COHERENCE IN FINITE GROUPS CONTAINING A FROBENIUS SECTION

BY

DAVID A. SIBLEY[1]

Let G be a finite group and N a subgroup of G. Let S be a set of irreducible characters of N and V the Z-module generated by S. Let V_0 be the submodule of V consisting of all generalized characters of degree 0. Suppose that there is a Z-linear isometry τ (i.e., preserving the usual inner product) from V_0 into the Z-module of generalized characters of G of degree 0. Following W. Feit [3] we say that the pair (S, τ) is coherent if $V_0 \neq (0)$ and there is an extension of τ to V which is also a Z-linear isometry into the Z-module of generalized characters of G. When τ is understood from the context, we will simply say S is coherent.

An example of this situation of particular interest occurs under the following hypothesis.

HYPOTHESIS (*). *The finite group G contains a subgroup of the form $M \times H$, $M \neq \{1\}$, satisfying the following conditions:*

(i) *If y is in $M \times H - H$, then $C_G(y) \subseteq M \times H$.*

(ii) *For every x in $G - N_G(M \times H)$, $(M \times H) \cap x(M \times H)x^{-1} \subseteq H$.*

(iii) *$N_G(M \times H) \neq M \times H$, and both H and M are normal subgroups of $N_G(M \times H)$.*

(iv) *M and H have coprime orders.*

Let $N = N_G(M \times H)$ and $C = M \times H$. Let $e = |N: C|$, whence $e \neq 1$. Let $\lambda_0, \lambda_1, \ldots$ be the irreducible characters of $(M \times H)/H$, where λ_0 is the principal character. Let β be any irreducible character of $(M \times H)/M$. Set $S = \{(\lambda_i\beta)^N \mid i \neq 0\}$. Let I denote the inertial group of β in N.

Groups satisfying Hypothesis (*) except for condition (iv) have been studied by W. Feit [2] and H. S. Leonard, Jr. [9]. Feit showed that N/H is a Frobenius group with Frobenius kernel $C/H \cong M$, that S is a set of irreducible characters of N (each arising from $|I: C|$ distinct λ_i), and that the usual inducing map τ is a Z-linear isometry on the Z-module V_0 described above, unless $V_0 = \{0\}$. Furthermore, he showed that unless M is a nonabelian p-group with $|M: M'| < 4e^2$, S is coherent. The question of whether S is coherent in all circumstances was left open. Of course, when G is a Suzuki group, M a Sylow 2-group and $H = \{1\}$, we do not have coherence [12]. However, if G is any group satisfying Hypothesis (*) and M is a 2-group, then M is a Sylow 2-group of G and a trivial intersection set in G. Such groups have been described by Suzuki [13]. Our main theorem settles all other cases.

Received March 4, 1975.

[1] This research was supported in part by the National Science Foundation.

THEOREM 1. *Suppose G satisfies Hypothesis (*) and M is not a 2-group. Then either S is coherent or $|S| = 1$.*

This theorem has several applications. Notably, as Feit [2] has observed, the theorem of N. Ito [8] and G. Glauberman [6] can now be deduced by Feit's original methods. Also, Theorem 2 of P. Ferguson [5] may be proved by the methods used earlier by M. Herzog [7]. Theorem 1 also allows the character-theoretic techniques from the classification of CA-groups of odd order [11] to be used in the classification of CN-groups of odd order [4]. All of the above are applications of the theorem when $H = \{1\}$, so that N itself is a Frobenius group.

We will apply Theorem 1 and a recent result of the author [10] to obtain information about the values of certain irreducible characters of G.

THEOREM 2. *Suppose G satisfies Hypothesis (*) and M is not a 2-group. Assume $2e < |M| - 1$. Let τ be the isometry for S guaranteed by Theorem 1 and let $\phi \in T_j$. Let $y \in M \times H - H$. Then $\phi^\tau(y) = \phi(y)$.*

Of course, either ϕ^τ or $-\phi^\tau$ is an irreducible character of G. Theorem 2 is an improvement of Corollary 2.4 of Feit [2].

1. Preliminaries

In this section we give a brief exposition of several results concerning the character theory of groups satisfying Hypothesis ($*$). The first is a crucial part of the proof of Theorem 1. It is implicit in Feit's original work [2], as Leonard [9] has pointed out.

