ILLINOIS JOURNAL OF MATHEMATICS
Volume 29, Number 3, Fall 1985

THE CLOSURE IN LIPα NORMS OF RATIONAL MODULES WITH THREE GENERATORS

BY

J.J. CARMONA DOMÉNECH

1. Introduction

Let X be a compact subset of the complex plane $\mathbf{C}$. We will denote by $R_0(X)$ the algebra of rational functions with poles off X, by $R(X)$ the uniform closure of $R_0(X)$ in $C(X)$ and by $\operatorname{lip}(\alpha, X)$ and $\operatorname{Lip}(\alpha, X)$, $0 < \alpha < 1$, the spaces of Lipschitzian functions with the usual norm for which they are Banach spaces [12]. If $g_1, \dots, g_n$ are functions on X, then we will denote by $R_0(X, g_1, \dots, g_n)$ the rational module

$$R_0(X) + R_0(X)g_1 + \cdots + R_0(X)g_n.$$

When $g_1, \dots, g_n$ are differentiable in a neighborhood of X we will write $Z_i = \{x \in X/\bar{\partial} g_i(x) = 0\}$, $i = 1, \dots, n$.

In the case that $g_1(z) = \bar{z}, \dots, g_n(z) = \bar{z}^n$, the closures in different norms of $R_0(X, g_1, \dots, g_n)$ have been studied by O'Farrell [5], in connection with problems of rational approximation in $\operatorname{Lip}(\alpha)$ norms, and by Wang [14], [15], [16]. Trent and Wang [10] have proved that if X is a compact subset of $\mathbf{C}$ with empty interior, then $R_0(X, \bar{z})$ is uniformly dense in $C(X)$. It was shown later (by Trent and Wang [11] and, independently, by the author [2]), under the same hypothesis about X, that $R_0(X, g_1)$ is uniformly dense in $C(X)$ if and only if $R(Z_1) = C(Z_1)$. It was reasonable to try to extend this approximation result to norms stronger than the uniform norm, for example the $\operatorname{Lip}(\alpha)$ norms. However Wang [17] has recently established relations between the $\operatorname{Lip}(\alpha)$ approximation by functions of $R_0(X, \bar{z})$ and the L^p rational approximation, showing the failure of the analogous to the former result in $\operatorname{Lip}(\alpha)$ norm. At this point we pose the following question: If g is a differentiable function, what functions $g_1, \dots, g_n$ have to be added to $R_0(X, g)$ to make $R_0(X, g, g_1, \dots, g_n)$ dense in $\operatorname{lip}(\alpha, X)$?. Theorem 1, which is the main result of this work, gives a solution to above problem. Theorem 1 states that it is enough to consider $g_1 = F \circ g$, F being a holomorphic function. If we want to weaken the hypothesis to analyticity of F, assuming only differentiability (of class C^2),

Received March 29, 1983.

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we have to assume that $g = \bar{h}$ and $g_1 = F \circ h$, with h holomorphic. This is Theorem 2. Both theorems are stated in §2. In §3 we state some lemmas used in the proofs of Theorem 1 and 2, presented in §4 and §5 respectively. Section 6 is devoted to analyzing the uniform closure of $R_0(X, g_1, \dots, g_n)$. In Theorem 3 a general result is achieved, without assuming functional relations among $g_1, \dots, g_n$.

This work gives information on the problem proposed in III of [11], which consists of the study of the closures (in different norms) of $R_0(X, g_1, \dots, g_n)$.

Some of these results are taken from the author's doctoral dissertation [3].

Let us introduce some notations. If U is an open subset of $\mathbf{C}$ and $k \in \mathbf{N}$, we denote by $C^k(U)$ (resp. $D^k(U)$) the set of functions of class C^k (resp. of class C^k with compact support included in U). When $k = \infty$ we write $D(U)$. The Lebesgue two-dimensional measure in $\mathbf{C}$ is denoted by m. Throughout,

$$R^\alpha(X) \quad (\text{resp. } R^\alpha(X, g_1, \dots, g_n))$$

stands for the Lip(α) closure of $R_0(X)$ (resp. $R_0(X, g_1, \dots, g_n)$), $0 < \alpha < 1$.

2. Formulation of the theorems

In this section we are going to assume that X is a compact subset of $\mathbf{C}$ with empty interior.

