ if $j \in X_i$, and $H = \bigcup_{j \in X_i} W_i w_j$, $i = 1, \ldots, m$. From now on, we will always assume that

$$\{w_j \mid j \in X_i, \; i = 1, \ldots, m\}$$

is a fixed set satisfying these conditions with $w_i = 1$ for all $i = 1, \ldots, m$, and thus if $j \in X_i$ and $h \in H$ then

$$W_i w_{h^{-1}(j)} = W_i(w_j h) \quad \text{and} \quad G_j = G_i^{w_j} = w_j^{-1} G_i w_j.$$

If $G^{(i)} = \times_{j \in X_i} G_j$, then $G^* = \times_{i=1}^{m} G^{(i)}$, and each $x \in G^*$ may be written (uniquely) as a product $x = \prod_{i=1}^{m} x^{(i)}$, $x^{(i)} \in G^{(i)}$. Further, each $x^{(i)} \in G^{(i)}$, $i = 1, \ldots, m$, may be expressed in the form $x^{(i)} = \prod_{j \in X_i} x_j^{w_j}$, where each x_j, $j \in X_i$ is an uniquely defined element of G_i called the jth *component* of $x^{(i)}$. At this stage, it is convenient to introduce some standard notation which will often be used without further reference. h, h', h'' will denote arbitrary elements of H, h_i an element of $H \backslash W_i$, h'_i, h''_i elements of W_i, and g_i, g'_i, g''_i elements of G_i, $i = 1, \ldots, m$.

We now derive a set of generators and relations for $G \wr H$.

THEOREM 1. *Let $\{v(h) \mid h \in H\}$, $\{v(g_i) \mid g_i \in G_i\}$ be sets in* 1-1 *correspondence with H and G_i, $i = 1, \ldots, m$, respectively, and let F be the free group generated by $\{v(h), v(g_i)\}$, with $v(1_H) = v(1_{G_i}) = 1$, $i = 1, \ldots, m$. If R is the normal closure in F of the elements*

$$b_i(g_i, g'_i) = v(g_i g'_i)^{-1} v(g_i) v(g'_i), \qquad c(h, h') = v(hh')^{-1} v(h) v(h')$$

$$d_i^{h_i}(g_i, g'_i) = [v(g_i)^{v(h_i)}, v(g'_i)], \qquad e_i(h'_i, g_i) = [v(h'_i), v(g_i)]$$

$$f_{ij}^{h}(g_i, g_j) = [v(g_i)^{v(h)}, v(g_i)], \quad j \neq i, \; i = 1, \ldots, m,$$

then $F/R \cong G \wr H$.

Proof. From the above work, it is easy to see that $G \wr H$ is a homomorphic image of F/R. For $h \in H$, $g_i \in G_i$, $j \in X_i$, $i = 1, \ldots, m$, we define

$$u_j(g_i) = v(g_i)^{v(w_j)} R, \qquad u(h) = v(h)R.$$

Then any element of F/R may be expressed as a product $\prod_{j=1}^{n} u_j(g_i) u(h)$, where $h \in H$ and $g_i \in G_i$ whenever $j \in X_i$, and thus $|F/R| \leq |G|^n |H|$.

Section 2

Let F, R be as above. We now consider the group $R/[F, R]$; the Schur multiplier of $G \wr H$ (denoted by $H^2(G \wr H; C^*)$) is then isomorphic to the torsion subgroup of $R/[F, R]$. (See [3, p. 631].)

We shall use $\bar{r}$ to denote the left coset of $[R, F]$ containing $r \in R$. Thus $R/[F, R]$ is generated by $\bar{b}_i(g_i, g'_i)$, $\bar{c}(h, h')$, $\bar{d}_i^{h_i}(g_i, g'_i)$, $\bar{e}_i(h'_i, g_i)$, $\bar{f}_{ij}^{h}(g_i, g_j)$, $j \neq i$.