LEMMA 1. (Feit [3], Lemma 31.2). *Suppose G is a finite group and N is a subgroup of G. Suppose U is a set of irreducible characters of N and V is the Z-module generated by U. Let V_0 be the submodule of V consisting of all generalized characters of degree 0. Assume there is a Z-linear isometry τ mapping V_0 into the Z-module of all generalized characters of G of degree 0. Suppose all of the following hold.*

(i) *$U = \bigcup_{i=1}^k U_i$, a disjoint union, where $U_i = \{X_{is} \mid s = 1, \dots, n_i\}$, and for each i either U_i is coherent or all the characters in U_i have the same degree.*

(ii) *There are integers a_{is} such that $X_{is}(1) = a_{is}X_{11}(1)$ and $a_{i1} \mid a_{is}$ for* $1 \leq i \leq k$ and $1 \leq s \leq n_i$.

(iii) *$n_1 \geq 2$. For any integer m with $1 < m \leq k$,*

$$\sum_{i=1}^{m-1} \sum_{s=1}^{n_i} a_{is}^2 > 2a_{m1}. \tag{1}$$

Then U is coherent. If U contains more than two characters the extension of τ to V is uniquely determined. Otherwise there are exactly two such extensions.

The next lemma is probably not crucial in our arguments, but it makes several calculations neater.

LEMMA 2. *Suppose E is a Frobenius complement of odd order, and p is a prime not dividing $|E|$. Then all $GF(p)[E]$-modules on which E has faithful irreducible Frobenius action have the same $GF(p)$-dimension.*

Proof. E is metacyclic. Let A be a maximal normal cyclic subgroup of E. Since any abelian subgroup of E is cyclic, we have $C_E(A) = A$. It follows from Clifford's theorem that the faithful irreducible complex representations of E are exactly those induced from faithful linear characters of A. Thus the faithful irreducible complex representations of E are algebraic conjugates of one another. In particular they all have the same degree. If F is the algebraic closure of $GF(p)$, all faithful irreducible $F[E]$-modules have the same F-dimension, and their traces generate the same finite extension of $GF(p)$, Now the lemma follows from Theorems 24.10 and 24.14 of [1].

2. Coherence of subsets of S

We begin the proof of Theorem 1. Assume that G satisfies Hypothesis (*). As we remarked earlier, in view of Feit's work [2], we may assume M is a nonabelian p-group for some odd prime p. Thus, e is odd. The main results of this section are implicit in Feit [2].

Fix an N-chief series

$$M = M_0 \supset M_1 \supset \cdots \supset M_m \supset M_{m+1} = 1$$

for M. Since M is nonabelian, $m \geq 1$, and since e is odd Lemma 2 shows $|M_i/M_{i+1}| = q$ is independent of i. Thus $|M/M_i| = q^i$. Let S_i be that subset of S whose kernels contain M_{i+1} but not M_i. We will show that each S_i is coherent.

LEMMA 3. *Let $U_j = \bigcup_{i=0}^{j} S_i$ for $j = 0, 1, \ldots, m$. Then for each j we have*

$$e\beta(1)^2|N: I| + \sum_{\phi \in U_j} \phi(1)^2 = eq^{j+1}|N: I|\beta(1)^2.$$

Proof. Say $\phi \in U_j$. Then $\phi = (\lambda\beta)^N$ for some irreducible character λ of M, $\lambda \neq 1_M$. Each such ϕ arises in this way from exactly $|I: C|$ such λ. The degree of ϕ is $e\lambda(1)\beta(1)$. Hence,

$$\sum_{\phi \in U_j} \phi(1)^2 = \frac{e^2\beta(1)^2}{|I: C|} \sum{}' \lambda(1)^2 = e|N: I|\beta(1)^2(q^{j+1} - 1),$$

where the sum $\sum'$ is over all non-1 irreducible characters of M/M_{j+1}. The lemma follows.

LEMMA 4. *For each $i = 0, 1, \ldots, m$, the set S_i is coherent.*

Proof. Let $\beta(1)ep^{k_1} < \beta(1)ep^{k_2} \cdots$ be the distinct degrees of the characters in S_i. (The lemma is automatic if all members of S_i have the same degree.) Let n_j denote the number of characters in S_i of degree $e\beta(1)p^{k_j}$.