THEOREM 1. *Let g be a function of class C^1 in a neighborhood of X and let F be a holomorphic function in an open subset V such that $g(X) \subset V$. Suppose $F'' \neq 0$ in each component of V intersecting $g(X)$. Let $Z = \{w \in X / \bar{\partial} g(w) = 0\}$. Then, for each α,*

$$R^\alpha(X, g, F \circ g) = \{ f \in \operatorname{lip}(\alpha, X) / f|_Z \in R^\alpha(Z) \}.$$

COROLLARY 1. *Under the hypothesis of Theorem 1, $R^\alpha(X, g, F \circ g) = \operatorname{lip}(\alpha, X)$ if and only if $R^\alpha(Z) = \operatorname{lip}(\alpha, Z)$.*

COROLLARY 2. *If $n \geqq 2$, then $R^\alpha(X, g, g^n) = \operatorname{lip}(\alpha, X)$ if and only if $R^\alpha(Z) = \operatorname{lip}(\alpha, Z)$.*

When $g(z) = \bar{z}$ and $n = 2$, Corollary 2 gives a generalization of a result of Wang [14]. The condition $F'' \neq 0$ is necessary. In fact, if we take $g(z) = \bar{z}$ and F linear, then there is a suitable compact subset X such that $R_0(X, g, F \circ g) = R_0(X, \bar{z})$ is not dense in lip(α, X) (see [17]).

THEOREM 2. *Let h be an analytic function in a neighborhood U of X, non-constant in each component of U and let F be a function of class C^2 in a* *neighborhood of $h(X)$. Let*

$$T = \{x \in X/\bar{\partial}^2 F(h(x)) = 0\}.$$

Then the following propositions hold.

(a) *If $R^\alpha(T) = \mathrm{lip}(\alpha, T)$, then $R^\alpha(X, \bar{h}, F \circ h) = \mathrm{lip}(\alpha, X)$.*
(b) *If $R^\alpha(X, \bar{h}, F \circ h) = \mathrm{lip}(\alpha, X)$, then $R^\alpha(T, \bar{h}) = \mathrm{lip}(\alpha, T)$.*

Let us point out that necessary and sufficient conditions for the approximation in $\mathrm{lip}(\alpha, Z)$ by functions of $R_0(Z)$ are well known [7].

An interesting application of the above theorems is obtained when $m(Z) = 0$ or $m(T) = 0$.

The proofs of Theorems 1 and 2 are by duality. If $\phi \in \mathrm{lip}(\alpha, X)^*$, then the restriction of ϕ to $D|_X$ is a distribution ϕ_1. We associate to ϕ_1 a distribution $\tilde{\phi}_1$ (defined by (2) with T replaced by ϕ_1). It can be shown that $\tilde{\phi}_1$ is a dm-absolutely continuous measure, the density function being continuous (Lemma 4). If, moreover, ϕ_1 is orthogonal to $R_0(X, g_1, g_2)$, then we deduce that $\tilde{\phi}_1 = 0$ on $\mathbf{C}$. From this it follows that ϕ_1 is supported in Z and is orthogonal to $\mathrm{lip}(\alpha, Z)$. The results will follow readily.

3. Preliminary lemmas

We will consider a general scheme which applies to the theorems above. We assume that $g_1, g_2 \in C^1(U)$, U an open neighborhood of X. Also assume:

(1) There exists $u \in C^1(U)$ such that $\bar{\partial} g_2 = u \cdot \bar{\partial} g_1$ in U.

In this case we define the kernel:

$$k(z,w) = (g_2(z) - g_2(w) - u(w) \cdot (g_1(z) - g_1(w)))/(z - w), \quad z, w \in U,$$
$$k(z,z) = 0, \qquad z \in U.$$

The function k is locally bounded in $U \times U$. We write

$$\tilde{\psi}(z) = \int k(z,w)\psi(w)\,dm(w), \qquad \psi \in D^1(U).$$

This function is called the transform of ψ relative to g_1, g_2. It is easy to see that $\tilde{\psi} \in C^1(U)$. If T is a distribution of order 1 with compact support in U, then we define the transform of T, relative to g_1, g_2, by

$$\tilde{T}(\psi) = T(\tilde{\psi}), \qquad \psi \in D^1(U). \tag{2}$$

Let us recall the definition of the Cauchy transform of a compactly supported distribution in $\mathbf{C}$ [5]:

$$\hat{T}(\psi) = -T(\hat{\psi}) \quad \text{if } \psi \in D^1(\mathbf{C}),$$

where

$$\hat{\psi}(w) = \int \frac{1}{z-w}\psi(z)\, dm(z).$$

We will also need the definition of the transform of ψ, relative to g_1 [11], [3];

$$\check{\psi}(w) = \int \frac{g_1(z) - g_1(w)}{z-w}\psi(z)\, dm(z), \qquad \psi \in D^1(U),$$

and the corresponding one for compactly supporting distributions,

$$\check{T}(\psi) = T(\check{\psi}), \qquad \psi \in D^1(U).$$

LEMMA 1. *Let g_1 and g_2 satisfy* (1). *Let T be a distribution of order* 1 *with compact support in U, such that $\check{T}$ and $\hat{T}$ are measures. Then*
(a) $\bar{\partial}\tilde{T} = \bar{\partial}u \cdot \check{T}$ *and*
(b) $\bar{\partial}\check{T} = -\bar{\partial}g_1 \cdot \hat{T}$.