THEOREM 2. *These elements satisfy the following relations:*

(1) $\bar{b}_i(g_i, 1) = \bar{b}_i(1, g_i) = 1, \quad \bar{b}_i(g_i g_i', g_i'')\bar{b}_i(g_i, g_i') = \bar{b}_i(g_i', g_i'')\bar{b}_i(g_i, g_i' g_i''),$

(2) $\bar{c}(h, 1) = \bar{c}(1, h) = 1, \quad \bar{c}(hh', h'')\bar{c}(h, h') = \bar{c}(h', h'')\bar{c}(h, h'h''),$

(3) $$\bar{d}_i^{h_i}(g_i g_i', g_i'') = \bar{d}_i^{h_i}(g_i, g_i'')\bar{d}_i^{h_i}(g_i', g_i''), \quad \bar{d}_i^{h_i}(g_i, g_i' g_i'') = \bar{d}_i^{h_i}(g_i, g_i')\bar{d}_i^{h_i}(g_i, g_i''),$$
$$\bar{d}_i^{h_i}(g_i, g_i')\bar{d}^{h_i^{-1}}(g_i', g_i) = 1,$$

(4) $$\bar{d}_i^{h_i' h_i h_i''}(g_i, g_i') = \bar{d}_i^{h_i}(g_i, g_i')$$

(5) $\bar{e}_i(h_i' h_i'', g_i) = \bar{e}_i(h_i', g_i)\bar{e}_i(h_i'', g_i), \quad \bar{e}_i(h_i', g_i g_i') = \bar{e}_i(h_i', g_i)\bar{e}_i(h_i', g_i')$

(6) $$\bar{f}_{ij}^h(g_i, g_j g_j') = \bar{f}_{ij}^h(g_i, g_j)\bar{f}_{ij}^h(g_i, g_j'), \quad \bar{f}_{ij}^h(g_i g_i', g_j) = \bar{f}_{ij}^h(g_i, g_j)\bar{f}_{ij}^h(g_i', g_j)$$
$$\bar{f}_{ij}^h(g_i, g_j)\bar{f}_{ji}^{h^{-1}}(g_j, g_i) = 1, \quad \bar{f}_{ij}^{h_i' h h_j'}(g_i, g_j) = \bar{f}_{ij}^h(g_i, g_j),$$

for all $i = 1, \ldots, m, j \neq i$.

Proof. (1), (2), (3) are proved in a similar manner to (7)–(10) in [1, p. 120]. For (4) we need the following result:

LEMMA 1. $(v(g_i)^{v(h_i)})^{-1}v(g_i)^{v(h_i' h_i)} \in R$, $i = 1, \ldots, m$.

Proof. $v(h_i' h_i) = v(h_i')v(h_i)c(h_i', h_i)^{-1}$, where $c(h_i', h_i) \in R$, and thus,

$$\begin{aligned}(v(g_i)^{v(h_i)})^{-1}v(g_i)^{v(h_i' h_i)} &\\ &= v(h_i)^{-1}v(g_i)^{-1}v(h_i)c(h_i', h_i)v(h_i)^{-1}v(h_i')^{-1}v(g_i)v(h_i')v(h_i)c(h_i', h_i)^{-1}\\ &= v(h_i)^{-1}v(g_i)^{-1}v(h_i)c(h_i', h_i)v(h_i)^{-1}rv(g_i)v(h_i)c(h_i', h_i)^{-1}\end{aligned}$$

where $r \in R$, which gives the result.

Then we have

$$\begin{aligned}\bar{d}_i^{h_i}(g_i, g_i')\bar{d}_i^{h_i' h_i}(g_i, g_i')^{-1} &= [v(g_i)^{v(h_i)}, v(g_i')][v(g_i'), v(g_i)^{v(h_i' h_i)}][F, R]\\ &= [v(g_i'), (v(g_i)^{v(h_i)})^{-1}v(g_i)^{v(h_i' h_i)}][F, R]\\ &= [F, R]\end{aligned}$$

by Lemma 1, and thus, $\bar{d}_i^{h_i}(g_i, g_i') = \bar{d}^{h_i' h_i}(g_i, g_i')$. Further,

$$\begin{aligned}\bar{d}_i^{h_i h_i''}(g_i, g_i') &= (\bar{d}_i^{h_i''^{-1} h_i^{-1}}(g_i', g_i))^{-1} \quad \text{by (3)},\\ &= (\bar{d}_i^{h_i^{-1}}(g_i', g_i))^{-1}\\ &= \bar{d}_i^{h_i}(g_i, g_i') \quad \text{by (3)}.\end{aligned}$$

This proves (4).

(5) is proved as in [3], p. 650, and (6) is proved as (3) and (4) above.