The sum of the squares of degrees of characters in U_i is that for U_{i-1} plus that for S_i. Thus,

$$eq^{i+1}|N: I|\beta(1)^2 = eq^i|N: I|\beta(1)^2 + \sum_j n_j e^2\beta(1)^2 p^{2k_j}.$$

Cancelling $e\beta(1)^2$, and noting that $p^{2k_j} \mid q^i$ (for $M_i/M_{i+1} \subseteq Z(M/M_{i+1})$) we see that for each index s,

$$\sum_{j=1}^{s=1} n_j e p^{2k_j} \equiv 0 \pmod{p^{2k_s}}$$

so that

$$\sum_{j=1}^{s-1} n_j p^{2k_j} \equiv 0 \pmod{p^{2k_s}}.$$

Thus,

$$\sum_{j=1}^{s-1} n_j p^{2k_j} > p^{2k_s}$$

Dividing by p^{2k_1}, this becomes

$$\sum_{j=1}^{s-1} n_j p^{2k_j - 2k_1} \geq p^{2k_s - 2k_1} > 2p^{k_s - k_1} \tag{2}$$

since $p > 2$.

Writing S_i as the disjoint union of sets of characters of the same degree, we see that (2) is the inequality (1) of Lemma 1. All other hypotheses of Lemma 1 are easily verified, thus establishing the coherence of S_i.

We let τ_i denote the extension of τ to S_i. When $|S_i| = 2$, τ_i is not uniquely determined. This will be important later. If $\phi \in S_i$ then either $\pm\phi^{\tau_i}$ is irreducible.

Let $e\beta(1)f_i$ be the least degree among the members of S_i. Thus, f_i is a power of p. For each $i = 0, 1, \ldots, m$ we define the character α_i of N to be

$$\alpha_i = \sum_{\phi \in S_i} \frac{\phi(1)}{ef_i\beta(1)} \phi.$$

Because the next lemma is a standard application of coherence, we omit the proof.

LEMMA 5. *There are integers c_{ik}, $0 \leq i \leq m$, $0 \leq k \leq m$, such that for any $\phi \in S_i$ and $g \in M^{\#}$,*

$$\phi^{\tau_i}(g) = \phi(g) + \frac{\phi(1)}{ef_i\beta(1)} \sum_{k=0} c_{ik}\alpha_k(g).$$

Now that we know the values of ϕ^{τ_i} on $M^{\#}$, we can obtain a congruence on the degree of ϕ^{τ_i}. This is only needed for $\phi \in S_0$.

LEMMA 6. *For any $\phi \in S_0$ and $g \in M_m^{\#}$, we have*

$$\phi^{\tau o}(1) \equiv \phi^{\tau o}(g) + C_{0m} \frac{q^{m+1}}{f_m} \beta(1) \frac{e}{|I: C|} \pmod{q^{m+1}}.$$

Proof. Since $\phi \in S_0$, $\phi(1) = e\beta(i)$. Let $\hat{\phi} = \phi + \sum_{k=0}^{m} C_{0k}\alpha_k$. By Lemma 5, $\phi^{\tau o} \mid M - \hat{\phi}$ vanishes on $M^{\#}$, and so is a multiple of the regular representation of M. Thus, $\phi^{\tau o}(1) \equiv \hat{\phi}(1) \pmod{q^{m+1}}$ now for $g \in M_m^{\#}$ we have $\hat{\phi}(1) - \hat{\phi}(g) = C_{0m}(\alpha_m(1) - \alpha_m(g))$ since g is in the kernel of each member of S_i for $0 \leq i \leq m-1$. We see that

$$\begin{aligned}
\alpha_m(1) &= \sum_{\psi \in S_m} \frac{\psi(1)^2}{ef_m\beta(1)} \\
&= \frac{1}{ef_m\beta(1)} \sum_{\psi \in S_m} \psi(1)^2 \\
&= \frac{1}{ef_m\beta(1)} \beta(1)^2 \frac{e^2}{|I: C|} (|M| - |M: M_m|) \\
&= \frac{\beta(1)}{f_m} \frac{e}{|I: C|} (q^{m+1} - q^m) \\
&= \frac{\beta(1)}{f_m} \frac{q-1}{|I: C|} q^m e
\end{aligned}$$

and that

$$\begin{aligned}
\alpha_m(g) &= \sum_{\psi \in S_m} \frac{\psi(1)}{ef_m\beta(1)} \psi(g) \\
&= \frac{1}{ef_m\beta(1)} \sum_{\psi \in S_m} \psi(1)\psi(g) \\
&= \frac{1}{ef_m\beta(1)} (0 - |M: M_m|)\beta(1)^2 \frac{e^2}{|I: C|} \\
&= -\frac{(\beta 1)}{f_m} \frac{e}{|I: C|} q^m.
\end{aligned}$$

