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REDUCED Σ-SPACES

BY

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Σ-spaces and reduced Σ-spaces were first introduced by O. Loos [10] in 1972 as a generalisation of reflection spaces and symmetric spaces. Loos proved that if Σ is a compact Lie group then any Σ-space is a fibre bundle over a reduced Σ-space which, in turn, is homogeneous. This raises several questions about such spaces when the compactness assumption is removed. Our purpose in this paper is to introduce and study other classes of Σ-spaces, essentially for the reduced case. We hope that the terminology introduced here will prove to be acceptable and so lead to some standardisation of language within the subject.

Basic properties of any reduced Σ-space M (defined by $(\Sigma 1)$–$(\Sigma 6)$) are given in Section 1. Most of these are contained in [10] but, for completeness, we have selected what is needed for our purpose and given a self-contained account. Thus, properties of the group G_M are due to Loos, as are Lemma 1·6 and Theorem 1·7 which are fundamental. We have introduced tensor fields S^σ as a natural extension of the tensor field S defined on s-manifolds [2].

If M is a reduced Σ-space and Σ is compact then, by [10], G_M is a Lie transformation group of M on which Σ acts by automorphisms, and the associated coset space is a reduced Σ-space isomorphic to M. An important step in the proof is to show that M admits a particular affine connection. This connection can be characterised by two properties. In Section 2 we select one of these, $(\Sigma 7)$, to define a reduced affine Σ-space, and the space is then called canonical if the second property, $(\Sigma 8)$, holds. We consider such spaces with Σ possibly non-compact; in particular, Theorem 2·7 gives a coset space presentation for the canonical case. It follows, for the same case, that the group G_M is a connected Lie transformation group of M; this result is basic for later applications.

In Section 3 we study the case when Σ is cyclic, and show that, as for Σ-compact, such a space always admits the canonical connection. Furthermore, it is then, essentially, just an affine s-manifold. Other versions of these results can be found in [7] and [13]. It should be noted, however, that the compact and cyclic cases differ in their coset space presentation and we give some examples in that direction.

Finally, Section 4 deals with reduced Riemannian Σ-spaces and, in Theorem 4·4, we obtain a coset space presentation which shows that the existence of an invariant metric implies the existence of the canonical connection. Then by Corollary 4·6 we see that Σ can always be compactified so as to extend its

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action on M. By means of this result we obtain a de Rham decomposition theorem which relates to similar theorems for symmetric spaces and regular s-manifolds.

1 Reduced Σ-spaces

Following O. Loos [10] we have:

DEFINITION 1.1. Let M be a smooth connected manifold, Σ a Lie group, and μ: $M \times \Sigma \times M \to M$ a smooth map. Then the triple (M, Σ, μ) is a (differentiable) Σ-space if it satisfies

(Σ1) $\mu(x, \sigma, x) = x$;
(Σ2) $\mu(x, e, y) = y$;
(Σ3) $\mu(x, \sigma, \mu(x, \tau, y)) = \mu(x, \sigma\tau, y)$;
(Σ4) $\mu(x, \sigma, \mu(y, \tau, z)) = \mu(\mu(x, \sigma, y), \sigma\tau\sigma^{-1}, \mu(x, \sigma, z))$;

where $x, y, z \in M$, $\sigma, \tau \in \Sigma$, and e is the identity element of Σ.

The triple (m, Σ, μ) is usually just replaced by M. For each $x \in M$ and $\sigma \in \Sigma$, a diffeomorphism σ_x: $M \to M$ is defined by $\sigma_x(y) = \mu(x, \sigma, y)$, and a smooth map σ^x: $M \to M$ is defined by $\sigma^x(y) = \sigma_y(x)$. With respect to the first of these two maps the above conditions become

(Σ1)′ $\sigma_x(x) = x$;
(Σ2)′ $e_x = \mathrm{id}_M$;
(Σ3)′ $\sigma_x \tau_x = (\sigma\tau)_x$;
(Σ4)′ $\sigma_x \tau_y \sigma_x^{-1} = (\sigma\tau\sigma^{-1})\sigma_x(y)$.

For each $x \in M$ write Σ_x for the image of Σ under the map $\Sigma \to \Sigma_x$; $\sigma \to \sigma_x$; then from (Σ2)′, and (Σ3)′, Σ_x is a subgroup of Diff M and the map is a homomorphism.

For Σ-spaces M and M', we say a smooth map ϕ: $M \to M'$ is a homomorphism if

$$\phi(\mu(x, \sigma, y)) = \mu(\phi(x), \sigma, \phi(y)) \quad \text{for } x, y \in M, \sigma \in \Sigma,$$

or, equivalently, $\phi \circ \sigma_x = \sigma_{\phi(x)} \circ \phi$. If ϕ has a smooth inverse then it is an isomorphism, and if, in addition, $M = M'$ then ϕ is an automorphism of M; we write Aut M for the group of automorphisms of M.[1]

DEFINITION 1.2. For $p \in M$, any subgroup G of Diff M is Σ_p-stable if $\sigma_p g \sigma_p^{-1} \in G$ for all $g \in G$ and $\sigma \in \Sigma$; if this relation holds for all $p \in M$ then G is called Σ-stable. Similarly a (possibly non-effective) Lie transformation group of M is Σ_p (resp. Σ)-stable if its image in Diff M is Σ_p (resp. Σ)-stable.

[1] We remark that if, as in later sections, M and M' carry additional structures then the above definitions must be modified in the obvious way.

LEMMA 1.3. *Let $p \in M$ and denote by G_M the subgroup of* Diff M *generated by $\{\sigma_x \sigma_p^{-1}: x \in M$ and $\sigma \in \Sigma\}$. Then G_M is independent of the choice of p. Moreover G_M is a normal subgroup of* Aut M *and the groups G_M and* Aut M *are Σ-stable.*

Proof. For $q \in M$ we have $\sigma_x \sigma_q^{-1} = \sigma_x \sigma_p^{-1}(\sigma_q \sigma_p^{-1})^{-1}$ so G_M does not depend on p. Now each $\sigma_x \sigma_p^{-1} \in$ Diff M and by $(\Sigma 4)'$

$$\begin{aligned}\sigma_x \sigma_p^{-1} \tau_y(z) &= \sigma_x(\sigma^{-1}\tau\sigma)_{\sigma_p^{-1}(y)} \sigma_p^{-1}(z)\\ &= \tau_{\sigma_x \sigma_p^{-1}(y)} \sigma_x \sigma_p^{-1}(z).\end{aligned}$$

Thus G_M is a subgroup of Aut M. Moreover, if $\phi \in$ Aut M then

$$\phi \sigma_x \sigma_p^{-1} \phi^{-1} = \sigma_{\phi(x)} \sigma_{\phi(p)}^{-1} \in G_M$$

so G_M is a normal subgroup of Aut M. It follows easily that G_M and Aut M are Σ-stable.

For any smooth map ϕ between manifolds we again write ϕ (or occasionally ϕ_*) for its differential. We also write $\mathscr{X}(M)$ for the Lie algebra of smooth vector fields on a manifold M and denote by TM (resp. $T_x M$) the tangent bundle over M (resp. the tangent space to M at x).

LEMMA 1.4. *For each $\sigma \in \Sigma$ define a $(1, 1)$ tensor field S^σ on the Σ-space M by*

$$S^\sigma X_x = \sigma_x X_x \quad \textit{for all } x \in M \textit{ and } X_x \in T_x M.$$

Then

(i) *S^σ is smooth;*
(ii) *$\tau_x(S^\sigma X) = S^{\tau\sigma\tau^{-1}}(\tau_x X)$ for $\sigma, \tau \in \Sigma$, $X \in \mathfrak{X}(M)$, $x \in M$;*
(iii) *S^σ is* Aut *M-invariant;*
(iv) *$\sigma^x X_x = (I - \sigma_x) X_x = (I - S^\sigma) X_x$.*

Proof. (i) Define the smooth map $\mu_\sigma: M \times M \to M$ by

$$\mu_\sigma(x, y) = \sigma_x(y).$$

Then for any $X \in \mathfrak{X}(M)$ we have $S^\sigma X = \mu_{\sigma *} \circ (0, X)$, where X and $(0, X)$ are regarded as smooth maps from M into TM (and $T(M \times M)$) respectively. It follows that $S^\sigma X$ is smooth and so S^σ is smooth. Now, (ii) and (iii) are easily proved, and (iv) is a consequence of $(\Sigma 1)$.