Section 3

Let A be the abelian group generated by

$$\{\underline{b}(g_i, g'_i), \underline{c}(h, h'), \underline{d}_i^{h_i}(g_i, g'_i), \underline{e}_i(h'_i, g_i), \underline{f}_{ij}^{h}(g_i, g_j), i = 1, \ldots, m, j \neq i\},$$

with relations given by inserting $\underline{b}_i$, $\underline{c}$, $\underline{d}_i$, $\underline{e}_i$, $\underline{f}_{ij}$ for $\bar{b}_i$, $\bar{c}$, $\bar{d}_i$, $\bar{e}_i$, $\bar{f}_{ij}$ respectively in (1)–(6) of Theorem 2. The map $\Phi\colon A \to R/[F, R]$ given by $\Phi(\underline{b}_i) = \bar{b}_i$, $\Phi(\underline{c}) = \bar{c}$, $\Phi(\underline{d}_i) = \bar{d}_i$, $\Phi(\underline{e}_i) = \bar{e}_i$, $\Phi(\underline{f}_{ij}) = \bar{f}_{ij}$, $i = 1, \ldots, m, j \neq i$, is an epimorphism. We now show that Φ is an isomorphism.

Definition. Let $z \in G^{(i)}$ for some $i = 1, \ldots, m$, $h \in H$. We define z_h to be the $h^{-1}(i)$th component of z. In other words, if $z = \prod_{j \in X_i} x_j^{w_j}$, $x_j \in G_i$, then $z_h = x_{h^{-1}(i)}$.

Lemma 2. *Let $z \in G^{(i)}$, $h, h' \in H$. Then $(z^h)_{h'} = z_{h'h^{-1}}$.*

Proof. Let $z = \prod_{j \in X_i} x_j^{w_j}$. Then $z^h = \prod_{j \in X_i} x_j^{w_j h} = \prod_{j \in X_i} x_j^{w_{h^{-1}(j)}} = \prod_{j \in X_i} x_{h(j)}^{w_j}$. Thus $(z^h)_{h'} = x_{h(h')^{-1}(i)} = x_{(h'h^{-1})^{-1}(i)} = z_{h'h^{-1}}$.

Let $x, y, z \in G^*$, $h \in H$. We define mappings $\sigma, \rho, \lambda\colon G^* \times G^* \to A$, and $\tau_h, \kappa_h\colon G^* \to A$ as follows:

$$\sigma(x, y) = \prod_{i=1}^{m} \prod_{h \in H} \underline{b}_i(x_h^{(i)}, y_h^{(i)}),$$

$$\rho(x, y) = \prod_{i=1}^{m} \prod_{\substack{j, k \in X_i, \\ j<k}} \underline{d}_i^{w_k w_j^{-1}}(x_{w_k}^{(i)}, y_{w_j}^{(i)}),$$

$$\lambda(x, y) = \prod_{i<j} \prod_{\substack{k \in X_i, \\ l \in X_j}} \underline{f}_{ij}^{w_k w_l^{-1}}(x_{w_k}^{(i)}, y_{w_l}^{(j)}),$$

$$\tau_h(z) = \prod_{i=1}^{m} \prod_{\substack{j, k \in X_i, \\ j<k, \\ h^{-1}(j) > h^{-1}(k)}} \underline{d}_i^{w_j w_k^{-1}}(z_{w_j}^{(i)}, z_{w_k}^{(i)}),$$

$$\kappa_h(z) = \prod_{i=1}^{m} \prod_{j \in X_i} \underline{e}_i(w_{h^{-1}(j)} h^{-1} w_j^{-1}, z_{w_j}^{(i)}),$$

Lemma 3.

$$\sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z), \qquad \sigma(x^h, y^h) = \sigma(x, y),$$

$$\rho(xy, z) = \rho(x, z)\rho(y, z), \qquad \rho(x, yz) = \rho(x, y)\rho(x, z),$$

$$\tau_h(xy)\tau_h(x)^{-1}\tau_h(y)^{-1} = \rho(x^h, y^h)\rho(x, y)^{-1}, \qquad \tau_{hh'}(x) = \tau_h(x)\tau_{h'}(x^h),$$

$$\kappa_{hh'}(x) = \kappa_h(x)\kappa_{h'}(x^h),$$

$$\lambda(x, yz) = \lambda(x, y)\lambda(x, z), \qquad \lambda(xy, z) = \lambda(x, z)\lambda(y, z), \qquad \lambda(x^h, y^h) = \lambda(x, y),$$

for all $x, y, z \in G^*$, $h \in H$.