Hence,

$$\hat{\phi}(1) - \hat{\phi}(g) = C_{0m} \frac{q^{m+1}}{f_m} \beta(1).$$

The lemma is proved.

3. Class multiplication constants

For any finite group K and any $a, b, c \in K$ define

$$A(a, b, c; K) = \{(x, y) \mid x_K{\sim}a,\ y_K{\sim}b, \text{ and } xy = c\}.$$

Then $\gamma(a, b, c; K) = |A(a, b, c; K)|$ is the usual class multiplication constant. We fix $g \in M_m^\#$. Note that g is not conjugate to g^{-1}.

LEMMA 7. *Suppose $c \in M \times H$. Then*

$$\gamma(g, g, c; G) \equiv \gamma(g, g, c; N) \pmod{|C_M(c)|}.$$

Proof. $C_M(c)$ acts on $A(g, g, c; G)$ by conjugation. If Γ is an orbit of length less than $|M| = |C_M(c)|$, then some $y \in M^\#$ centralizes both of $a, b \in g^G$ with $ab = c$. Since M is a T.I. set, a and b are both in M and are N-conjugate to g. Since N controls G-fusion in M, the lemma is proved.

COROLLARY. *If $c \in C - M_m^\#$, $\gamma(g, g, c; G) \equiv 0 \pmod{|C_M(c)|}$.*

Proof. If $c \in C - M_m$ we have $\gamma(g, g, c; N) = 0$ as $g \in M_m$. If $c = 1$, $\gamma(g, g, 1; G) = 0$ as g is not conjugate to g^{-1}.

LEMMA 8. *We have $\sum' \gamma(g, g, c; N) = e$, where the sum is over a set of representatives c of the N-classes of M_m.*

Proof. Clearly $|g^N|^2 = \sum' \gamma(g, g, c; N)|c^N|$. Since $\gamma(g, g, 1; N) = 0$, this is $e^2 = \sum' \gamma(g, g, c; N)e$, for any element $x \in M_m^\#$ has $C_N(x) = M \times H$.

LEMMA 9. *Let $\phi \in S_0$ and $g \in M_m^\#$ as above. Then*

$$\phi^{\tau_0}(1) \equiv \phi^{\tau_0}(g) \pmod{q^{m+1}}.$$

Proof. Let ω be the irreducible representation of the complex class algebra of G associated with $\pm\phi^{\tau_0}$ (whichever is irreducible). Since the sign cancels we have

$$\omega(x) = \frac{|G|}{|C_G(x)|}\frac{\phi^{\tau_0}(x)}{\phi^{\tau_0}(1)},$$

for all $x \in G$. Multiply the usual multiplication formula for ω by $\phi^{\tau_0}(1)$ to get

$$\phi^{\tau_0}(1)\omega(g)^2 = \phi^{\tau_0}(1) \textstyle\sum' \gamma(g, g, x; G)\omega(x), \tag{3}$$

where the sum is over a set of representatives x for the conjugacy classes of G.

If x is not conjugate to an element of $M \times H$, then no conjugate of x centralizes any element of $M^\#$. Thus, $\omega(x)\phi^{\tau_0}(1)$ is an algebraic integer divisible by q^{m+1}. Now applying Lemma 7 and its corollary, we see that (2) becomes

$$\phi^{\tau_0}(1)\omega(g)^2 \equiv \phi^{\tau_0}(1) \textstyle\sum'' \gamma(g, g, x; N)\omega(x) \pmod{q^{m+1}},$$

where the sum is over a set of representatives x of the N-classes of M_m. Since $g \in M_m^\#$ and $m \neq 0$, we find that $\omega(x)$ is independent of $x \in M_m^\#$. Also $\gamma(g, g, 1; N) = 0$. Using this fact and Lemma 8, we have

$$\phi^{\tau_0}(1)\omega(g)^2 \equiv \phi^{\tau_0}(1)\omega(g)e \pmod{q^{m+1}}.$$