Proof. We will only prove (a), the proof of (b) being similar. Since

$$\bar{\partial}\tilde{T}(\psi) = T(-(\bar{\partial}\psi)^{\tilde{}}) \quad \text{and} \quad \bar{\partial}u \cdot \check{T}(\psi) = T((\bar{\partial}u \cdot \psi)^{\check{}}),$$

it is sufficient to show that

$$(3) \qquad (\bar{\partial}\psi)^{\tilde{}} = -(\bar{\partial}u \cdot \psi)^{\check{}} \quad \text{in } U, \quad \text{for } \psi \in D^2(U).$$

Let $z_0 \in U$. We consider an open set G with $G \subset U$, ∂G piecewise of class C^1, $\operatorname{supp}\psi \subset G$ and $z_0 \in G$. Let us choose an $\varepsilon > 0$ in such a way that the closed disc $D_\varepsilon = D(z_0, \varepsilon)$ is included in G. If $G_\varepsilon = G - D_\varepsilon$, then G_ε is an open bounded set and its boundary is piecewise of class C^1. We suppose this boundary endowed with the orientation induced by the usual one in G_ε. We consider the following differential form of class C^1 in a neighborhood of $\overline{G}_\varepsilon$:

$$k(z_0, z)\psi(z)\, dz.$$

Applying Stokes' Theorem and using (1), we get

$$(4) \quad \int_{G_\varepsilon} k(z_0, z)\,\bar{\partial}\psi(z)\, d\bar{z} \wedge dz - \int_{G_\varepsilon} \frac{g_1(z_0) - g_1(z)}{z_0 - z}\,\bar{\partial}u(z)\psi(z)\, d\bar{z} \wedge dz$$
$$= -\int_{C(z_0,\varepsilon)} k(z_0, z)\psi(z)\, dz.$$

Note that

$$|k(z_0, z)\,\bar\partial\psi(z)| = O(|z_0 - z|^{-1}),$$

$$\left|\frac{g_1(z_0) - g_1(z)}{z_0 - z}\,\bar\partial u(z)\psi(z)\right| = O(|z_0 - z|^{-1}), \qquad z \in \operatorname{supp}\psi,$$

and

$$k(z_0, z)\psi(z) = O(1), \qquad z \in C(z_0, \varepsilon).$$

Letting $\varepsilon \to 0$ in (4), we obtain (3).

From now on we will restrict ourselves to two cases:

(i) $g_1 = g$, $g_2 = F \circ g$, with F and g satisfying the hypothesis of Theorem 1. In this situation $\bar\partial g_2 = (F' \circ g) \cdot \bar\partial g$; so (1) holds with $u = F' \circ g$.

(ii) $g_1 = \bar h$, $g_2 = F \circ h$, with h and F as in Theorem 2. Here

$$\bar\partial g_2 = (\bar\partial F \circ h) \cdot \bar\partial g_1,$$

i.e., $u = \bar\partial F \circ h$.

The respective kernels for $z \neq w$, $z, w \in U$, are:

$$k_1(z,w) = (F(g(z)) - F(g(w)) - F'(g(w))(g(z) - g(w)))/(z - w),$$
$$k_1(z,z) = 0,$$
$$k_2(z,w) = (F(h(z)) - F(h(w)) - \bar\partial F(h(w))(\bar h(z) - \bar h(w)))/(z - w),$$
$$k_2(z,z) = \partial F(h(z)) \cdot h'(z).$$

Let us study the properties of the above kernels.

LEMMA 2. (a) *The functions k_1, k_2 are continuous on $U \times U$.*

(b) *For each compact $K \subset U$, the functions $k_i(\cdot, w)$, $w \in K$, are locally uniformly Lipschitzian*; *i.e., for each compact L,*

$$|k_i(x,w) - k_i(y,w)| \leqq C(g, K, L)|x - y|,$$
$$x, y \in L, \quad w \in K, \quad i = 1,2.$$

Proof. (a) It is sufficient to prove continuity at the points

$$(z_0, z_0) \in U \times U.$$

We will denote by g both the function g of (i) and the function h of (ii). Let V be an open neighborhood of $g(X)$ on which F is defined and such that $g(U) \subset V$.

