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THE HOMOTOPY INVARIANCE OF THE KURANISHI SPACE

BY

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In memory of Kuo-Tsai Chen

This paper is our second paper devoted to applying the ideas of rational homotopy theory as developed by Chen, Quillen and Sullivan to deformation problems from analytic geometry. Our first paper [GM1] studied the deformations of flat connections and holomorphic structures on principal bundles (for the most part over compact Kähler manifolds). In this paper we study the deformation spaces of complex structures on compact manifolds. The "controlling differential graded Lie algebra" (L, d) of Deligne, [GM1, p. 48], is now the Kodaira-Spencer algebra

$$(L, d) = \left(\bigoplus_{q=0}^{\infty} \mathscr{A}^{0,q}(M, T^{1,0}), \bar{\partial} \right)$$

where $\mathscr{A}^{0,q}(M, T^{1,0})$ denotes the space of C^{∞} exterior differential forms on M of type $(0, q)$ with values in the holomorphic tangent bundle.

In [Ku1] and [Ku2], Kuranishi constructed the versal deformation of a compact complex manifold M (see the appendix of this paper for definitions and terminology). The parameter (base) space of this deformation is an analytic germ in $H^1(L)$ with base point $0 \in H^1(L)$ which we will denote $(\mathscr{K}, 0)$ or $\mathscr{K}$ and will call the Kuranishi space. Although over twenty five years have passed since [Ku1] appeared many basic questions concerning $\mathscr{K}$ remain unanswered. In this paper we prove that $\mathscr{K}$ is a "homotopy invariant" of L and use this principle to compute $\mathscr{K}$ in the examples detailed below. We now explain precisely what we mean by the "homotopy invariance" of $\mathscr{K}$.

Let (L, d) be a differential graded Lie algebra over a field $\mathbf{k}$ (either $\mathbf{C}$ or $\mathbf{R}$ in what follows). Choose a complement $C^1(L)$ to the 1-coboundaries $B^1(L) \subset L^1$. We define a functor $A \to Y_L(A)$ on the category of Artin local $\mathbf{k}$-algebras by

$$Y_L(A) = \{\eta \in C^1(L) \otimes \mathfrak{m} \colon d\eta + \tfrac{1}{2}[\eta, \eta] = 0\}$$

Here $\mathfrak{m}$ is the maximal ideal of the Artin local $\mathbf{k}$-algebra A. It is proved in §1 that the functor Y_L satisfies the hypotheses of Theorem 2.11 of [Sc] and is

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consequently pro-representable by a complete local $\mathbf{k}$-algebra R_L (we will see later as a consequence of our main theorem that the isomorphism class of R_L does not depend on the complement $C^1(L)$). In §2 we further assume that L has a suitable topological structure i.e. the graded pieces are Banach, the complement $C^1(L)$ is closed etc. so that we can apply the method of Kuranishi [Ku1], [Ku2] to construct an analytic germ $\mathscr{K}_L$. In §3 we prove that the analytic local ring $\mathscr{O}_{\mathscr{K}_L}$ also represents Y_L. Hence the completion of $\mathscr{O}_{\mathscr{K}_L}$ is isomorphic to R_L. It is this isomorphism which is the link between the homotopy theory of differential graded Lie algebras and analytic geometry. We call a differential graded Lie algebra with the above topological structure an *analytic* differential graded Lie algebra and we call $\mathscr{K}_L$ the Kuranishi space of L.

In case L has zero differential the space $\mathscr{K}_L$ is the quadratic cone:

$$\mathscr{K}_L = \{\eta \in L^1 \colon [\eta, \eta] = 0\}.$$

The germ $\mathscr{K}_L$ is always Banach analytically isomorphic to the germ at 0 of

$$Y = \{\eta \in L^1 \colon d\eta + \tfrac{1}{2}[\eta, \eta] = 0 \text{ and } \eta \text{ lies in a fixed closed complement to } B^1(L) \text{ in } L^1\}.$$

Thus in case L^1 is finite dimensional we may replace $\mathscr{K}_L$ by Y.