Now $(p, \omega(g)) = 1$, since the same is true in the associated p-block of N, so we may cancel $\omega(g)$ without disturbing the modulus. Hence,

$$\frac{|G|}{|M \times H|}\,\phi^{\tau_0}(g) \equiv e\phi^{\tau_0}(1) \pmod{q^{m+1}}.$$

Since M is a T.I. set, $|G\colon M \times H| \equiv e \pmod{q^{m+1}}$. Thus,

$$\phi^{\tau_0}(g) \equiv \phi^{\tau_0}(1) \pmod{q^{m+1}}$$

as required.

LEMMA 10. *We have* $f_m \mid c_{0m}$.

Proof. Compare Lemmas 6 and 9.

4. Proof of Theorem 1

We first show $S_0 \cup S_m$ is coherent. Fix $\phi \in S_0$ and $X \in S_m$ with $X(1) = e\beta(1)f_m$. The generalized character $\mu = f_m\phi - X$ has support on $M^{\#}$. It suffices to show $\mu^G = f_m\phi^{\tau_0} - X^{\tau_m}$. Note that $\|\mu^G\|_G^2 = \|\mu\|_N^2 = f_m^2 + 1$.

Let ψ be any character in S_0 and define $\hat{\psi} = \psi + \sum_{k=0}^{m} c_{0k}\alpha_k$ as before. Since μ has support on $M^{\#}$, we have by Lemma 5,

$$(\mu^G, \psi^{\tau_0})_G = (\mu, \psi^{\tau_0} \mid N)_N = (\mu, \hat{\psi})_N = f_m(\phi, \psi) + f_m c_{00} - c_{0m}.$$

Thus,

$$\mu^G = f_m\phi^{\tau_0} + (f_m c_{00} - c_{0m})\alpha_0^{\tau_0} + \theta,$$

where θ has no constituents in $S_0^{\tau_0}$. Now $(\alpha_0^{\tau_0}, \alpha_0^{\tau_0})_G = (\alpha_0, \alpha_0)_N = (q - 1)/|I\colon C|$. We let $t = (q - 1)/|I\colon C|$. Since $|I\colon C|$ is odd and q is odd, t is even. We have

$$\|\mu^G\|_G^2 = f_m^2 + 1 \geq (f_m + f_m c_{00} - c_{0m})^2 + (f_m c_{00} - c_{0m})^2(t - 1).$$

First suppose $t > 2$. Since f_m divides c_{0m} by Lemma 10, we must have $f_m c_{00} - c_{0m} = 0$. Thus, $\mu^G = f_m\phi^{\tau_0} + \theta$ where $\|\theta\|_G^2 = 1$. It is easy to show that $\theta = -X^{\tau_m}$, as required.

Now suppose $t = 2$. Here it is also possible that $f_m c_{00} - c_{0m} = -f_m$. However, in this case, there is a second isometry on S_0 extending τ. It is clear that by changing τ_0 if necessary we again get $\mu^G = f_m\phi^{\tau_0} + \theta$ with $\|\theta\|_G^2 = 1$. It may also be necessary to change τ_m to show $\theta = -X^{\tau_m}$.

We have now shown that $S_0 \cup S_m$ is coherent. The proof of Theorem 1 is now an easy application of Lemma 1. Take $U_1 = S_0 \cup S_m$ and for $2 \leq i \leq m$ let $U_i = S_{i-1}$.

5. Proof of Theorem 2

If Hypothesis (*) holds, M is not a 2-group, and $2e < |M| - 1$. If M is an abelian p-group, Theorem 2 is true by Theorem 1 of [10]. Hence, we assume M is not an abelian p-group.

Because of coherence of S, we have an immediate improvement of Lemma 5. Using Brauer's second main theorem, we can also extend Lemma 5 to elements of $C - H$. (Compare Feit [3], Theorem 31.7). Here τ denotes the extended isometry on S.