Proof. These results are mostly proved as in Lemma 2 of [1]. We give two proofs.

$$\begin{aligned}
\kappa_{hh'}(z) &= \prod_{i=1}^{m} \prod_{j \in X_i} \underline{e}_i(w_k(hh')^{-1}w_j^{-1}, z_{w_j}^{(i)}) \quad \text{where } k = (hh')^{-1}(j) \\
&= \prod_{i=1}^{m} \prod_{j \in X_i} \underline{e}_i(w_k(h')^{-1}w_l^{-1}w_l h^{-1}w_j^{-1}, z_{w_j}^{(i)}) \quad \text{where } l = h^{-1}(j) \\
&= \prod_{i=1}^{m} \left(\prod_{j \in X_i} \underline{e}_i(w_k h'^{-1} w_l^{-1}, z_{w_j}^{(i)}) \right) \left(\prod_{j \in X_i} \underline{e}_i(w_l h^{-1} w_j^{-1}, z_{w_j}^{(i)}) \right) \\
&= \prod_{i=1}^{m} \left(\prod_{j \in X_i} \underline{e}_i(w_k h'^{-1} w_{h'(k)}^{-1}, (z^{(i)})^h_{w_{h'(k)}}) \right) \left(\prod_{j \in X_i} \underline{e}_i(w_l h^{-1} w_{h(l)}^{-1}, z_{w_{h(l)}}^{(i)}) \right), \\
&= \kappa_{h'}(z^h)\kappa_h(z).
\end{aligned}$$

$$\begin{aligned}
\tau_{h'}(z^h) &= \prod_{i=1}^{m} \prod_{\substack{j,k \in X_i, \\ j<k, \\ h'^{-1}(j) > h'^{-1}(k)}} \underline{d}_i^{w_j w_k^{-1}}((z^{(i)})^h_{w_j}, (z^{(i)})^h_{w_k}) \\
&= \prod_{i=1}^{m} \prod_{\substack{j,k \in X_i, \\ j<k, \\ h'^{-1}(j) > h'^{-1}(k)}} \underline{d}_i^{w_j w_k^{-1}}(z^{(i)}_{w_{h(j)}}, z^{(i)}_{w_{h(k)}}) \\
&= \prod_{i=1}^{m} \prod_{\substack{j,k \in X_i, \\ h^{-1}(j) < h^{-1}(k), \\ (hh')^{-1}(j) > (hh')^{-1}(k)}} \underline{d}_i^{w_j w_k^{-1}}(z^{(i)}_{w_j}, z^{(i)}_{w_k})
\end{aligned}$$

Thus

$$\begin{aligned}
&\tau_{h'}(z^h)\tau_h(z) \\
&\quad = \prod_{i=1}^{m} \left(\prod_{\substack{j,k \in X_i, \\ h^{-1}(j) < h^{-1}(k), \\ (hh')^{-1}(j) > (hh')^{-1}(k)}} \underline{d}_i^{w_j w_k^{-1}}(z^{(i)}_{w_j}, z^{(i)}_{w_k}) \right) \left(\prod_{\substack{j,k \in X_i, \\ j<k, \\ h^{-1}(j) > h^{-1}(k)}} \underline{d}_i^{w_j w_k^{-1}}(z^{(i)}_{w_j}, z^{(i)}_{w_k}) \right) \\
&\quad = \tau_{hh'}(z)
\end{aligned}$$

(using Equation (3) of Theorem 2).

We now define a mapping $\alpha\colon G \wr H \times G \wr H \to A$ as follows:

$$\alpha(xh, x'h') = \rho(x, x'^{h^{-1}})\sigma(x, x'^{h^{-1}})\lambda(x, x'^{h^{-1}})\underline{c}(h, h')\tau_{h^{-1}}(x')\kappa_{h^{-1}}(x'),$$

where $x, x' \in G^*$, $h, h' \in H$. Lemma 3 implies that $\alpha(r, s)\alpha(rs, t) = \alpha(r, st)\alpha(s, t)$, for all $r, s, t \in G \wr H$. Let K be the extension of A by $G \wr H$ with factor set α. Thus, there exists an injective mapping $\theta\colon G \wr H \to K$ such that $\theta(r)\theta(s) = \theta(rs)\alpha(r, s)$ for all $r, s \in G \wr H$, and we may easily prove;

LEMMA 4.

$$\theta(g_i g_i')^{-1}\theta(g_i)\theta(g_i') = \underline{b}_i(g_i, g_i'), \qquad \theta(hh')^{-1}\theta(h)\theta(h') = \underline{c}(h, h)',$$
$$[\theta(g_i)^{\theta(h_i)}, \theta(g_i')] = \underline{d}_i^{h_i}(g_i, g_i'), \qquad [\theta(h_i'), \theta(g_i)] = \underline{e}_i(h_i', g_i),$$
$$[\theta(g_i)^{\theta(h)}, \theta(g_j)] = \underline{f}_{ij}^{h}(g_i, g_j)$$

for all $i = 1, \ldots, m, j \neq i$.