In §4 we state and prove our main theorem, Theorem 4.1. Before stating it we recall that two differential graded Lie algebras L and $\bar{L}$ are quasi-isomorphic if there is a chain of homomorphisms

$$L = L_0 \to L_1 \leftarrow L_2 \to \cdots \to L_n = \bar{L}$$

all of which induce isomorphisms of cohomology. Our main theorem is then the following.

THEOREM. *If L_1 and L_2 are quasi-isomorphic then R_{L_1} and R_{L_2} are isomorphic.*

We will apply this theorem in §5 and §6 as follows. Let L be the Kodaira-Spencer algebra of a compact complex manifold M and $\bar{L}$ a differential graded Lie algebra with $\bar{L}^1$ finite dimensional. Suppose we have a quasi-isomorphism of differential graded Lie algebras

$$L = L_0 \to L_1 \leftarrow L_2 \to \cdots \to L_n = \bar{L}.$$

Choosing complements $C^1(L_i)$ for $0 \le i \le n$ and applying the main theorem repeatedly we obtain an isomorphism from R_L to $R_{\bar{L}}$ and consequently by [A] an isomorphism of analytic germs from $\mathscr{K}_L$ to $\mathscr{K}_{\bar{L}}$. The latter germ is described by explicit algebraic equations in a finite dimensional vector space. We emphasize the fact that the intermediate L_i's need not have an analytic structure and the arrows do not have to preserve the splittings.

If L is quasi-isomorphic to a differential graded Lie algebra with zero differential (one says L is formal) then it follows that

$$\mathscr{K}_L = \{\eta \in H^1(L) : [\eta, \eta] = 0\}.$$

Thus if the Kodaira-Spencer algebra is formal and the cup-product on $H^1(M, T^{1,0})$ is zero then the Kuranishi space of M is all of $H^1(M, T^{1,0})$ even if $H^2(M, T^{1,0}) \neq \{0\}$. We use this observation in §5 to explain Bogomolov's Theorem that if M admits a nowhere-zero top degree holomorphic form then the Kuranishi space of M is all of $H^1(M, T^{1,0})$. The key step in verifying the formality of the Kodaira-Spencer algebra in this case is a lemma of Tian and Todorov. Our proof merely places their proofs in a conceptual framework. In the light of this example it would be interesting to find other sufficient conditions implying the formality of the Kodaira-Spencer algebra analogous to the results of [Si] for the twisted de Rham algebra.

In §6 we apply Theorem 4.1 to determine the Kuranishi space of a compact complex parallelizable nilmanifold. Let $M = \Gamma \backslash N$, N a nilpotent complex Lie group with Lie algebra $\mathfrak{n}$ defined over $\mathbf{R}$ and Γ a cocompact lattice. Let L be the Kodaira-Spencer algebra and $\bar{L} \subset L$ the image of the left N-invariants. The inclusion $\bar{L} \to L$ is a quasi-isomorphism so $\mathscr{K}_L = \mathscr{K}_{\bar{L}}$. It is easy to see that

$$\mathscr{K}_{\bar{L}} = \mathrm{End}_{\mathrm{alg}}(\mathfrak{n}),$$

the affine variety of Lie algebra endomorphisms of $\mathfrak{n}$. If we describe $\mathfrak{n}$ by generators and relations we can produce a very large number of germs that are Kuranishi spaces of complex manifolds. For example let $\mathfrak{n}$ be the free Lie algebra on two generators X and Y subject to the relations

(i) all $(n + 1)$-fold commutators $= 0$

(ii) $ad^{n-1}X(Y) = ad^{n-1}Y(X)$.

Then $\mathscr{K}_L = \{(X', Y') \in \mathfrak{n} \colon X', Y' \text{ satisfy (ii)}\}$ so $\mathscr{K}_L$ is a homogeneous cone of degree n.