Let $z_0 \in U$ and let $(z_n, w_n)_n$ be a sequence converging to (z_0, z_0). By applying Taylor's formula to F, we get

$$|F(z) - F(z') - \bar{\partial}F(z')(\bar{z} - \bar{z}') - \partial F(z')(z - z')| = O(|z - z'|^2),$$

for every z and z' in a closed disc centered at z_0 and included in V. By the continuity of g,

$$\begin{aligned}|F(g(z_n)) - F(g(w_n)) - \bar{\partial}F(g(w_n))(\bar{g}(z_n) - \bar{g}(w_n)) \\ - \partial F(g(w_n))(g(z_n) - g(w_n))| = O\left(|g(z_n) - g(w_n)|^2\right),\end{aligned}$$

for n large enough.

In case (i) we have $\bar{\partial}F = 0$ and

$$|k_1(z_n, w_n)| \leqq M|g(z_n) - g(w_n)|^2 \big/ |z_n - w_n| = O(|z_n - w_n|).$$

On the other hand, in case (ii) we obtain

$$(5) \left|k_2(z_n, w_n) - \partial F(g(w_n))(g(z_n) - g(w_n))(z_n - w_n)^{-1}\right| = O(|z_n - w_n|).$$

Since g is holomorphic, the function $(g(z) - g(w))(z - w)^{-1}$ is continuous on $U \times U$. Thus by taking limits in (5) we obtain (a).

(b) Let K and L be two compact subsets of U. We will show that

$$(6) \qquad |k_i(x,w) - k_i(y,w)| \leqq C(g, K, L)|x - y|, \qquad w \in K, \quad x, y \in L, \quad i = 1,2,$$

for x and y sufficiently close.

Let $M = K \cup L$ and let x, y, w be three fixed points in M with

$$|x - y| \leqq \tfrac{1}{2}d(M, \mathbf{C} - U).$$

By (a) we can assume in (6) that $w \notin [x, y]$. In this case, $k_i(\cdot, w)$ is of class C^1 in a neighborhood of $[x, y]$. If we estimate $\bar{\partial}k_i(\cdot, w)$ and $\partial k_i(\cdot, w)$ uniformly, then (6) follows. Let us calculate these derivatives.

Case (i). By the uniform continuity of g on M and by developing F in a power series, we conclude that there exists δ, $0 < \delta < d(M, \mathbf{C} - U)$, such that

$$F(g(z)) - F(g(w)) - F'(g(w))(g(z) - g(w)) = (g(z) - g(w))^2 H(z,w),$$

for $|z - w| < \delta$, $z, w \in M$, where H is a differentiable function defined on a neighborhood of $M \times M$. A straightforward calculation shows that

$$\bar{\partial} k_1(s,w) = 2\frac{g(s)-g(w)}{s-w}\bar{\partial} g(s) H(s,w) + \frac{(g(s)-g(w))^2}{s-w}\bar{\partial} H(s,w),$$

$$\partial k_1(s,w) = 2\frac{(g(s)-g(w))\,\bar{\partial} g(s) H(s,w) + (g(s)-g(w))^2\,\partial H(s,w)}{s-w} - \frac{(g(s)-g(w))^2 H(s,w)}{(s-w)^2}, \qquad s \in [x,y].$$

If $|s-w| < \frac{1}{2}\delta$ we can apply the mean value theorem, and if $|s-w| \geqq \frac{1}{2}\delta$ we can estimate directly to obtain

$$\bar{\partial} k_1(s,w) = O(1), \qquad \partial k_1(s,w) = O(1).$$

Case (ii). We have

$$\bar{\partial} k_2(s,w) = (\bar{\partial} F(h(s)) - \bar{\partial} F(h(w)))\overline{h'(s)}/(s-w) = O(1),$$

$$\tag{7} \partial k_2(s,w) = \frac{-F(h(s)) + F(h(w)) + \bar{\partial} F(h(w))(\bar{h}(s) - \bar{h}(w)) + \partial F(h(s))(h'(s)(s-w))}{(s-w)^2},$$

for $s \in [x,y]$.

The absolute value of the numerator of (7) will be less than

$$\Big|-F(h(s)) + F(h(w)) + \bar{\partial} F(h(w))\big(\overline{h(s)} - \overline{h(w)}\big) + \partial F(h(w))(h(s)-h(w))\Big| + |(\partial F(h(w)) - \partial F(h(s)))(h(s)-h(w))| + |\partial F(h(s))(h'(s)(s-w) - h(s) - h(w))||$$

It is sufficient to estimate these terms for s and w sufficiently close. They turn out to be $O(|s-w|^2)$. The first is estimated using Taylor's formula, the second by the mean value theorem and the third recalling that h is a holomorphic function. As a consequence we obtain $\partial k_2(s,w) = O(1)$ for $s \in [x,y]$ and $w \in M$.