Thus, K is generated by $\{\theta(g_i), \theta(h) \mid g_i \in G_i, h \in H\}$, and as F is free, there exists an epimorphism $\chi: F \to K$ such that $\chi(v(g_i)) = \theta(g_i)$, $g_i \in G_i$, $i = 1, \dots, m$, $\chi(v(h)) = \theta(h)$, $h \in H$. Further, χ maps R onto A and vanishes on $[F, R]$, and thus χ gives rise to an epimorphism $\bar{\chi}: R/[F, R] \to A$ such that

$$\bar{\chi}(\bar{b}_i(g_i, g'_i)) = \underline{b}_i(g_i, g'_i), \qquad \bar{\chi}(\bar{c}(h, h')) = \underline{c}(h, h'),$$

$$\bar{\chi}(\bar{d}_i^{h_i}(g_i, g'_i)) = \underline{d}_i^{h_i}(g_i, g'_i), \qquad \bar{\chi}(\bar{e}_i(h'_i, g_i)) = \underline{e}_i(h'_i, g_i),$$

$$\bar{\chi}(\bar{f}_{ij}^{h}(g_i, g_j)) = \underline{f}_{ij}^{h}(g_i, g_j),$$

for all $i = 1, \dots, m, j \neq i$. Hence $\bar{\chi}\Phi$ is the identity map, and $A \cong R/[F, R]$.

Section 4

In order to determine the torsion subgroup of A, we consider the following groups:

$$B_i(G) = \langle \underline{b}_i(g_i, g'_i) \rangle, i = 1, \dots, m, \qquad C(H) = \langle \underline{c}(h, h') \rangle,$$

$$D_i(G, H) = \langle \underline{d}_i^{h_i}(g_i, g'_i) \rangle, i = 1, \dots, m, \qquad E_i(G, H) = \langle \underline{e}_i(h'_i, g_i) \rangle, i = 1, \dots, m,$$

$$F(G, H) = \langle \underline{f}_{ij}^{h}(g_i, g_j), i = 1, \dots, m, j \neq i \rangle.$$

Then $A \cong (\mathsf{X}_{i=1}^{m} (B_i(G) \times D_i(G, H) \times E_i(G, H))) \times C(H) \times F(G, H)$.

If we denote the torsion subgroup of a group J by Tor (J), then Tor $(B_i(G)) \cong H^2(G; C^*)$, $i = 1, \dots, m$, and Tor $(C(H)) = H^2(H; C^*)$. (See [3, p. 652.]) $E_i(G, H) \cong G \otimes W_i(H)$ (see [3, p. 650]) where $\otimes$ denotes the tensor product of groups, and is a finite group. Thus Tor $(E_i(G, H)) = G \otimes W_i(H)$. Let $F_{ij} = \langle \underline{f}_{ij}^{h}(g_i, g_j) \mid i \neq j \rangle$. Then $F_{ji} = F_{ij}$ $(j \neq i)$, and if p_{ij} is the number of (W_i, W_j) double cosets in H, $F_{ij} \cong \mathsf{X}^{p_{ij}} (G \otimes G)$ (see [3, p. 650]) and hence, $F(G, H) \cong \mathsf{X}^{q} (G \otimes G)$, where $q = \sum_{i<j} p_{ij}$. Finally we consider $D_i(G, H)$. Let a_i be the number of nontrivial, self inverse (W_i, W_i) double cosets in H, and let $2b_i$ be the number of (W_i, W_i) double cosets which are not self-inverse. If $T(G)$ is the subgroup of $G \otimes G$ generated by elements of the form

$$(g \otimes g')(g' \otimes g), \quad g, g' \in G,$$

then $D_i(G, H) = \mathsf{X}^{a_i} (G \otimes G)/T(G) \, \mathsf{X}^{b_i} (G \otimes G)$ (argument as in [1, p. 119]). The following result enables us to determine $D_i(G, H)$ more explicitly.