We conclude the paper by showing how our ideas can be used to show that certain properties of a complex manifold are preserved by small deformation. For the examples just discussed above we show that any small deformation of the locally bi-invariant complex structure is isomorphic to a locally left N-invariant structure. This follows because the algebra $\bar{L}$ above is the controlling differential graded Lie algebra for locally left N-invariant complex structures on M.

It is a pleasure to thank Pierre Deligne whose ideas led to this paper. In a letter to us he outlined the construction of the functors

$$A \mapsto \mathscr{C}(L; A)$$

of [GM1] and the key Theorem 2.4 of [GM1] that a quasi-isomorphism f: $L_1 \to L_2$ induces an equivalence of groupoids $\mathscr{C}(L_1; A) \to \mathscr{C}(L_2; A)$ which is natural in A. (A very similar theorem was also proved by Schlessinger-Stasheff [SS].) In this paper we do not use these functors nor do we use Theorem 2.4 of [GM1]. However, Theorem 2.4 motivated Theorem 1 of this paper and implies it if $H^0(L) = 0$. We have abandoned the Deligne functors $\mathscr{C}(L, A)$ for this paper for two reasons. The action of the diffeomorphism group on $\mathscr{A}^{0,1}(M, T^{0,1})$ appears to be too complicated to construct groupoids analogous to those of [GM1]. Second there do not exist finite dimensional augmentations of the Kodaira-Spencer algebra and consequently the method used in [GM1] to treat those L for which $H^0(L) \neq \{0\}$ is not available here. The Kuranishi method gets around both problems. We should mention also that there is some overlap between [NR] and §2 of this paper. We have borrowed the term "analytic differential graded Lie algebra" from them (but changed its meaning). We have profited from conversations with a large number of mathematicians while working on this project. They are listed in [GM1] and we take this opportunity to thank them again. We wish to thank Richard Penney for the proof of Lemma 6.5 and Robert Steinberg and Jim Carrell for helpful conversations concerning the material in §6. Finally, the second author would like to thank Steve Halperin and Richard Hain for patiently explaining many of the basic constructions of rational homotopy theory.

1. The complete local k-algebra associated to a differential graded Lie algebra

Let L be a differential graded Lie algebra over the field $\mathbf{k}$ with $\dim H^1(L) < \infty$. Let $C^1(L)$ be a complement to the 1-coboundaries $B^1(L) \subset L^1$. We define a functor $A \to Y_L(A)$ from the category $\mathscr{A}$ of Artin local $\mathbf{k}$-algebras to the category of sets by

$$Y_L(A) = \left\{\eta \in C^1(L) \otimes \mathfrak{m}: d\eta + \tfrac{1}{2}[\eta, \eta] = 0\right\}.$$

Here $\mathfrak{m}$ is the maximal ideal of the ring A.

1.1 THEOREM. *The functor Y_L is pro-representable; that is, there exists a complete local* $\mathbf{k}$*-algebra R_L and a natural isomorphism of functors*

$$Y_L \to \operatorname{Hom}_{\mathrm{alg}}(R_L, \cdot)$$

Proof. We verify the axioms of [Sc]. Theorem 2.11. The tangent space, $Y_L(\mathbf{k}[\varepsilon])$, to the functor Y_L is clearly given by the space of solutions to $d\eta = 0$ with $\eta \in C^1(L) \otimes \mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal in $\mathbf{k}[\varepsilon]$. Clearly, this vector space is isomorphic to $H^1(L)$.

Schlessinger's other axioms all follow from the observation that the natural map

$$Y_L(A' \times_A A'') \to Y_L(A') \times_{Y_L(A)} Y_L(A'')$$

is a bijection for any triple of Artin local $\mathbf{k}$-algebras, A', A'', A and homomorphisms $A' \to A$ and $A'' \to A$.