DEFINITION 1.5. A Σ-space M is a reduced Σ-space if, for each $x \in M$,

$(\Sigma 5)$ $T_x M$ is generated by the set of all $\sigma^x(X_x)$, that is

$$\begin{aligned}T_x M &= \text{gen}\,\{\sigma^x(X_x): X_x \in T_x M \text{ and } \sigma \in \Sigma\}\\ &= \text{gen}\,\{(I - S^\sigma)X_x: X_x \in T_x M \text{ and } \sigma \in \Sigma\};\end{aligned}$$

$(\Sigma 6)$ if $X_x \in T_x M$ and $\sigma^x X_x = 0$ for all $\sigma \in \Sigma$ then $X_x = 0$, and thus no non-zero vector in $T_x M$ is fixed by all S^σ.

We now establish two key properties of M which are special cases of 10, Lemma 2·5 and Theorem 2·6].

LEMMA 1.6. *Let M be a reduced Σ-space and $p \in M$. Then there exists an open neighbourhood V of p and a map $\theta\colon V \to G_M$ such that*

(i) $\theta(p) = \mathrm{id}_M$;

(ii) $\theta(x)(p) = x$ *for* $x \in V$;

(iii) *the maps* Φ, $\Psi\colon V \times M \to M$ *defined by* $\Phi(x, y) = \theta(x)(y)$ *and* $\Psi(x, y) = (\theta(x))^{-1}(y)$ *are smooth.*

Proof. It follows from $(\Sigma 5)$ that there exists a finite set $\sigma_1, \sigma_2, \ldots, \sigma_m$ such that $T_p M = \mathrm{gen}\,\{\mathrm{im}\,(I - S^{\sigma_i})_p,\ i = 1, 2, \ldots, m\}$. Now, define a map $\alpha\colon M^m \to G_M$ by

$$\alpha(x_1, x_2, \ldots, x_m) = ((\sigma_1)_{x_1}(\sigma_1^{-1})_p)((\sigma_2)_{x_2}(\sigma_2^{-1})_p) \cdots ((\sigma_m)_{x_m}(\sigma_m^{-1})_p),$$

and a smooth map $\alpha_p\colon M^m \to M$ by

$$\alpha_p(x_1, x_2, \ldots, x_m) = \alpha(x_1, x_2, \ldots, x_m)(p).$$

Writing $q = (p, p, \ldots, p)$, we have $\alpha_p(q) = p$, and it follows, using Leibniz's formula [4, Chapter I, Proposition 1.4], that

$$\begin{aligned}\alpha_p(X_1, X_2, \ldots, X_m) &= \sum_{i=1}^{m} \sigma_i^p(X_i)\\ &= \sum_{i=1}^{m} (I - S^{\sigma_i})X_i \quad \text{for } (X_1, X_2, \ldots, X_m) \in T_q M^m.\end{aligned}$$

Thus $\alpha_p(T_q M^m) = T_p M$. Consequently, there is a submanifold U of M^m, with $q \in U$, and an open neighbourhood V of p in M such that $\beta = \alpha_p | U\colon U \to V$ is a diffeomorphism. Define $\theta\colon V \to G_M$ by $\theta = \alpha \circ \beta^{-1}$. Clearly $\theta(p) = \mathrm{id}_M$, and for $x \in V$, $\theta(x)(p) = x$. Finally, since $\mu\colon M \times \Sigma \times M \to M$ is smooth, the two maps of $M^{m+1} \to M$ defined by

$$(x_1, x_2, \ldots, x_m, y) \to \alpha(x_1, x_2, \ldots, x_m)(y),$$

and

$$(x_1, x_2, \ldots, x_m, y) \to (\alpha(x_1, x_2, \ldots, x_m))^{-1}(y)$$

are smooth, hence Φ and Ψ are smooth as required.

THEOREM 1.7. *G_M is transitive on M and is the smallest subgroup of* Aut M *transitive on M and Σ-stable.*

Proof. With the notation of the previous lemma, we have $V \subset G_M(p)$ and it follows that $G_M(p)$ is open for each $p \in M$. Hence $G_M(p)$ is also closed and so $G_M(p) = M$ since M is connected. Next suppose a subgroup G of Aut M acts transitively on M and is Σ-stable. Then for each $x \in M$ there exists $g \in G$ such that $g(p) = x$, thus, for each $\sigma \in \Sigma$,

$$\sigma_x \sigma_p^{-1} = \sigma_{g(p)} \sigma_p^{-1} = g\sigma_p g^{-1} \sigma_p^{-1} = g(\sigma_p g^{-1} \sigma_p^{-1}) \in G,$$

so $G_M \subset G$ and the assertion follows.

2. Reduced affine Σ-spaces

DEFINITION 2.1. An affine Σ-space is a Σ-space M together with an (affine) connection ∇ such that

(Σ7) ∇ is Σ_M-invariant; that is, each σ_x is an affine transformation.

Furthermore, ∇ is called a canonical connection if

(Σ8) $\nabla S^\sigma = 0$ for all $\sigma \in \Sigma$.

We shall consider such spaces only for the reduced case and write (M, ∇) for a reduced affine Σ-space. We say (M, ∇) (or just M) is canonical if ∇ also satisfies (Σ8). This terminology is justified by the following lemma.

LEMMA 2.2. *A connection on a reduced Σ-space M satisfying* (Σ7) *and* (Σ8) *is unique and* Aut *M-invariant.*

Proof. Let ∇ and ∇' be two such connections on M, and define the smooth tensor field D of type (1, 2) on M by

$$D(X, Y) = \nabla_X Y - \nabla'_X Y \quad \text{for } X, Y \in \mathscr{X}(M).$$

Writing D_X for the derivation defined by $D_X = \nabla_X - \nabla'_X$, we have, by ($\Sigma$8),

$$D_X S^\sigma = 0 \quad \text{for } X \in \mathscr{X}(M), \sigma \in \Sigma. \tag{2.1}$$

Also, by (Σ7), $\sigma_x(D_X Y)_x = (D_{\sigma_x X} \sigma_x Y)_x$, so

$$S^\sigma(D_X Y) = D_{S^\sigma X}(S^\sigma Y). \tag{2.2}$$

Then (2.1) and (2.2) imply

$$D_{(I - S^\sigma)X}(S^\sigma Y) = 0. \tag{2.3}$$

Now S^σ is invertible and it follows, using (Σ5), that $D_X Y = 0$ for all X, $Y \in \mathscr{X}(M)$. Thus $D = 0$ and $\nabla = \nabla'$ as required.

The invariance of the canonical connection ∇ by Aut M is now easily proved by showing that if $\phi \in \text{Aut } M$ then the connection ∇' defined by $\nabla'_X Y = \phi^{-1} \nabla_{\phi X} \phi Y$ satisfies (Σ7) and (Σ8) and so is just ∇. This completes the proof.

For any reduced affine Σ-space (M, ∇) write $A(M, \nabla)$ for the Lie transformation group of all affine transformations of (M, ∇) with respect to the compact-open topology [4, Chapter VI, Theorem 1·5].[2] Unless otherwise stated, the Lie algebra $a(M, \nabla)$ of $A(M, \nabla)$ will be considered as the Lie algebra of complete infinitesimal affine transformations of (M, ∇) (cf. [4, Chapter VI]). We denote by Aut (M, ∇) the group $A(M, \nabla) \cap$ Aut M and consider some of its properties in the next lemma. First we remark that a diffeomorphism ϕ of M, will be called S^{Σ}-preserving if $\phi(S^{\sigma}X) = S^{\sigma}(\phi X)$ for each $X \in \mathscr{X}(M)$ and $\sigma \in \Sigma$.