Let us see a characterization of the dual space of $\mathrm{lip}(\alpha, X)$, $0 < \alpha < 1$. The original idea belongs to de Leew [4].

LEMMA 3. *Let $\phi \in \mathrm{lip}(\alpha, X)^*$. Then there are two regular Borel measures μ and ν, on X and $X \times X$ respectively, such that for any $f \in \mathrm{lip}(\alpha, X)$ we have*

$$\tag{8} \phi(f) = \int_X f\,d\mu + \int_{X\times X} \frac{f(x)-f(y)}{|x-y|^\alpha}\,d\nu(x,y).$$

Proof. Let us consider the space $E = X \cup (X \times X)$ with the sum topology. We define the linear mapping

$$T\colon \operatorname{lip}(\alpha, X) \to C(E), \qquad T(f) = \tilde{f}$$

where $\tilde{f}(x) = f(x)$ if $x \in X$,

$$\tilde{f}(x, y) = \frac{f(x) - f(y)}{|x - y|^\alpha} \quad \text{if } x \neq y,$$
$$\tilde{f}(x, x) = 0, \quad x \in X.$$

It is clear that T is an isometry. The linear form ϕ can be defined on $T(\operatorname{lip}(\alpha, X))$ and by the Hahn-Banach theorem we can extend it to $C(E)$. From the Riesz representation theorem we infer that there is a regular Borel measure ω on E such that $\phi(f) = \int_E \tilde{f}\, d\omega$. Defining $\mu = \omega|_X$ and $\nu = \omega|_{X\times X}$ we obtain (8).

If $\phi \in \operatorname{lip}(\alpha, X)^*$ we denote by ϕ_1 the distribution of order 1, with compact support, defined by

$$\phi_1(\psi) = \phi(\psi|_X), \qquad \psi \in C^1(\mathbf{C}).$$

LEMMA 4. *Let $\phi \in \operatorname{lip}(\alpha, X)^*$. Then the transform $\tilde{\phi}_1$ of ϕ_1 (both with respect to k_1 and k_2) is a dm-absolutely continuous measure, with a continuous density function.*

Proof. Let μ and ν be the measures of Lemma 3 satisfying (8). If $\psi \in DU^1(\mathrm{U})$, then

$$(*) \qquad \tilde{\phi}_1(\psi) = \int_X \tilde{\psi}(z)\, d\mu(z) + \int_{X\times X} \frac{\tilde{\psi}(x) - \tilde{\psi}(y)}{|x - y|^\alpha}\, d\nu(x, y)$$
$$= \int_X \left(\int k(z,w)\psi(w)\, dm(w) \right) d\mu(z)$$
$$+ \int_{X\times X} \left(\int \frac{k(x,w) - k(y,w)}{|x - y|^\alpha} \psi(w)\, dm(w) \right) d\nu(x, y)$$

According to Lemma 2 we can apply Fubini's Theorem to $(*)$ to obtain

$$\int \psi(w)\big(\mu^*(w) + \nu_\alpha^*(w)\big)\, dm(w),$$

with

$$\mu^*(w) = \int k(z,w)\, d\mu(z)$$

and

$$\nu_\alpha^*(w) = \int \frac{k(x,w) - k(y,w)}{|x-y|^\alpha}\, d\nu\,(x,y).$$

Now we are going to show that μ^* and ν_α^* are continuous functions. Let $w_0 \in U$ and let (w_n) be a sequence converging to w_0. By Lemma 2,

$$k(z,w_n) \to k(z,w_0), \qquad z \in U,$$

and

$$\frac{k(x,w_n) - k(y,w_n)}{|x-y|^\alpha} \to \frac{k(x,z_0) - k(y,z_0)}{|x-y|^\alpha}, \qquad x, y \in X,\ x \neq y.$$

If $x = y$, both expressions are equal to 0. Moreover $k(z,w_n) = O(1)$ and

$$\frac{|k(x,w_n) - k(y,w_n)|}{|x-y|^\alpha} = O(|x-y|^{1-\alpha}), \qquad x, y \in X,\ n \in N.$$

By the Lebesgue convergence theorem, $\mu^*(w_n) \to \mu^*(w_0)$ and $\nu_\alpha^*(w_n) \to \nu_\alpha^*(w_0)$.