1.2 *Remark.* We will see later in (4.1) that the isomorphism class of R_L does not depend on the choice of complement.

1.3 *Remark.* R. Hain has shown that another description of the complete, local, $\mathbf{k}$-algebra R_L can be obtained using the chain functor $L \to \mathscr{C}(L)$ of [Q] from the category of connected cochain Lie algebras to the category of cocommutative differential graded coalgebras. We must modify $\mathscr{C}$ slightly because $L^0 \neq \{0\}$ for the differential graded Lie algebras of interest to us here. We define the *reduced* cochain functor $\mathscr{C}_{\text{red}}$ as follows. Let L be a differential graded Lie algebra. Let $C^1(L)$ be a complement to the 1-coboundaries $B^1(L)$ and define $\overline{L} = C^1(L) \oplus \oplus_{i \geq 2} L^i$. We then define

$$\mathscr{C}_{\text{red}}(L) = \mathscr{C}(\overline{L}).$$

We let $\hat{\mathscr{C}}_{\text{red}}(L)$ denote the completion of the algebra $\mathscr{C}_{\text{red}}(L)$ with respect to its augmentation ideal. It can then be shown that the complete local $\mathbf{k}$-algebra $H_0(\hat{\mathscr{C}}_{\text{red}}(L))$ also represents Y_L whence

$$R_L \approx H_0(\hat{\mathscr{C}}_{\text{red}}(L)).$$

2. The Kuranishi space associated to an analytic differential graded Lie algebra

In this section we show that if L has further topological structure then we can construct an analytic germ $(\mathscr{K}_L, 0)$, the Kuranishi space of L. The point is that in case L is the Kodaira-Spencer algebra of a compact complex manifold M then $(\mathscr{K}_L, 0)$ is the parameter space for the versal deformation of the complex structure of M constructed by Kuranishi. In §3 we will show that the completion of $\mathscr{O}_{\mathscr{K}_L}$ is the $\mathbf{k}$-algebra R_L. The only material in this section to be used later is the definition of $\mathscr{K}_L$ and the splitting $(*)$ of L^j below.

We now define a normed differential graded algebra to be a differential graded Lie algebra equipped with a norm $\| \|_i$ on each L^i making L^i into a normed vector space such that

(i) $d^i: L^i \to L^{i+1}$ is continuous,

(ii) $[\ ,\ ]: L^1 \otimes L^1 \to L^2$ is continuous.

We let $\hat{L}^i$ be the completion of L^i for $i = 0, 1, 2, \ldots$. The vector space L^i is the analogue of the smooth i-forms and $\hat{L}^i$ of the Sobolev i-forms.

An *analytic differential graded Lie algebra* is a normed differential graded Lie algebra with finite-dimensional cohomology in degrees 0 and 1, equipped with continuous splittings of the short exact sequences

$$0 \to Z^j(\hat{L}) \to \hat{L}^j \xrightarrow{d} B^{j+1}(\hat{L}) \to 0$$

and

$$0 \to B^j(\hat{L}) \to Z^j(\hat{L}) \to H^j(\hat{L}) \to 0.$$

We let $\hat{\mathscr{A}}^j \subset \hat{L}^j$ be the image of the splitting $B^{j+1} \to \hat{L}^j$ so that $\hat{\mathscr{A}}^j$ is a closed subspace such that

$$\hat{L}^j = Z^j(\hat{L}) \oplus \hat{\mathscr{A}}^j$$

where the projections on both summands are continuous. We let $\mathscr{H}^j \subset Z^j(\hat{L})$ be the image of the splitting $H^j(\hat{L}) \to Z^j(\hat{L})$ so that $\mathscr{H}^j$ is a closed finite-dimensional subspace such that

$$Z^j(\hat{L}) = B^j(\hat{L}) \oplus \mathscr{H}^j \quad \text{for } j = 0, 1, \ldots$$