LEMMA 2.3. *Let (M, ∇) be a reduced affine Σ-space. Then*

(i) Aut (M, ∇) *is the group of all S^{Σ}-preserving affine transformations of* (M, ∇);

(ii) Aut (M, ∇) *is a closed subgroup of $A(M, \nabla)$ and then acts as a transitive Lie transformation group of M*;

(iii) Aut (M, ∇) *has Lie algebra* aut (M, ∇) *defined by*

$$\mathrm{aut}\,(M, \nabla) = \{X \in a(M, \nabla)\colon \mathscr{L}_X S^{\sigma} = 0 \textit{ for all } \sigma \in \Sigma\},$$

where $\mathscr{L}_X$ denotes Lie derivation with respect to X.

Proof. (i) Clearly, by (iii) of Lemma 1·4, each $\phi \in$ Aut (m, ∇) is S^{Σ}-preserving.

Conversely, suppose $\phi \in A(M, \nabla)$ and is S^{Σ}-preserving. Then

$$(\phi^{-1} \circ \sigma_{\phi(x)} \circ \phi)X_x = \phi^{-1}(S^{\sigma}_{\phi(x)}(\phi X_x)) = S^{\sigma}_x X_x = \sigma_x X_x.$$

Thus $\phi^{-1} \circ \sigma_{\phi(x)} \circ \phi$ and σ_x are affine transformations of M which fix x and have the same differential at x, hence they are equal and $\phi \in$ Aut (M, ∇).

(ii) $A(M, \nabla)$ acts as a Lie transformation group of M and hence of the $(1, 1)$ tensor bundle $\mathscr{T}^1_1(M)$. Since S^{σ} is a closed section of $\mathscr{T}^1_1(M)$ then the subgroup of $A(M, \nabla)$ preserving S^{σ} is closed. Now Aut (M, ∇) is the intersection of such subgroups for all $\sigma \in \Sigma$ and so is closed in $A(M, \nabla)$. With its induced Lie group structure Aut (M, ∇) then acts as a Lie transformation group of M. Finally, Aut (M, ∇) is transitive on M because, by Theorem 1·7, G_M is transitive on M and, by Lemma 1.3 and (Σ7), $G_M \subset$ Aut (M, ∇).

(iii) This is an immediate consequence of (i) and (ii).

Remark. If a reduced Σ-space admits a canonical connection ∇' then, by Lemma 2.2, Aut M = Aut (M, ∇') and so in that case Lemma 2·3 will apply to Aut M; in particular, Aut M will be a Lie transformation group of M.

The next two lemmas describe some basic properties of Aut (M, ∇) and aut (M, ∇). Note that we call any subspace $\mathfrak{m}$ of aut (M, ∇) Σ_p-stable if $\sigma_p X \in \mathfrak{m}$ for each $X \in \mathfrak{m}$, $\sigma \in \Sigma$, and if this property holds for all $p \in M$ then $\mathfrak{m}$ is Σ-stable.

[2] We remark that $A(M, \nabla)$ is second countable and hence, by [12, Chapter IV, Theorem VI], $A(M, \nabla)$ has a unique topology as a Lie transformation group of M.

LEMMA 2.4. Aut (M, ∇) *is Σ-stable (cf. Definition 1.2) and, with respect to a base point $p \in M$, Σ then acts as a Lie transformation group of automorphisms of* Aut (M, ∇) *or of any connected Σ_p-stable Lie subgroup G of* Aut (M, ∇) *by $\sigma(g) = \sigma_p g \sigma_p^{-1}$. The induced action of Σ on* aut (M, ∇) *(and on the Lie algebra $\mathcal{G}$ of G) is given by $X \to \sigma_p X$, so that $\mathcal{G}$ is Σ_p-stable and* aut (M, ∇) *is Σ-stable, p being arbitrary.*

Proof. Since $A(M, \nabla)$ and Aut M are Σ-stable then so is Aut (M, ∇). Now choose $p \in M$; each $\sigma \in \Sigma$ acts as an automorphism of Aut (M, ∇) by $\phi \to \sigma_p \phi \sigma_p^{-1}$ and the differential of this map acts on aut (M, ∇) by $X \to \sigma_p X$ (recalling that elements of aut (M, ∇) are infinitesimal affine transformations). Thus aut (M, ∇) is Σ_p-stable. Since dim aut (M, ∇) is finite it follows easily that Σ acts as a Lie transformation group of aut (M, ∇). Then, for any neighbourhood U of the identity in Aut (M, ∇) on which $\exp^{-1}$ is defined (as a map), the diagram

$$\begin{array}{ccc} \Sigma \times U & \longrightarrow & \mathrm{Aut}\,(M, \nabla) \\ \downarrow \mathrm{id} \times \exp^{-1} & & \uparrow \exp \\ \Sigma \times \mathrm{aut}\,(M, \nabla) & \longrightarrow & \mathrm{aut}\,(M, \nabla) \end{array}$$

is commutative; hence Σ acts as a Lie transformation group of Aut (M, ∇).

If G is any Σ_p-stable Lie subgroup of Aut (M, ∇) then by restricting the action of Σ on Aut (M, ∇) to G we have a smooth map $\Sigma \times G \to$ Aut (M, ∇) with values in G. If G is connected then it is second countable and hence $\Sigma \times G \to G$ is smooth [1, Chapter III, Section IX, Proposition 1]. Thus Σ acts as a Lie transformation group of G. The last part of the lemma is obvious.

LEMMA 2.5. *Choose $p \in (M, \nabla)$ and let $\mathfrak{g}$ be any Lie subalgebra of* aut (M, ∇) *which is Σ_p-stable and for which the map $\mathfrak{g} \to T_p M$ defined by evaluation at p is surjective. Define a Lie subalgebra $\mathfrak{h}$ and a subspace $\mathfrak{m}$ of $\mathfrak{g}$ by.*

$$\mathfrak{h} = \{X \in \mathfrak{g}\colon X_p = 0\},$$

$$\mathfrak{m} = \mathrm{gen}\,\{X - \sigma_p X\colon \sigma \in \Sigma, X \in \mathfrak{g}\}.$$

Then

(i) $\mathfrak{h} = \{X \in \mathfrak{g}\colon \sigma_p X = X$ *for all* $\sigma \in \Sigma\}$;

(ii) $\mathfrak{m}$ *is Σ_p-stable*;

(iii) $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

Furthermore, if (M, ∇) is canonical then

(iv) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *(vector space direct sum).*

Proof. Following the method of Nomizu [11], let $X \in a(M, \nabla)$ and define the tensor field A_X of type (1, 1) on M by

$$A_X Y = [X, Y] - \nabla_X Y, \quad Y \in \mathfrak{X}(M).$$

Along any smooth curve γ on M with tangent vector field V, the pair (X, A_X) satisfies the system of first order linear differential equations

$$\nabla_V X = T(V, X) - A_X V, \qquad \nabla_V A_X = R(X, V).$$

Hence (X, A_X) on γ is determined by its value at any point of γ. Since M is connected it follows that X is determined on M by $(X_p, (A_X)_p)$.

We now show that if $X \in \operatorname{aut}(M, \nabla)$ then

$$(A_{X-\sigma_p X} + (\nabla_X S^\sigma) S^{\sigma^{-1}})_p = 0 \quad \text{for all } \sigma \in \Sigma. \tag{2.4}$$

Thus, for $Y \in \mathfrak{X}(M)$,

$$\begin{aligned} A_{\sigma_p X} Y &= [\sigma_p X, Y] - \nabla_{\sigma_p X} Y \\ &= -\nabla_Y \sigma_p X - T(\sigma_p X, Y) \\ &= -\sigma_p \nabla_{\sigma_p^{-1} Y} X - \sigma_p T(X, \sigma_p^{-1} Y). \end{aligned}$$

Then, using (iii) of Lemma 2·3,

$$\begin{aligned} (A_{\sigma_p X} Y)_p &= -S^\sigma(\nabla_{S^{\sigma^{-1}} Y} X + T(X, S^{\sigma^{-1}} Y))_p \\ &= S^\sigma([X, S^{\sigma^{-1}} Y] - \nabla_X(S^{\sigma^{-1}} Y))_p \\ &= ([X, Y] - \nabla_X Y - S^\sigma(\nabla_X S^{\sigma^{-1}}) Y)_p \\ &= (A_X Y + (\nabla_X S^\sigma) S^{\sigma^{-1}} Y)_p, \end{aligned}$$

which proves (2.4).