4. Proof of Theorem 1

Let us write $B = \{f \in \mathrm{lip}(\alpha, X)/f|_Z \in R^\alpha(Z)\}$. As $\bar\partial g = \bar\partial(F \circ g) = 0$ on Z we have, as a consequence of a theorem of O'Farrell [8], that $R^\alpha(X, g, F \circ g) \subset B$. Let ϕ be a continuous linear functional on $\mathrm{lip}(\alpha, X)$ which is orthogonal to $R_0(X, g, F \circ g)$. We must prove that ϕ is orthogonal to B. We consider the distribution ϕ_1. By Lemma 4, it follows that

$$\tilde\phi_1(\psi) = \int \psi(\mu^* + \nu_\alpha^*)\, dm, \qquad \psi \in C^1(U).$$

On the other hand, the representation of Lemma 3 gives

$$\phi_1(k_1(\cdot, w)) = (\mu^* + \nu_\alpha^*)(w), \qquad w \in U.$$

If $w \notin X$, then $k_1(\cdot, w) \in R_0(X, g, F \circ g)$. Therefore $\mu^* + \nu_\alpha^* = 0$ on $U - X$. By Lemma 4, $\mu^* + \nu_\alpha^* = 0$ on U, so $\check\phi_1 = 0$ on U. Since $\check\phi_1$ satisfies

$$|\check\phi_1(\psi)| \leqq c\|\psi\|_\infty, \qquad \psi \in D^1(U), d(\operatorname{supp}\psi, \mathbf{C} - U) \geqq \delta,$$

$\check\phi_1$ is a measure with support in X. The same is true of $\hat\phi_1$ [5]. The hypothesis of Lemma 1 is satisfied, with $u = F' \circ g$, and so

$$\bar{\partial}\tilde{\phi}_1 = \bar{\partial}g \cdot (F'' \circ g) \cdot \check{\phi}_1, \qquad \bar{\partial}\check{\phi}_1 = -\bar{\partial}g \cdot \hat{\phi}_1. \tag{9}$$

Let $Z_1 = Z \cup \{x \in X/F''(g(x)) = 0\}$. By assumption F'' has finitely many zeros in $g(X)$. Applying the implicit function theorem we obtain that $Z_1 - Z$ is contained in a finite union of subsets of C^1-curves, so that $m(Z_1 - Z) = 0$. From (9) and the fact that $\phi_1 \perp R_0(X)$ it follows that $\hat{\phi}_1 = 0$ on $\mathbf{C} - Z_1$. Since $\hat{\phi}_1$ is a dm-absolutely continuous measure, we have $\hat{\phi}_1 = 0$ on $\mathbf{C} - Z$ and so ϕ_1 is orthogonal to $R_0(Z)$.

Now let us show that

$$|\phi_1(h)| \leqq c\|h\|_{\mathrm{lip}(\alpha, Z)}, \qquad h \in C^1(\mathbf{C}), \tag{10}$$

where $c > 0$ does not depend on h.

We need the following property:

(11) If $f \in \mathrm{lip}(\alpha, X)$ and $f = 0$ on Z, then $\phi(f) = 0$.

This means that ϕ depends only on the values of f on Z. When $f \in C^1(\mathbf{C})$ and $f = 0$ on a neighborhood of Z, (11) holds because $\operatorname{supp} \phi_1 \subset Z$. The general case is proved as follows. We suppose that f has been extended to a function of $\mathrm{lip}(\alpha, X)$ [13, p. 174]. Let K be a compact neighborhood of X. By a result of Sherbert [12] there is a sequence (f_n) such that $f_n \in \mathrm{lip}(\alpha, K)$, $f_n = 0$ on a neighborhood of Z and $f_n \to f$ in $\mathrm{lip}(\alpha, K)$. Considering convolutions of the functions f_n with an approximate identity we can also assume that $f_n \in C^1(\mathbf{C})$. Therefore

$$\phi(f) = \lim \phi(f_n|_X) = 0,$$

and (11) holds.

We denote by E: $\mathrm{lip}(\alpha, Z) \to \mathrm{lip}(\alpha, X)$ the linear continuous extension operator defined in [13, p. 174]. If $h \in C^1(\mathbf{C})$, then

$$\phi_1(h) = \phi(h|_X) = \phi(E(h|_Z)),$$

because the functions $h|_X$ and $E(h|Z)$ are equal on Z. It follows that

$$|\phi_1(h)| \leqq \|\phi\| \, \|E\| \, \|h|_Z\|_{\mathrm{lip}(\alpha, Z)} = C\|h|_Z\|_{\mathrm{lip}(\alpha, Z)},$$

and so (10) is proved.