LEMMA 5. *Let G/G' (derived factor) $\cong C_{r_1} \times C_{r_2} \times \cdots \times C_{r_t}$ where C_{r_j} is the cyclic group of order r_j generated by x_j, $j = 1, \dots, t$. ($r_1, r_2, \dots, r_t$ are called the invariants of G/G'.) Then:*

(i) *$G \otimes G = \mathsf{X}_{i,j=1}^{t} C_{(r_i, r_j)}$ where $C_{(r_i, r_j)}$ is generated by $x_i \otimes x_j$.*

(ii) *$(G \otimes G)/T(G) \cong \mathsf{X}_{i<j} C_{(r_i, r_j)} \mathsf{X}^{s} C_2$ where s is the number of even r_i, $i = 1, \dots, t$.*

Proof. (i) See [3, p. 649].

(ii) Let J_{ij} $(i < j)$ be the subgroup of $C_{(r_i, r_j)} \times C_{(r_j, r_i)}$ generated by

$(x_i \otimes x_j, x_j \otimes x_i)$, and let J_i be the subgroup of $C_{(r_i, r_i)}$ generated by $(x_i \otimes x_i)^2$, $i = 1, \ldots, t$. Then

$$T(G) \cong \mathop{\mathsf{X}}_{i=1}^{t} J_i \mathop{\mathsf{X}}_{j<k} J_{jk}$$

and the result now follows since $(C_{(r_i, r_j)} \times C_{(r_j, r_i)})/J_{ij} \cong C_{(r_i, r_i)}$, and $C_{(r_i, r_i)}/J_i \cong \{1\}$ if r_i is odd, and $\cong C_2$ if r_i is even.

Since $G \otimes G$ is a finite group, $F(G, H)$ and $\mathsf{X}_{i=1}^{m} D_i(G, H)$ are both torsion groups and we have our main result:

THEOREM 3. *Let the notation be as above. Then*

$$H^2(G \wr H; C^*) \cong H^2(H; C^*)$$
$$\times \left(\mathop{\mathsf{X}}_{i=1}^{m} (H^2(G; C^*) \times D_i(G, H) \times (G \otimes W_i(H))) \mathop{\mathsf{X}}^{q} (G \otimes G) \right)$$

Applications

(i) The regular or complete wreath product $G \wr_r H$ (G, H arbitrary finite groups), is defined to be the set $\{(f, h) \mid f\colon H \to G, h \in H\}$, together with the product

$$(f, h)(f', h') = (ff'_h, hh'), \quad \text{where } f'_h(h'') = f'(h''h)$$

for all $h, h'' \in H$. (See [3, p. 95].) Let $h \in H$. We define $h^+ : H \to H$ by $(h^+)(h') = h'h^{-1}$ for all $h' \in H$. Then h^+ permutes the elements of H, and $+ : H \to \mathrm{Sym}_H$ is a monomorphism. Routine checking gives the following result.

LEMMA 6. *$G \wr_r H \cong G \wr H^+$ where H^+ is now thought of as a subgroup of* Sym_H.

We can now derive Blackburn's result [1, Theorem 1]. H^+ is a transitive subgroup of Sym_H, and thus $m = 1$. $W_1(H^+) = \{h^+ \mid h^+(1) = 1\} \cong \{1\}$. Hence $G \otimes W_1(H^+) \cong \{1\}$, and $D_1(G, H)$ reduces to Blackburn's group $C(H; G)$.

(ii) $G \wr \{1\} \cong \mathsf{X}^n G$, where $\{1\}$ represents the identity subgroup of S_n. In this case, $m = n$, and

$$D_i(G, \{1\}) \cong G \otimes W_i(\{1\}) \cong \{1\}, \quad i = 1, \ldots, n,$$

and thus $H^2(\mathsf{X}^n G; C^*) \cong \mathsf{X}^n H^2(G; C^*) \mathsf{X}_{i=1}^{n(n-1)/2} (G \otimes G)$, which is a simple generalization of the well-known result on the Schur multiplier of a direct product. (See [3, p. 650].)

(iii) Before proceeding further, we list some well-known properties of the groups C_n, S_n, and A_n. Proofs of those results which are not immediate may be found in [6], [7], and [8].

LEMMA 7.