(i) Clearly $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, and if $X \in \mathfrak{h}$ then

$$(X - \sigma_p X)_p = (A_{X - \sigma_p X})_p = 0$$

so $X - \sigma_p X = 0$. Conversely if $X \in \mathfrak{g}$ and $X - \sigma_p X = 0$ for all $\sigma \in \Sigma$ then, by (Σ6), $X_p = 0$ so $X \in h$.

(ii) $\mathfrak{m}$ is Σ_p-stable because $\mathfrak{g}$ is Σ_p-stable and

$$\tau_p(X - \sigma_p X) = \tau_p X - (\tau\sigma\tau^{-1})_p \tau_p X.$$

(iii) If $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$ then, by (i),

$$[X, Y - \sigma_p Y] = [X, Y] - \sigma_p[X, Y] \in m,$$

from which (iii) follows.

(iv) If ∇ is the canonical connection then (2.4) is just $(A_{X-\sigma_p X})_p = 0$ since $\nabla S^\sigma = 0$. Then if $X \in \mathfrak{h} \cap \mathfrak{m}$ we have $X_p = (A_X)_p = 0$ and so $X = 0$. Also, by (Σ5), if $X \in \mathfrak{g}$ then

$$X_p = \sum_{i=1}^{k} (X_i - \sigma_{i_p} X_i)_p \quad \text{for some } \sigma_{i_p} \in \Sigma_p,\ X_i \in \mathfrak{g},\ i = 1, 2, \ldots, k.$$

Hence

$$\begin{aligned} X &= \left(X - \sum_{i=1}^{k} (X_i - \sigma_{i_p} X_i)\right) + \sum_{i=1}^{k} (X_i - \sigma_{i_p} X_i) \\ &= Y + Z \end{aligned}$$

where $Y \in \mathfrak{h}$ and $Z \in \mathfrak{m}$, which proves (iv).

Before obtaining an alternative characterisation of a canonical reduced affine Σ-space we make the following definition.

DEFINITION 2.6. Let G be a connected Lie group, Σ a Lie transformation group of automorphisms of G, and H a closed subgroup of G such that $(G^\Sigma)_0 \subset H \subset G^\Sigma$, where

$$G^\Sigma = \{g \in G : \sigma(g) = g \quad \text{for all } \sigma \in \Sigma\}.$$

Then (G, H, Σ) is a Σ-triple if $\mathscr{G} = \mathscr{H} \oplus \mathfrak{m}$ (direct sum) where m is the subspace of G generated by

$$\{X - \sigma(X) : X \in G \quad \text{and} \quad \sigma \in \Sigma\}.$$

This definition implies that G/H is reductive with respect to the decomposition $G = H + \mathfrak{m}$ since, as an easy consequence of $H \subset G^\Sigma$, we have $\mathrm{ad}_G H(\mathfrak{m}) \subset \mathfrak{m}$. Then G/H admits the connection of the second kind [5, Chapter X] which we use in the next theorem.

THEOREM 2.7. (a) *Let* (G, H, Σ) *be a* Σ*-triple. Define* $M = G/H$; *let* $\nu : G \to M$ *be the natural projection, and set* $p = \nu(H)$. *Then* M *admits a unique reduced* Σ*-space structure with the two properties*:

(i) $\sigma_p \circ \nu = \nu \circ \sigma$ *for each* $\sigma \in \Sigma$;

(ii) *with the standard action of* G *on* M *as a Lie transformation group, each element of* G *is an automorphism of* M *and* G *is* Σ*-stable.*

Moreover, (M, ∇) *is then canonical where* ∇ *is the connection of the second kind on* M.[3]

(b) *Conversely, let* (M, ∇) *be a canonical reduced affine* Σ*-space with base point* p. *Let* G *be any connected* Σ*-stable Lie subgroup of* Aut M *transitive*[4] *on* M, *and let* H *be the isotropy subgroup of* G *at* p. *Then* (G, H, Σ) *is a* Σ*-triple for which the map* $aH \to a(p)$, $a \in G$ *is an isomorphism of the corresponding reduced* Σ*-space structure on* G/H *with that on* M.

Proof. (a) Suppose given (G, H, Σ). Then (i) can be written as $\sigma_H(bH) = \sigma(b)H$, and, by (ii), we require $\sigma_{aH}(bH) = a\sigma(a^{-1}b)H$ for all $a, b \in G$ and $\sigma \in \Sigma$.

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[3] If, for a given (G, H, Σ), G acts effectively on $M = G/H$ then clearly it acts as a Lie subgroup of Aut M. However, by analogy, with the definition of a symmetric pair [3, Chapter IV] it is preferable not to make this assumption.

[4] For example the identity component $(\mathrm{Aut}\ M)_0$.

Thus we have uniqueness of the Σ-space structure and accordingly define a map

$$\mu: M \times \Sigma \times M \to M$$

by

$$\mu(aH, \sigma, bH) = a\sigma(a^{-1}b)H$$

(this is well defined because $H \subset G^{\Sigma}$). Now it is easy to see that μ is a smooth map with properties (Σ1)–(Σ4) so M becomes a Σ-space satisfying (i) and (ii). Note that G is Σ-stable since, by (i), it is Σ_p-stable and hence Σ-stable by transitivity.

To prove (Σ5) and (Σ6), we first note that $\mathfrak{m}$ is stable with respect to the induced action of Σ on $\mathscr{G}$; for if $\sigma, \tau \in \Sigma$ and $X \in \mathscr{G}$, then

$$\tau(X - \sigma(X)) = \tau(X) - \tau\sigma\tau^{-1}(\tau X) \in \mathfrak{m}.$$

Next, if $Y \in \mathfrak{m}$ satisfies $\sigma(Y) = Y$ for all $\sigma \in \Sigma$ then $Y \in \mathscr{H}$ and so $Y = 0$. It now follows, using (i) and the above remarks, that (Σ5) and (Σ6) hold at p and hence on M by the transitivity of G. Thus M is a reduced Σ-space.

It remains to prove (Σ7) and (Σ8) where ∇ is the connection of the second kind on M. For each $X \in \mathscr{G}$ write $\bar{X}$ for the corresponding infinitesimal affine transformation of M. Then ∇ is characterised by the condition

$$\nabla_{\bar{X}_H} Y = [\bar{X}, Y]_H \quad \text{for all } X \in \mathfrak{m}, \qquad Y \in \mathfrak{X}(M).$$

Since $\sigma_H(\bar{X}) = \overline{\sigma(X)}$, it is clear that each σ_H preserves ∇ at the base point $p = H$, and it follows from the action of G on M that each σ_{aH} is an affine transformation. Finally, by (ii) above and (iii) of Lemma 1·4, each S^{σ} is G-invariant and hence $\nabla S^{\sigma} = 0$ since ∇ is the canonical connection of the second kind.

(b) Conversely, suppose (M, ∇) is canonical with G and H as given in (b). Let Σ act as a Lie transformation group of G by $\sigma(g) = \sigma_p g \sigma_p^{-1}$ as in Lemma 2·4. Clearly Lemma 2·5 applies with $\mathscr{G}, \mathscr{H}$ replacing $\mathfrak{g}, \mathfrak{h}$ respectively. Then, in particular, $\mathscr{H}$ is the fixed point subset of $\mathscr{G}$ under the induced action of Σ on $\mathscr{G}$, hence $H \supset (G^{\Sigma})_0$. Also

$$\sigma_p h \sigma_p^{-1} = \sigma_p \sigma_{h(p)}^{-1} h = h \quad \text{for } h \in H$$

so $H \subset G^{\Sigma}$; thus $(G^{\Sigma})_0 \subset H \subset G^{\Sigma}$.

Now define $\mathfrak{m}$ as in Lemma 2.5. For $X \in \mathscr{G}$ and $h \in H$ we have

$$h(X - \sigma_p X) = h(X) - \sigma_p h(X) \in \mathfrak{m},$$

and it follows, using (iv) of Lemma 2.5, that $M = G/H$ is reductive with $\mathscr{G} = \mathscr{H} \oplus m$. Thus (G, H, Σ) is a Σ-triple. The isomorphism property is clear by construction of the Σ-space structure on G/H; it is an affine transformation since the canonical connection is unique. This completes the proof.