Next we claim that ϕ is orthogonal to B. To see this, let $f \in B$ and $h_n \in R_0(Z)$ such that $f|_Z = \lim h_n|_Z$. Then by (10) and (11) it follows that

$$\phi(f) = \phi(E(f|_Z)) = \lim \phi(E(h_n|_Z)) = 0.$$

5. Proof of Theorem 2

(a) Let $\phi \in \mathrm{lip}(\alpha, X)^*$. In this case $u = \bar{\partial}(F \circ h)$. Lemma 1 shows that

$$\bar{\partial}\tilde{\phi}_1 = \left(\bar{\partial}^2 F \circ h\right) \cdot \overline{h'} \cdot \check{\phi}_1 \quad \text{and} \quad \bar{\partial}\check{\phi}_1 = -\overline{h'} \cdot \hat{\phi}_1.$$

Let $T_1 = \{x \in X / \bar{\partial}^2 F(h(x))\overline{h'(x)} = 0\}$. By the hypothesis, we have

$$R^\alpha(T_1) = \mathrm{lip}(\alpha, T_1).$$

The procedure of §5 can be repeated now to show that $\hat{\phi}_1$ is supported on T_1 and is orthogonal to $R^\alpha(T_1)$. Then (a) follows.

(b) We assume that $R^\alpha(X, \bar{h}, F \circ h) = \mathrm{lip}(\alpha, X)$. It is enough to show that $F \circ h \in R^\alpha(T, \bar{h})$. First we remark that the integral formula of Proposition 1 of [2] is also true with the following weaker hypothesis: There exists $u \in C^1$ and $\bar{\partial} f = u \cdot \bar{\partial} g$.

In our case $\bar{\partial}(F \circ h) = (\bar{\partial} F \circ h) \cdot \bar{\partial} g$ on U. Let ϕ be an element of $\mathrm{lip}(\alpha, T)^*$ which is orthogonal to $R^\alpha(T, \bar{h})$. An application of the aforesaid formula (for a convenient subset G) gives

$$\phi_1(F \circ h) = \phi_1\left(\frac{1}{2\pi i} \int_{\partial G} \frac{F(h(z))}{z - w}\, dz\right) - \phi_1\left(\frac{1}{2\pi i} \int_{\partial G} F(h(z)) \frac{\bar{h}(z) - \bar{h}(w)}{z - w}\, dz\right) + \check{\phi}_1\left(1/\pi\left(\bar{\partial}^2 F \circ h\right)\overline{h'}\right).$$

As in [5] (see [3]) it can be shown that $\check{\phi}_1$ is a dm-absolutely continuous measure with support on T. Since

$$\int_{\partial G} \frac{F(h(z))}{z - w}\, dz \in R_0(T)$$

and

$$\int_{\partial G} \bar{\partial} F(h(z)) \frac{\bar{h}(z) - \bar{h}(w)}{z - w}\, dz \in R_0(T, \bar{h}),$$

it follows that $\phi_1(F \circ h) = 0$. Thus (b) holds.

6. Further results

When dealing with the uniform approximation by functions of

$$R_0(X, g_1, \ldots, g_n)$$

a more precise information can be achieved. Let $R(X, g_1, \ldots, g_n)$ denote the uniform closure of $R_0(X, g_1, \dots, g_n)$. To simplify notation we are going to state the following theorem only for $n = 2$.

THEOREM 3. *Let X be a compact subset of* $\mathbf{C}$, *and g_1, g_2 two functions of class C^3 in a neighborhood of X. Then the following assertions are equivalent*:

(a) $R(X, g_1, g_2) = \{f \in C(X)/f|_{Z_1 \cap Z_2} \in R(Z_1 \cap Z_2)\}$.

(b) $\mathring{X} \subset Z_1 \cap Z_2$.

LEMMA 5. *Let X be a compact subset of* $\mathbf{C}$ *and g a function of class C^2 in a neighborhood of X. Given a measure μ on X, the following conditions are equivalent*:

(i) $\mu \in R(X, g)^{\perp}$.

(ii) *μ is concentrated on $\mathring{X} \cup Z$ and* $\mu \in R(\overline{\mathring{X} \cup Z}, g)^{\perp}$.

We omit the proof of Lemma 5. It can be obtained by using the ideas of the proof of Theorem of [2] (see [3]).