We assume $\mathscr{H}^j \subset L^j$. (In the usual Hodge decomposition L^j is the complex of C^∞ differential forms, $\mathscr{A}^j$ is the subspace of coexact differential forms and $\mathscr{H}^j$ is the subspace of harmonic forms.) Let β: $\hat{L}^j \to B^j(\hat{L})$, α: $\hat{L}^j \to \hat{\mathscr{A}}^j$ and $\mathbf{H}$: $\hat{L}^j \to \mathscr{H}^j$ be the corresponding projections. We assume that all three projections carry L^j into itself and that $\beta(L^j) = dL^{j-1}$. We put $\mathscr{A}^j = \hat{\mathscr{A}}^j \cap L^j$ and $B^j(L) = B^j(\hat{L}) \cap L^j$. It is immediate that $\hat{\mathscr{A}}^j = \alpha(L^j)$, $B^j(L) = \beta(L^j)$ and that there is an algebraic direct sum decomposition

$$(*) \qquad L^j = B^j(L) + \mathscr{H}^j + \mathscr{A}^j.$$

Clearly $\beta \circ d$ maps $\hat{\mathscr{A}}^j$ isomorphically onto $B^j(\hat{L})$. Let γ: $B^{j+1}(\hat{L}) \to \hat{\mathscr{A}}^j$ denote the inverse to $\beta \circ d$: $\hat{\mathscr{A}}^j \to B^{j+1}(\hat{L})$ and define δ: $\hat{L}^{j+1} \to \hat{L}^j$ to be the composition

$$\delta = \iota \circ \gamma \circ \beta$$

where ι: $\hat{\mathscr{A}}^j \hookrightarrow \hat{L}^j$ denotes inclusion. (In the usual Hodge theory what we are calling δ is actually $d^* \circ G$ where d^* is the adjoint to d and G is the Green's operator.) It is clear that $\delta\colon \hat{L}^{j+1} \to \hat{L}^j$ maps $B^{j+1}(\hat{L})$ isomorphically onto $\hat{\mathscr{A}}^j$, that $\delta \circ \delta = 0$ and that $\delta(L^{j+1}) \subset L^j$.

Since $\delta \circ d = \alpha$, $d \circ \delta = \beta$ and $\alpha + \beta = I - \mathbf{H}$ the following is immediate:

2.1. Lemma (The Hodge Decomposition). $d \circ \delta + \delta \circ d = I - \mathbf{H}$.

We define $Y \subset \hat{L}^1$ by

$$Y = \left\{ \xi \in \hat{\mathscr{A}}^1 + \mathscr{H}^1 \colon d\xi + \tfrac{1}{2}[\xi, \xi] = 0 \right\}.$$

We now show that there exists a neighborhood of 0 in Y which has the structure of a finite-dimensional analytic space by using the method of Kuranishi.

The *Kuranishi* map $F\colon \hat{L}^1 \to \hat{L}^1$ is the quadratic map defined by

$$F(\xi) = \xi + \tfrac{1}{2}\delta[\xi, \xi].$$

We observe that $F(L^1) \subset L^1$.

2.2. Lemma. *There exist balls B and B' around* 0 *in $\hat{L}^1$ such that F is an analytic diffeomorphism $B \to B'$.*

Proof. The differential of F at the origin is the identity and $\hat{L}^1$ is a Banach space. The lemma follows from the inverse function theorem. ∎

We now define a **k**-analytic subset $\mathscr{K}_L \subset \mathscr{H}^1$ (the Kuranishi space) by

$$\mathscr{K}_L = \left\{ \eta \in B' \cap \mathscr{H}^1 \mid \mathbf{H}\left(\left[F^{-1}(\eta), F^{-1}(\eta)\right]\right) = 0 \right\}.$$

Observe that F carries the space $\hat{\mathscr{A}}^1 + \mathscr{H}^1$ into itself since $\delta F(\xi) = \delta\xi$. Also

$$F^{-1}\left((\hat{\mathscr{A}}^1 + \mathscr{H}^1) \cap B'\right) \subset (\hat{\mathscr{A}}^1 + \mathscr{H}^1) \cap B.$$

2.3. Theorem (Kuranishi). *F induces a homeomorphism from a neighborhood of* 0 *in Y to a neighborhood of* 0 *in $\mathscr{K}_L$.*

The proof will follow from the next two lemmas.