We remark that the canonical connection is complete since it agrees with the connection of the second kind.

COROLLARY 2.8. *Suppose (M, ∇) is canonical with base point p and associated Σ-triple (G, H, Σ) as in* (b) *of Theorem 2.7. Let G' be the subgroup of G generated by $\{g\sigma(g^{-1})\colon g \in G$ and $\sigma \in \Sigma\}$, let G_M be the subgroup of* Aut M *generated by all elements of the form $\sigma_x \sigma_p^{-1}$ for $x \in M$ and $\sigma \in \Sigma$ as in Lemma* 1.3, *and let G'' be the connected Lie subgroup of G with Lie algebra $\mathscr{G}''$ generated by $\mathfrak{m}$. Then $G' = G_M = G''$. Thus we may consider G_M as a connected Lie subgroup of* Aut M.

Proof. We identify M with G/H and use (a) of Theorem 2·7, the Lie algebra of G being regarded as the tangent space to G at the identity e. First note that $G' = G_M$, for if $a, b \in G$ and $\sigma \in \Sigma$ then

$$a\sigma(a^{-1})(bH) = a\sigma(a^{-1}\sigma^{-1}(bH)) = \sigma_{aH}\sigma_H^{-1}(bH).$$

Next, G' is a Lie subgroup of G. Thus, for any (piecewise smooth) curve γ in G,

$$\gamma(t)\sigma((\gamma(t))^{-1}) \subset G'$$

and, as an easy consequence, G' is an arcwise connected subgroup of G, hence a Lie subgroup [4, Appendix 4]. Moreover, $\mathscr{G}''$ is a subalgebra of $\mathscr{G}'$, for if γ is a curve in G with prescribed tangent vector X at e then the curve $t \to \gamma(t)\sigma((\gamma(t))^{-1})$ in G' has tangent vector $X - \sigma(X)$ at e. Consequently, G'' is a Lie subgroup of G'.

Finally, we show that $G_M \subset G''$. Now $\mathfrak{m}$ projects onto T_pM so the orbit $G''(p)$ is open in M. But G'' is a normal subgroup of G since $\mathscr{G}''$, being ad H-stable, is an ideal in $\mathscr{G}$; it results that every orbit of G'' is open and G'' is transitive on M. Furthermore, $\mathfrak{m}$ is stable under the action of Σ on $\mathscr{G}$, so G'', as a subgroup of Aut M, is Σ_p-stable and hence Σ-stable being transitive. Then, by Theorem 1·7, $G_M \subset G''$.

We complete this section with an example of a Σ-space of the form $M = G/H$, where Σ acts as a Lie transformation group of automorphisms of G with $(G^\Sigma)_0 \subset H \subset G^\Sigma$ and M admits a Σ_M-invariant (and G-invariant) connection but is not a reduced Σ-space.

Example. Let G be the identity component of

$$\left\{ \begin{pmatrix} x & y & z \\ 0 & u & v \\ 0 & t & w \end{pmatrix} \in SL(3, \mathbf{R}) \right\}$$

and H its Lie subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Define Σ to be the group of inner automorphisms I_h of G for all $h \in H$, then Σ is isomorphic to H and $(G^{\Sigma})_0 = H$. It can be seen that

$$\mathscr{H} \subset \{X - \sigma(X) \colon X \in \mathscr{G} \text{ and } \sigma \in \Sigma\}$$

so (G, H, Σ) is not a Σ-triple. However, as the proof of Theorem 2·7(a) shows, μ is well defined so that $M = G/H$ is a Σ-space with G acting effectively. Moreover, a G-invariant connection can be defined on M as follows. Define a direct sum decomposition of $\mathscr{G}$ by $\mathscr{G} = \mathscr{H} \oplus \mathfrak{m}$ where

$$\mathfrak{m} = \left\{ \begin{pmatrix} x & y & 0 \\ 0 & u & 0 \\ 0 & t & w \end{pmatrix} : x + u + w = 0 \right\},$$

and identity $\mathfrak{m}$ with $\mathbf{R}^4$ by

$$\begin{pmatrix} x & y & 0 \\ 0 & u & 0 \\ 0 & t & w \end{pmatrix} \to \begin{pmatrix} x \\ y \\ u \\ t \end{pmatrix}.$$

The isotropy representation $\lambda \colon H \to GL(4, \mathbf{R})$ is then given by

$$\lambda(h)(X) = (\mathrm{Ad}_G\,(h)X) \quad \text{for } X \in \mathfrak{m},$$

and the induced Lie algebra homomorphism, also denoted by λ, is just $\lambda(X)(Y) = [XY]_{\mathfrak{m}}$ for $X \in \mathscr{H}$, $Y \in \mathfrak{m}$. Define $\Lambda \colon G \to \mathrm{gl}\,(4, \mathbf{R})$ by

$$\Lambda(X) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & X & \\ 0 & & & \end{pmatrix}.$$

Then

$$\Lambda(X) = \lambda(X) \quad \text{for } X \in \mathscr{H},$$

and

$$\Lambda([X, Y]) = [\Lambda(X), \Lambda(Y)] \quad \text{for } X \in \mathscr{H},\ Y \in \mathscr{G}.$$

Hence, by [5, Chapter X, Theorem 1.2], there exists a G-invariant connection on M. Such a connection is also Σ_M-invariant. For let $\sigma = I_h$ and $a \in G$. Then for $b \in G$, $\sigma_{aH}(bH) = a\sigma(a^{-1}b)H = aha^{-1}bH$, so $\sigma_{aH} = t_{aha^{-1}}$ where for $x \in G$, t_x is the affine transformation $M \to M$; $bH \to xbH$. The Σ-space M is not reduced, since

$$(\mathrm{Ad}_G\,(h)X)_{\mathfrak{m}} = X \quad \text{for } X = \begin{pmatrix} x & y & 0 \\ 0 & u & 0 \\ 0 & 0 & w \end{pmatrix} \in \mathfrak{m},$$

so ($\Sigma 6$) is not satisfied. Also, in this case the connection is not canonical, for it is easily seen that if ($\Sigma 8$) is satisfied then

$$(\mathrm{Ad}_G\, h)(\Lambda(X)) = (\Lambda(X))(\mathrm{Ad}_G\, h) \quad \text{for } h \in H,\ X \in \mathfrak{m};$$

but the equation is not satisfied when, for example,

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and } X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

3. Cyclic reduced Σ-spaces

In this section we consider the case when Σ is cyclic with a generator denoted by σ, unless otherwise stated, and we call such spaces cyclic reduced Σ-spaces. They are shown to be in one-to-one correspondence with affine regular s-manifolds. Moreover, as for the case of Σ compact, it results that a cyclic reduced Σ-space always admits a canonical connection. We first recall the following:

DEFINITION 3.1. A regular s-manifold is a connected manifold M together with a map s from M into Diff M such that

(i) each point $x \in M$ is an isolated fixed point of the corresponding diffeomorphism $s(x)$ (written as s_x);

(ii) $s_x \circ s_y = s_{s_x(y)} \circ s_x$ for all $x, y \in M$;

(iii) the (1, 1) tensor field S on M defined by

$$SX_x = s_x X_x \quad \text{for } x \in M,\ X_x \in T_xM$$

is smooth.

For each $x \in M$, s_x is called the symmetry at x and S is the symmetry tensor field. If, in addition, a connection ∇ exists on M such that each symmetry is affine then M is an affine regular s-manifold [2], [9].

We now prove a lemma for arbitrary reduced Σ-spaces although it will be used here only for the cyclic case.