Proof of Theorem 3. (b) $\Rightarrow$ (a). Let

$$B = \left\{f \in C(X)/f|_{Z_1 \cap Z_2} \in R(Z_1 \cap Z_2)\right\}.$$

Since $\bar{\partial} g_i = 0$ on $Z_1 \cap Z_2$, then $R(X, g_1, g_2) \subset B$. Let $\mu \in R(X, g_1, g_2)^{\perp}$ and $f \in B$. We will show that $\int f\, d\mu = 0$. By Lemma 5, μ is supported in $\overline{\mathring{X} \cup Z_1}$ and $\overline{\mathring{X} \cup Z_2}$. Thus (b) implies that μ is supported in $Z_1 \cap Z_2$ and also that $\mu \in R(Z_1 \cap Z_2)^{\perp}$. Therefore $\int f\, d\mu = 0$.

(a) $\Rightarrow$ (b). We suppose that $\mathring{X} \not\subset Z_1 \cap Z_2$. There is an open disc D such that $D \subset \bar{D} \subset \mathring{X}$ and, let us say, that $\bar{D} \cap Z_1 = \varnothing$. If $f \in R(X, g_1, g_2)$, then f satisfies (in the distribution sense) the following equation on D:

$$\bar{\partial}\left(\frac{\bar{\partial} f}{\bar{\partial} g_1}\right) = \bar{\partial}\left(\frac{\bar{\partial} g_2}{\bar{\partial} g_1}\right) \cdot k, \quad \text{with } \bar{\partial} k = 0 \text{ on } D.$$

Let

$$A = \left\{z \in D/\bar{\partial}\left(\frac{\bar{\partial} g_2}{\bar{\partial} g_1}\right)(z) = 0\right\}.$$

We distinguish two cases.

(i) $\mathring{A} \neq \varnothing$. For every function $f \in R(X, g_1, g_2)$, we have

$$\bar{\partial}\left(\frac{\bar{\partial} f}{\bar{\partial} g_1}\right) = 0 \quad \text{on } \mathring{A}. \tag{12}$$

By choosing a function that vanishes in a neighborhood of $Z_1 \cap Z_2$ and does not satisfy (12), we infer that $B \not\subset R(X, g_1, g_2)$.

(ii) $\mathring{A} = \varnothing$. There is an open disc $D_1 \subset D$ such that

$$\bar{\partial}\left(\frac{1}{\psi}\bar{\partial}\left(\frac{\bar{\partial} f}{\bar{\partial} g_1}\right)\right) = 0 \quad \text{on } D_1, \tag{13}$$

where

$$\psi = \bar{\partial}\left(\frac{\bar{\partial} g_2}{\bar{\partial} g_1}\right) \quad \text{and} \quad f \in R(X, g_1, g_2).$$

We can always find a function that vanishes in a neighborhood of $Z_1 \cap Z_2$ and does not satisfy (13); thus $B \not\subset R(X, g_1, g_2)$.

Remark. We observe that Lemma 5 is true for g only of class C^1 on X. Therefore (b) $\Rightarrow$ (a) of Theorem 3 holds when g_1, g_2 are C^1 on X. The stronger differentiability of g_1 and g_2 was only required to prove (a) $\Rightarrow$ (b) in Theorem 3.

Theorem 3 provides a generalization of a theorem by Trent and Wang [17]. Next we present an application of Theorem 3 to a classical problem. Several authors, for example Wermer [18] and Preeskenis [9], have studied the following question: If D is the closed unit disc on $\mathbf{C}$, for which functions $g \in C(D)$ is the closed algebra generated by z and g equal to $C(D)$? When we consider this problem for another compact X of $\mathbf{C}$, it is necessary to replace z by $R(X)$, and Theorem 3 provides a solution in the case $\mathring{X} = \varnothing$.

COROLLARY 4. *Let X be a compact subset of $\mathbf{C}$ and g a function of class C^1 in a neighborhood of X. We suppose that $\mathring{X} \subset Z$. Then the closed algebra generated by $R_0(X)$ and g is $R(X, g) = \{f \in C(X)/f|_Z \in R(Z)\}$.*

Notice the similarity with the theorem of [18, p. 9].

Finally, we point out two problems whose solution we do not know.

Problem 1. Characterize the compact subsets X of $\mathbf{C}$ such that

$$R^{\alpha}(X, \bar{z}) = \operatorname{lip}(\alpha, X).$$

Problem 2. Characterize the compact subsets X of $\mathbf{C}$ such that $R_0(X, \bar{z})$ is dense in $D^1(X)$ in the Lip(1) norm (see [5] for the definition of $D^1(X)$).

Notice that it is well known when $R_0(X)$ is dense in $D^1(X)$ [6].

Acknowledgment. The author would like to thank Prof. Joan Verdera for the suggestion of generalizing Corollary 2 to Corollary 1, as well as for other valuable remarks.

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UNIVERSITAT AUTÒNOMA DE BARCELONA
BARCELONA, SPAIN