(i)
$$S_n/S_n' \cong C_2 \quad \textit{if } n \geq 2,$$
$$\cong \{1\} \quad \textit{if } n = 1.$$

(ii)
$$A_n/A_n' \cong C_3 \quad \textit{if } n = 3, 4,$$
$$\cong \{1\} \quad \textit{if } n \neq 3, 4.$$

(iii)
$$H^2(S_n; C^*) \cong C_2 \quad \textit{if } n \geq 4,$$
$$= \{1\} \quad \textit{if } n \leq 3.$$

(iv)
$$H^2(A_n; C^*) \cong C_2 \quad \textit{if } n \geq 4, n \neq 6, 7,$$
$$\cong C_6 \quad \textit{if } n = 6, 7,$$
$$\cong \{1\} \quad \textit{if } n \leq 3.$$

(v)
$$H^2(C_n; C^*) \cong \{1\} \textit{ for all } n.$$

LEMMA 8. *Let* $n > 1$.

(i) S_n *is transitive on* $\{1, \ldots, n\}$, *and* $W_1(S_n)$ *is the symmetric group on* $\{2, \ldots, n\}$.

(ii) $G \otimes W_1(S_n) \cong \mathsf{X}^s C_2$ *if* $n > 2$ *where* s *is the number of even invariants of* G/G' *and* $G \otimes W_1(S_2) \cong \{1\}$.

(iii) *There is precisely one nontrivial, and thus self inverse,* $(W_1(S_n), W_1(S_n))$ *double coset in* S_n.

LEMMA 9. *Let* $n > 2$.

(i) A_n *is transitive on* $\{1, \ldots, n\}$ *and* $W_1(A_n)$ *is the alternating group on* $\{2, \ldots, n\}$.

(ii)
$$G \otimes W_1(A_n) \cong \mathop{\mathsf{X}}^{t} C_3 \quad \textit{if } n = 4, 5,$$
$$\cong \{1\} \quad \textit{if } n \neq 4, 5$$

where t *is the number of invariants of* $G/G' \equiv 0 \pmod 3$.

(iii) *If* $n \geq 4$, *there is one nontrivial, and thus self inverse,* $(W_1(A_n), W_1(A_n))$ *double coset in* A_n. *If* $n = 3$, *there are two nontrivial* $(W_1(A_3), W_1(A_3))$ *double cosets in* A_3 *which are inverses of each other.*

Write

$$U(G, H) = \mathop{\mathsf{X}}_{i=1}^{m} D_i(G, H) \mathop{\mathsf{X}}_{i=1}^{m} G \otimes W_i(H) \mathop{\mathsf{X}}^{q} G \otimes G.$$

We may now determine $H^2(G \wr H; C^*)$ $(G = S_n, C_n, A_n, H = S_n, A_n)$ by determining $U(G, H)$ in each case, and then applying Theorem 3 and Lemma 7. We firstly consider the trivial cases.

LEMMA 10.

(i) $U(G, S_1) \cong U(G, A_1) \cong U(S_1, H) \cong U(A_1, H) \cong U(A_2, H) \cong \{1\}$.
(ii) $U(G, A_2) \cong G \otimes G$.

Proof. $S_1 \cong A_1 \cong A_2 \cong \{1\}$, and (i) follows from Lemma 5 and the fact that $\{1\} \otimes J \cong J \otimes \{1\} \cong \{1\}$ for all finite groups J. To prove (ii), we simply note that A_2 has two orbits on $\{1, 2\}$.

Henceforth, we will only consider S_n for $n \geq 2$, and A_n for $n \geq 3$.

THEOREM 4. *$U(C_l, S_n) \cong \mathsf{X}^r C_2$ where*

$$r = 2 \quad \text{if } l \text{ is even, } n \geq 2,$$
$$= 1 \quad \text{if } l \text{ is even, } n = 2,$$
$$= 0 \quad \text{otherwise.}$$

Proof.

$$D({}_lC_l, S_n) \cong C_l \otimes C_l/T(C_l) \text{ (by Lemma 8 (iii))} \cong C_2 \quad \text{if } l \text{ is even,}$$
$$\cong \{1\} \quad \text{if } l \text{ is odd}$$
$$\text{(by Lemma 5 (ii))}$$

If $n > 2$,

$$C_l \otimes W_1(S_n) \cong C_2 \quad \text{if } l \text{ is even}$$
$$\cong \{1\} \quad \text{if } l \text{ is odd} \quad \text{(by Lemma 8 (ii)).}$$
$$C_l \otimes W_1(S_2) \cong \{1\}.$$

Note. See [2] for an alternative derivation of $H^2(C_l \wr S_n; C^*)$.