LEMMA 3.2. *Let M be a reduced Σ-space for which there exists a map $L\colon TM \to \mathfrak{X}(M)$ such that, for each $\sigma \in \Sigma$, p, $x \in M$ and $X_p \in T_pM$,*

(i) *$L \mid T_pM$ is linear;*

(ii) *$\sigma_x L(X_p) = L(\sigma_x X_p)$, $(L(X_p))_p = X_p$.*

Then M admits a Σ_M-invariant connection. Moreover, M admits a canonical connection if and only if a map $L\colon TM \to \mathfrak{X}(M)$ exists satisfying (i), (ii) *and*

(iii) *$\mathscr{L}_{L(X_p)} S^\sigma = 0$, for each $X_p \in TM$.*

Proof. Suppose M satisfies (i) and (ii), and for each $X_p \in TM$ and $Y \in \mathfrak{X}(M)$, define

$$\nabla_{X_p} Y = [L(X_p), Y]_p. \tag{3.1}$$

Then

$$\begin{aligned}\sigma_x(\nabla_{X_p} Y) &= [\sigma_x L(X_p), \sigma_x Y]_{\sigma_x(p)}\\ &= [L(\sigma_x X_p), \sigma_x Y]_{\sigma_x(p)}\\ &= \nabla_{\sigma_x X_p}\sigma_x Y. \qquad (3.2)\end{aligned}$$

We now use the map Φ: $V \times M \to M$ defined in Lemma 1·6 to prove that a smooth connection ∇ is given by (3.1); the linear properties of ∇ are clear. Since Φ is smooth and $\Phi(x, p) = x$ for $x \in V$, it follows that for each $X_p \in T_pM$ a smooth vector field X' satisfying $X'_p = X_p$ is defined on V by $X'_x = \Phi(0_x, X_p) = \theta(x)X_p$. Then, from equation (3.2) and the definition of $\theta(x)$ we have,

$$\begin{aligned}\nabla_{X_{x'}} Y &= \theta(x)\nabla_{X_p}(\theta(x))^{-1}Y\\ &= \theta(x)[L(X_p), (\theta(x))^{-1}Y]_p\end{aligned}$$

which is clearly a smooth vector field on V. Also vector fields X' as above generate $\mathfrak{X}(V)$, and the smoothness of ∇ is an easy consequence. Finally ∇ is Σ_M-invariant by (3.2).

We next note that if the above map L also satisfies (iii) then (M, ∇) is canonical since, by (3.1), for each $X_p \in TM$ and $\sigma \in \Sigma$,

$$\nabla_{X_p} S^\sigma = (\mathscr{L}_{L(X_p)} S^\sigma)_p = 0.$$

Conversely, suppose (M, ∇) is canonical and let $X_p \in T_pM$. Then, by (Σ5) and the transitivity of Aut M, we have

$$X_p = \sum_{i=1}^{k} (X_i - \sigma_{i_p}(X_i))_p \quad \text{for some } \sigma_{i_p} \in \Sigma_p,\ X_i \in \text{aut } M,\ i = 1, 2, \ldots, k.$$

Writing

$$L(X_p) = \sum_{i=1}^{k} (X_i - \sigma_{i_p}(X_i))$$

we see that $L(X_p)$ is unique since $(L(X_p))_p = X_p$, $(A_{L(X_p)})_p = 0$, and $L(X_p)$ is an infinitesimal affine transformation. Thus L: $TM \to \mathfrak{X}(M)$ is defined and clearly satisfies (i). Property (ii) follows easily, and (ii) is a consequence of (iii) of Lemma 2·3 since $L(X_p) \in$ aut M. This completes the proof.

THEOREM 3.3. *Any cyclic reduced Σ-space admits a connection satisfying (Σ7) and (Σ8), and is then an affine regular s-manifold. Conversely, every affine regular s-manifold is a cyclic reduced Σ-space. In each case the correspondence is just $\sigma_x = s_x$.*

Proof. Let M be a cyclic reduced Σ-space. Then $I - S$ is invertible, for it follows from (Σ6) and the cyclic property of Σ that $(I - S^\sigma)X_p = 0$ implies $X_p = 0$; hence (Σ5) is redundant. Now define maps K, L: $TM \to \mathfrak{X}(M)$ by

$$(K(X_p))_x = \sigma^{\sigma_p^{-1}(x)} X_p$$

and

$$L(X_p) = K((I - S^\sigma)^{-1} X_p) \quad \text{for all } p, x \in M.$$

Clearly (i) of Lemma 3.2 is satisfied. Also (ii) holds because

$$\begin{aligned}\sigma_x(L(X_p))_q &= \sigma_x \sigma^{\sigma_p^{-1}(q)}(I - S^\sigma)^{-1} X_p \\ &= \sigma^{\sigma_x \sigma_p^{-1}(q)} \sigma_x (I - S^\sigma)^{-1} X_p \\ &= \sigma^{\sigma_x \sigma_p^{-1}(q)} (I - S^\sigma)^{-1} \sigma_x X_p \quad \text{(by Lemma 1.4)} \\ &= (L(\sigma_x X_p))_{\sigma_x(q)};\end{aligned}$$

and, secondly, $(L(X_p))_p = \sigma^p(I - S^\sigma)^{-1} X_p = X_p$. Hence, by Lemma 3.2, a connection ∇ is defined on M. We next show that, for $X_p \in T_pM$, $K(X_p) \in \text{aut}\,(M, \nabla)$. Now Aut (M, ∇) is transitive on M so each X_p can be extended to some $X \in \text{aut}\,(M, \nabla)$ generating a 1-parameter group ϕ_t in Aut M. Then for all $x \in M$, $\sigma_p \phi_t(x) = \phi_t \sigma_{\phi_{-t}(p)}(x)$. It follows from Leibniz's formula, that

$$\sigma_p X_x = X_{\sigma_p(x)} - \sigma^x X_p,$$

which implies

$$X - \sigma_p X = K(X_p).$$

Then, by Lemma 2.4, $K(X_p) \in \text{aut}\,(M, \nabla)$. It is immediate that $L(X_p) \in \text{aut}\,(M, \nabla)$ hence $\mathscr{L}_{L(X_p)} S^\sigma = 0$ proving (iii) of Lemma 3.2. Thus ∇ is the canonical connection on M.

To show that (M, ∇) is an affine regular s-manifold with $s_x = \sigma_x$, let $p \in M$ and let U be a normal neighbourhood of p. Since $I - S$ is invertible, it follows easily that p is the only point in U fixed by σ_p. Thus (i) of Definition 3.1 is satisfied. Also (ii) and (iii) are immediate by (Σ4)′ and Lemma 1.4.

Conversely, let M be an affine regular s-manifold, and let Σ be the discrete group isomorphic to the cyclic group generated by s_p for a fixed point $p \in M$. Note that Σ is independent of p since, by [8], if $x \in X$ then there is a diffeomorphism ϕ of M such that

$$\phi(p) = x \quad \text{and} \quad s_x = \phi \circ s_p \circ \phi^{-1}.$$

Now define μ: $M \times \Sigma \times M \to M$ by $\mu(x, \sigma^k, y) = s_x^k(y)$. The smoothness of μ is proved in [9] for the Riemannian case but the same proof applies here. Then (Σ1)–(Σ4) are immediate. Also $I - S$ is invertible, as follows easily by using (i) of Definition 3.1 for a normal neighbourhood, as above. (Σ5) and (Σ6) are clearly satisfied and the proof complete.

Let G be a connected Lie group, Σ a Lie transformation group of automorphisms of G and H a subgroup of G such that $(G^\Sigma)_0 \subset H \subset G^\Sigma$. Then it can be shown [10] that, if Σ is compact, the conditions of Theorem 2·7 always apply so that, in particular, G/H has the structure of a reduced Σ-space. However, this is not true when Σ is cyclic as we now show.

Example. Let $G = SL(2, \mathbf{R})$ and H its Lie subgroup of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbf{R}.$$

Let

$$\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and let Σ be the cyclic group of inner automorphisms of G generated by σ. Then $H = G^\Sigma = (G^\Sigma)_0$ and G acts effectively on G/H. As in the proof of Theorem 2.7, G/H is a Σ-space. However, it is not reduced; thus for example

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathscr{G}$$

is fixed modulo $\mathscr{H}$ by the induced action of Σ on $\mathscr{G}$ and hence its (non-zero) image at the point $H \in G/H$ is fixed by S^σ which clearly contradicts (Σ6).

Again, one might conjecture from Theorem 3.3 that if M is a reduced Σ-space there always exists $\sigma \in \Sigma$ such that, with respect to the group Σ' generated by σ, M is a cyclic reduced Σ'-space, or perhaps that M is a regular s-manifold with respect to some $\sigma \in \Sigma$. A counterexample to both conjectures is as follows.