LEMMA 11. *There is an integer c_0 such that for $g \in C - H$ and $\phi \in S$,*

$$\phi^\tau(g) = \phi(g) + \frac{\phi(1)}{e\beta(1)} c_0 \sum{}' \beta(g),$$

where the sum is over the N-conjugates β' of β.

To prove Theorem 2, we must show $c_0 = 0$. Say β has k N-conjugates and let $c = c_0 k$. We show $c = 0$.

Let p be a prime dividing $|M|$. Let B be the union of all p-blocks represented in S^τ. From Leonard [9], we know that any character in $B - S^\tau$ is constant on $M^\#$. Say these are $X_1, X_2, \ldots$ and $X_i(g) = d_i$ for $g \in M^\#$. Let $g, h \in M^\#$ with g and h in different p-sections. Then by block orthogonality

$$\begin{aligned} 0 &= \sum_i X_i(g)\overline{X_i(h)} + \sum_{\phi \in S} \phi^\tau(g)\overline{\phi^\tau(h)} \\ &= \sum_i d_i^2 + \sum_{\phi \in S} \left(\phi(g) + \frac{\phi(1)}{e} c\right)\left(\overline{\phi(h)} + \frac{\phi(1)}{e} c\right) \\ &= \sum_i d_i^2 + \sum_{\phi \in S} \left(\phi(g)\overline{\phi(h)} + (\phi(g) + \overline{\phi(h)})\frac{\phi(1)}{e} c + \frac{\phi^2(1)}{e^2} c^2\right) \\ &= \sum_i d_i^2 - e\beta^2(1) - 2c\beta^2(1) + \frac{1}{e}(|M| - 1)c^2\beta^2(1). \end{aligned}$$

Thus,

$$0 \geq -e - 2c + \frac{1}{e}(|M| - 1)c^2. \tag{4}$$

Now M has at least two N-chief factors, say of orders a and b. Then $e \mid (a - 1)$ and $e \mid (b - 1)$. Hence, $(e + 1)^2 \leq |M|$, and if $(e + 1)^2 = |M|$, M is a p-group. In the latter case p is odd so e is even, whence M is abelian, contrary to our assumptions. Thus, $(e + 1)^2 < |M|$.

Inequality (4) then implies that

$$0 > -e - 2c + \frac{1}{e}(e^2 + 2e)c^2$$

so that $e + 2|c| > (e + 2)c^2 \geq (e + 2|c|)|c|$, whence $1 > |c|$ so $c = 0$ as required. This proves Theorem 2.

References

1. L. Dornhoff, *Group representation theory*, Dekker, New York, 1971.
2. W. Feit, *On a Class of doubly transitive permutation groups*, Illinois J. Math., vol. 4 (1960), pp. 170–186.
3. ———, *Characters of finite groups*, Benjamin, New York, 1967.
4. W. Feit, M. Hall, Jr., and J. Thompson, *Finite groups in which the centralizer of any non-identity element is nilpotent*, Math. Zeit., vol. 24 (1960), pp. 1–17.
5. P. Ferguson, *A theorem on CC-subgroups*, J. Algebra, vol. 25 (1973), pp. 203–221.
6. G. Glauberman, *On a class of doubly transitive permutation groups*, Illinois J. Math., vol. 13 (1969), pp. 394–399.
7. M. Herzog, *On finite groups which contain a Frobenius subgroup*, J. Algebra, vol. 6 (1967), pp. 192–221.
8. N. Ito, *On a class of doubly transitive permutation groups*, Illinois J. Math., vol. 6 (1962), pp. 341–352.
9. H. Leonard, Jr., *On finite groups which contain a Frobenius factor group*, Illinois J. Math., vol. 9 (1965), pp. 47–57.
10. D. Sibley, *Finite linear groups with a strongly self-centralizing Sylow subgroup*, to appear in J. Algebra.
11. M. Suzuki, *The nonexistence of a certain type of simple group of odd order*, Proc. Amer. Math. Soc., vol. 8 (1957), pp. 686–695.
12. ———, *A new type of simple group of finite order*, Proc. Nat. Acad. Sci. U.S.A., vol. 46 (1960), pp. 868–870.
13. ———, *Finite groups of even order in which Sylow 2-groups are independent*, Ann. of Math. (2), vol. 80 (1964), pp. 58–77.

The Pennsylvania State University
University Park, Pennsylvania