THEOREM 5. *$U(S_l, S_n) \cong \mathsf{X}^r C_2$ where $r = 2$ if $n > 2$, and $r = 1$ if $n = 2$.*

Proof.

$$D_1(S_l, S_n) \cong S_l \otimes S_l/T(S_l) \quad \text{(by Lemma 8 (iii)).}$$
$$\cong C_2 \quad \text{(by Lemmas 5 (ii) and 7 (i)).}$$
$$S_l \otimes W_1(S_n) \cong C_2 \quad \text{if } n > 2,$$
$$\cong \{1\} \quad \text{if } n = 2 \quad \text{(by Lemmas 8 (ii) and 7 (i)).}$$

THEOREM 6. *$U(A_l, S_n) \cong \{1\}$.*

Proof.

$$D_1(A_l, S_n) \cong A_l \otimes A_l/T(A_l)$$
$$\cong \{1\} \quad \text{(by Lemmas 7 (ii) and 5 (ii)).}$$
$$A_l \otimes W_1(S_n) \cong \{1\} \quad \text{(by Lemmas 7 (ii) and 8 (ii)).}$$

THEOREM 7.

$$U(C_l, A_3) \cong C_l.$$

$$\begin{aligned} U(C_l, A_4) \cong U(C_l, A_5) &\cong C_2 \times C_3 && \textit{if } l \equiv 0 \pmod 6, \\ &\cong C_3 && \textit{if } l \equiv 3 \pmod 6, \\ &\cong C_2 && \textit{if } l \equiv 2, 4 \pmod 6, \\ &\cong \{1\} && \textit{if } l \equiv 1, 5 \pmod 6. \end{aligned}$$

$$\begin{aligned} U(C_l, A_n) &\cong C_2 && \textit{if } n > 5, l \textit{ even}, \\ &\cong \{1\} && \textit{if } n > 5, l \textit{ odd}. \end{aligned}$$

Proof. $D_1(C_l, A_3) \cong C_l \otimes C_l$ (by Lemma 9 (iii) $\cong C_l$ (by Lemma 5 (i)). If $n > 3$,

$$\begin{aligned} D_1(C_l, A_n) \cong C_l \otimes C_l/T(C_l) \text{ (by Lemma 9 (iii))} &\cong C_2 && \text{if } l \text{ is even}, \\ &\cong \{1\} && \text{if } l \text{ is odd} \end{aligned}$$
(by Lemma 5 (ii)).

$$\begin{aligned} C_l \otimes W_1(A_n) &\cong C_3 && \text{if } 3 \mid l, n = 4, 5, \\ &\cong \{1\} && \text{otherwise} \quad \text{(by Lemma 9 (ii))}. \end{aligned}$$

THEOREM 8. $U(S_l, A_n) \cong C_2$.

Proof. $D_1(S_l, A_3) \cong S_l \otimes S_l$ (by Lemmas 9 (iii) and 5 (i)) $\cong C_2$ (by Lemma 7 (i)). If $n > 3$, $D_1(S_l, A_n) \cong S_l \otimes S_l/T(S_l)$ (by Lemmas 9 (iii) and 5 (ii)) $\cong C_2$ (by Lemma 7 (i)). $S_l \otimes W_1(A_n) \cong \{1\}$ for all l, n (by Lemma 9 (ii)).

THEOREM 9.

$$\begin{aligned} U(A_l, A_n) &\cong C_3 && \textit{if } l = 3, 4, n = 3, 4, 5, \\ &\cong \{1\} && \textit{otherwise}. \end{aligned}$$

Proof.

$$\begin{aligned} D_1(A_1, A_3) \cong A_1 \otimes A_1 \text{ (by Lemma 9 (iii))} &\cong C_3 && \text{if } l = 3, 4, \\ &\cong \{1\} && \text{if } l \neq 3, 4 \end{aligned}$$
(by Lemma 7 (ii)).

If $n > 3$, $D_1(A_l, A_n) \cong A_l \otimes A_l/T(A_l) \cong \{1\}$ (by Lemmas 5 (ii) and 7 (ii)).

$$\begin{aligned} A_l \otimes W_1(A_n) &\cong C_3 && \text{if } l = 3, 4, n = 4, 5, \\ &\cong \{1\} && \text{otherwise} \quad \text{(by Lemmas 9 (ii) and 7 (ii))}. \end{aligned}$$

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