Example. Let $M = \mathbf{R}^3$, $\Sigma = \{e, \sigma, \tau\}$ where $\sigma^2 = \tau^2 = (\sigma\tau)^2 = e$, and define $M \times \Sigma \times M \to M$; $(x, \sigma, y) \mapsto \sigma_x(y)$ by

$$\begin{aligned} \sigma_x(y) &= (y_1, 2x_2 - y_2, 2x_3 - y_3) \\ \tau_x(y) &= (2x_1 - y_1, y_2, 2x_3 - y_3). \\ e_x(y) &= y,\ (\sigma\tau)_x(y) = \sigma_x \tau_x(y). \end{aligned}$$

It can be seen that M is then a Σ-space. Moreover, it is reduced because if $V = (v_1, v_2, v_3) \in T_p M$ then

$$4V = (V - S^\sigma V) + (V - S^\tau V) + (V - S^{\sigma\tau} V)$$

and no non-zero vector is fixed by S^σ, S^τ and $S^{\sigma\tau}$. However, S^σ, S^τ and $S^{\sigma\tau}$ each fix some vector at p. Also, taking p as the origin in $\mathbf{R}^3$, it follows from the linearity of Σ_p that p is not an isolated fixed point of σ_p, τ_p or $(\sigma\tau)_p$. Thus neither (Σ6) nor (i) of Definition 3.1 can be satisfied by selecting one element of Σ.

4. Reduced Riemannian Σ-spaces

DEFINITION 4.1. A Riemannian Σ-space is a Σ-space M together with a Σ_M-invariant (Riemannian) metric g, that is, a metric for which each σ_x is an isometry. An $R\Sigma$-space is a Σ-space which admits such a metric.[5]

We consider these spaces only for the reduced case, and (M, g) will denote a reduced Riemannian Σ-space.

Remark. (Σ5) is now a consequence of (Σ6). For at $p \in M$, let $V = \text{gen}\,\{X - S^\sigma X\colon X \in T_p M \text{ and } \sigma \in \Sigma\}$, and suppose $g(V, Y) = 0$ for some $Y \in T_p M$. Then for all $X \in T_p M$ and $\sigma \in \Sigma$, $g(Y - S^\sigma Y, X) = g(Y, X - S^\sigma X) = 0$ so $Y = 0$ by (Σ6). Thus $V = T_p M$ proving (Σ5).

We write $I(M, g)$ for the Lie transformation group of all isometries of (M, g) with respect to the compact-open topology, and define $\text{Aut}\,(M, g) = \text{Aut}\,M \cap I(M, g)$. Now $I(M, g)$ is a closed Lie subgroup of $A(M, \nabla)$ where ∇ is the Riemannian connection on (M, g), and clearly an affine reduced Σ-space structure is induced on (M, ∇). Then, with slight modifications, Lemma 2.3 and 2.4 apply to the Riemannian case. In particular (ii) of Lemma 2.3 is true for $\text{Aut}\,(M, g)$ and so it will be considered as a closed Lie subgroup of $I(M, g)$. Now write $i(M, g)$ and $\text{aut}\,(M, g)$ for the Lie algebras of $I(M, g)$ and $\text{Aut}\,(M, g)$ respectively. Then in order to avoid repetition we simply state:

LEMMA 4.2. *Lemmas* 2.3 *and* 2.4 *remain true with* $\text{Aut}\,(M, \nabla)$, $\text{aut}\,(M, \nabla)$, $a(M, \nabla)$ *replaced by* $\text{Aut}\,(M, g)$, $\text{aut}\,(M, g)$, $i(M, g)$ *respectively.*

We remark that $G_M \subset \text{Aut}\,(M, g)$ as a consequence of Definition 4.1.

DEFINITION 4.3. Let G be a connected Lie group, H a closed Lie subgroup of G, and Σ a Lie transformation group of automorphisms of G. We call (G, H, Σ) an $R\Sigma$-triple if

(i) $(G^\Sigma)_0 \subset H \subset G^\Sigma$;

(ii) the subgroup of Aut G generated by $\text{ad}_G\, H$ and Σ_* has compact closure in Aut G, where Σ_* is just the image of Σ under its differential representation on G.

We now obtain the following analogue of Theorem 2.7.

THEOREM 4.4. (a) *Let (G, H, Σ) be any $R\Sigma$-triple. Then it is a Σ-triple, and the corresponding reduced Σ-space $M = G/H$ is a reduced $R\Sigma$-space admitting a Σ_M-invariant and G-invariant metric.*

(b) *Conversely, suppose M is a reduced $R\Sigma$-space with base point p. Let g be any Σ_M-invariant metric on M, G any connected Σ-stable Lie subgroup of* $\text{Aut}\,(M, g)$ *transitive on M,*[6] *and H the isotropy subgroup of G at p. Then (G, H, Σ) is an $R\Sigma$-triple for which the isomorphism property in Theorem 2.7 is satisfied.*

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[5] Clearly, a similar distinction can be made for the affine case.

[6] For example, $(\text{Aut}\,(M, g))_0$.

Proof. (a) Let (G, H, Σ) be an $R\Sigma$-triple and let K be the closure of $\mathrm{ad}_G H$ and Σ_* in Aut $\mathscr{G}$. It then follows that $K(\mathscr{H}) = \mathscr{H}$. Also, since K is compact, there exists a K-invariant positive definite quadratic form $\langle\ ,\ \rangle$ on $\mathscr{G}$. Let

$$\mathfrak{m} = \text{gen}\,\{X - \sigma(X): X \in \mathscr{G} \text{ and } \sigma \in \Sigma\}.$$

We show that $\mathscr{G}$ has the orthogonal decomposition $\mathscr{G} = \mathscr{H} \oplus \mathfrak{m}$. Thus for all $X \in \mathscr{G}$, $Y \in \mathscr{H}$, and $\sigma \in \Sigma$, we have

$$\langle X - \sigma(X), Y\rangle = \langle X, Y\rangle - \langle X, \sigma^{-1}(Y)\rangle = 0$$

so $\mathscr{H}$ and $\mathfrak{m}$ are orthogonal subspaces of $\mathscr{G}$. Also, if $X \in G$ is orthogonal to $\mathscr{H}$ and $\mathfrak{m}$ then for all $Y \in \mathscr{G}$ and $\sigma \in \Sigma$,

$$\begin{aligned}\langle X - \sigma(X), Y\rangle &= \langle XY\rangle - \langle X, \sigma^{-1}(Y)\rangle \\ &= \langle X, Y - \sigma^{-1}(Y)\rangle \\ &= 0.\end{aligned}$$

Thus $X - \sigma(X) = 0$ for all $\sigma \in \Sigma$, and as X is orthogonal to $\mathscr{H}$ we have $X = 0$ by (i). This proves the orthogonal decomposition of $\mathscr{G}$ and shows that $K(\mathfrak{m}) = \mathfrak{m}$. In particular, $\mathrm{ad}_G H(\mathfrak{m}) = \mathfrak{m}$ so (G, H, Σ) is a Σ-triple. Now apply Theorem 2·7 to obtain $M = G/H$ as a reduced Σ-space. It is standard that the restriction of $\langle\ ,\ \rangle$ to $\mathfrak{m}$ gives rise to a G-invariant metric g on M and it follows easily that g is also Σ_M-invariant, the proof being similar to that for the invariance of ∇ in Theorem 2.7(a).

(b) Conversely, suppose M is a reduced $R\Sigma$-space with G, H and g as given in (b), and with Σ acting on G as in Lemma 2.4. Then (i) of Definition 4.3 follows as in Theorem 2.7(b). To prove (ii), let H' denote the closure in $I(M, g)$ of the group generated by H and Σ_p. Then H' is compact, and it is easy to see that $\mathscr{G}$ is invariant by $\mathrm{ad}_I H'$, the image of H' under its adjoint representation on $i(M, g)$. Then the restriction of $\mathrm{ad}_I H'$ to $\mathscr{G}$ is a compact subgroup of Aut $\mathscr{G}$ containing $\mathrm{ad}_G H$ and Σ_*, and (ii) of Definition 4.3 follows immediately. The isomorphism property holds as in Theorem 2.7 and the proof is complete.

Corollary 4.5. *Any reduced $R\Sigma$-space M is canonical and, for any Σ_M-invariant metric g, $\nabla g = 0$ where ∇ is the canonical connection.*

Proof. For a given g there exists an $R\Sigma$-triple (G, H, Σ) as in Theorem 4.4(b) and then, by (a), it is a Σ-triple. Hence, by Theorem 2·7, ∇ exists on M. Moreover, since it is the connection of the second kind and g is G-invariant, then $\nabla g = 0$ as required.

Corollary 4.6. *Let M be a reduced $R\Sigma$-space. Then there exists a compact Lie group Σ^c with respect to which M is a reduced $R\Sigma^c$-space such that*

(i) *for each $x \in M$ and $\sigma \in \Sigma$, $\sigma_x \in \Sigma^c_x$;*

(ii) *every Σ_M-invariant metric on M is Σ^c_M-invariant.*

Proof. Let $I = \bigcap_{g \in A} I(M, g)$ where A is the set of all Σ_M-invariant metrics on M. It is easy to see that I is a closed subgroup of each $I(M, g)$. Moreover, since each $I(M, g)$ has the compact-open topology the induced Lie group structure on I is independent of the choice of $I(M, g)$. Now $G_M \subset I$, hence I is a transitive Lie transformation group of M. Choose $p \in M$ and let K be the isotropy subgroup of I at p. Then K is compact and $\Sigma_p \subset K$. Hence the closure $\bar{\Sigma}_p$ of Σ_p in K is compact. In order to standardise notation, let Σ^c be the image of $\bar{\Sigma}_p$ in I under inclusion. Then we can write the inverse map as $\Sigma^c \to \bar{\Sigma}_p$; $\tau \to \tau_p$.

Now define the closed Lie subgroup G of I by $G = (I \cap \text{Aut } M)_0$; G is Σ_p-stable and hence $\bar{\Sigma}_p$-stable by continuity. Also G is transitive on M. Let H be the isotropy subgroup of G at p. Then H is compact since K is compact. Next choose a metric connection associated with some $g \in A$. Now the first part of the proof of Theorem 2·7(b) does not use the canonical property of a connection and so can be applied here since the metric connection is Σ_M-invariant. Thus we have $(G^\Sigma)_0 \subset H \subset G^\Sigma$. Then $G^{\Sigma^c} = \{a \in G\colon \sigma_p a \sigma_p^{-1} = a \text{ for all } \sigma \in \Sigma^c\}$ clearly satisfies $(G^{\Sigma^c})_0 \subset (G^\Sigma)_0$, and, by continuity $H \subset G^{\Sigma^c}$ as a consequence of $H \subset G^\Sigma$. Thus $(G^{\Sigma^c})_0 \subset H \subset G^{\Sigma^c}$.

Now G is a closed subgroup of I with Σ^c acting on G by $a \to \sigma_p a \sigma_p^{-1}$, also Σ^c and H are compact. Hence (G, H, Σ^c) is an $R\Sigma^c$-triple. Thus, by Theorem 4.4, M is a reduced $R\Sigma^c$-space for which (i) and (ii) are clearly true, and the proof is complete.

We now prove a generalisation of the de Rham decomposition theorem for symmetric spaces [5, Chapter XI, Theorem 6.6] and, more generally, Riemannian regular s-manifolds [6, Theorem 3].

THEOREM 4.7. *Let (M, g) be a simply connected Riemannian reduced Σ-space*[7] *and let $M = M_0 \times M_1 \times \cdots \times M_r$ be its de Rham decomposition where M_0 is Euclidean and $M_1, M_2, \ldots, M_r$ are irreducible. Then each factor is a Riemannian reduced Σ_i-space where Σ_i is a closed Lie subgroup of Σ of finite index.*

Proof. We consider the cases Σ compact and Σ non-compact separately. First some general remarks. Choose a base point $p \in M$ and let

$$T_p M = V_0 \oplus V_1 \oplus \cdots \oplus V_r$$

be the orthogonal decomposition of $T_p M$ with respect to the linear holonomy group $\Psi(p)$. Now each σ_p is an element of the isotropy group at p and so normalises $\psi(p)$, that is $S^\sigma \psi(p) S^{\sigma^{-1}} = \Psi(p)$. Thus each S^σ preserves the above decomposition of $T_p M$ except possibly its order, also $S^\sigma(V_0) = V_0$. For each $i = 0, 1, \ldots, r$, define

$$\Sigma_i = \{\sigma \in \Sigma\colon S^\sigma(V_i) = V_i\} \quad \text{and} \quad \Sigma_{ip} = \{\sigma_p \in \Sigma_p\colon \sigma \in \Sigma_i\}.$$

[7] (M, g) is complete since it is homogeneous [4, Chapter IV, Theorem 6.2].

Clearly, $\Sigma_0 = \Sigma$ and each Σ_i is a closed subgroup of Σ; moreover, it follows from the previous remarks on S^σ that each Σ_i has finite index in Σ. Let $\bar{M}_i$ be the totally geodesic submanifold of M with $T_p\bar{M}_i = V_i$. Then $\Sigma_{ip}(\bar{M}_i) = \bar{M}_i$. Now G_M is connected and transitive on M and for each i we set

$$G_{\bar{M}_i} = \{\phi \in G_M\colon \phi(\bar{M}_i) \subset \bar{M}_i\}.$$

Then $G_{\bar{M}_i} = G_M \cap I_0(\bar{M}_i)$ and is transitive on $\bar{M}_i$. Let $x \in \bar{M}_i$ and choose $\phi \in G_{\bar{M}_i}$ such that $\phi(p) = x$. Then for $\sigma \in \Sigma_i$, $\sigma_x = \phi\sigma_p\phi^{-1}$ and so $\sigma_x(M_i) \subset (M_i)$. It follows easily that, for each i, the map

$$\mu_i\colon \bar{M}_i \times \Sigma_i \times \bar{M}_i \to \bar{M}_i; \quad (x, \sigma, y) \to \sigma_x(y),$$

is smooth; thus each $\bar{M}_i$ is a Riemannian Σ_i-space with respect to the induced metric. It remains only to prove (Σ6) for each $\bar{M}_i$ (cf. the remark following Definition 4.1).

Case (i) Suppose Σ is compact. Let $d\sigma$ (resp. $d\sigma_i$) denote bi-invariant Haar measures on Σ (resp. Σ_i). Then, for any $X \in T_pM$ and $\tau \in \Sigma$,

$$S^\tau \int_\Sigma S^\sigma(X)\,d\sigma = \int_\Sigma S^{\tau\sigma}(X)\,d\sigma = \int_\Sigma S^\sigma(X)\,d\sigma,$$

and, by (Σ6), $\int_\Sigma S^\sigma(X)\,d\sigma = 0$. Now suppose Σ_i has index k in Σ. Thus there exist $\tau_1, \tau_2, \ldots, \tau_k \in \Sigma$ such that

$$S^\Sigma V_i = S^{\tau_1}V_i \oplus S^{\tau_2}V_i \oplus \cdots \oplus S^{\tau_k}V_i. \tag{4.1}$$

Let $X \in V_i$; then as a special case of [3, Chapter X, Theorem 1.7] we have, for suitable measures,

$$0 = \int_\Sigma S^\sigma(X)\,d\sigma = \sum_{\alpha=1}^{k} S^{\tau_\alpha}\int_{\Sigma_i} S^{\sigma_i}(X)\,d\sigma_i.$$

Hence, by (4.1), $\int_{\Sigma_i} S^{\sigma_i}(X)\,d\sigma_i = 0$, and it follows that if $S^{\sigma_i}(X) = X$ for all $\sigma_i \in \Sigma_i$ then $X = 0$. By using $G_{\bar{M}_i}$ we see that the same property holds at each point of $\bar{M}_i$ and hence (Σ6) is satisfied on $\bar{M}_i$, as required.

Case (ii) Suppose Σ is non-compact. Then, by Corollary 4.6, (M, g) is a reduced Riemannian Σ^c-space. Moreover, $\Sigma_p \subset \Sigma_p^c$ and Σ_p^c is the closure of Σ_p. Now, by Case (i), the action of Σ_{ip}^c on V_i has no fixed vectors and it follows immediately that the same must be true for Σ_{ip}, which completes the proof.

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