2.4. Lemma. $F(Y) \subset \mathscr{K}_L$.

Proof. Let $\xi \in Y$. We have seen that $\delta\xi = 0$ implies $\delta F(\xi) = 0$. Also $\mathbf{H}([\xi,\xi]) = 0$. It remains to check that $dF(\xi) = 0$. But

$$dF(\xi) = d\xi + \tfrac{1}{2}d\delta[\xi, \xi] = -\tfrac{1}{2}[\xi, \xi] + \tfrac{1}{2}d\delta[\xi, \xi].$$

By the Hodge decomposition we have

$$d\delta[\xi, \xi] = [\xi, \xi] - \delta d\,[\xi, \xi] - \mathbf{H}([\xi, \xi])$$

and

$$d[\xi, \xi] = 2[d\xi, \xi] = -[[\xi, \xi], \xi] = 0.$$

Hence $d\delta\,[\xi, \xi] = [\xi, \xi]$ and the lemma is proved. ■

2.5. LEMMA. *There exists a ball B' around 0 in $\mathscr{H}_L$ such that $F^{-1}(B') \subset Y$.*

Proof. We claim that there exists a ball B around 0 in $\hat{L}^1$ such that if $\xi \in B$ and

$$\delta d\,[\xi, \xi] = \delta[\delta d\,[\xi, \xi], \xi]$$

then

$$\delta d\,[\xi, \xi] = 0.$$

Indeed if we put $\psi = \delta d\,[\xi, \xi]$ then the above equation becomes

$$\psi = \delta[\psi, \xi].$$

Since δ and $[\ ,\ ]$ are continuous there exists C independent of ψ and ξ such that

$$\|\psi\| \leq C\|\psi\|\,\|\xi\|.$$

Thus if $\|\xi\| < 1/C$ then $\|\psi\| < \|\psi\|$ so $\psi = 0$. This establishes the claim.

We now claim that $B' = B \cap \mathscr{H}_L$ satisfies the conclusions of the lemma.

Let $\xi \in \hat{L}^1$ with $F(\xi) = \eta \in B'$. Then $\delta\xi = 0$. Moreover, since η is closed we have

$$d\xi + \tfrac{1}{2}d\delta\,[\xi, \xi] = 0.$$

The lemma is proved if we can establish

$$d\delta\,[\xi, \xi] = [\xi, \xi]$$

which follows (using the Hodge decomposition) once we establish that $\delta d\,[\xi, \xi] = 0$. But

$$\delta d\,[\xi, \xi] = 2\delta[d\xi, \xi]$$

and using the facts that $d\eta = 0$ and $[[\xi, \xi], \xi] = 0$, we obtain

$$\delta d\,[\xi, \xi] = -\delta[\delta d\,[\xi, \xi], \xi]$$

from which it follows $\delta d\,[\xi, \xi] = 0$ as desired. This concludes the proof of Lemma 2.10. ■

We have obtained the required $\mathbf{k}$-analytic structure on a neighborhood of 0 in Y.

The following theorem is obvious but important.

2.6. THEOREM. *If L^1 is finite dimensional then $(\mathscr{K}_L, 0)$ is analytically equivalent to the analytic germ $(Y, 0)$ where*

$$Y = \left\{ \xi \in L^1 \colon d\xi + \tfrac{1}{2}[\xi, \xi] = 0 \text{ and } \delta\xi = 0 \right\}.$$

2.7. REMARK. This result remains true in infinite dimensions but the analytic structure on Y induced by the embedding $Y \subset \hat{L}^1$ is no longer useful.

3. Formal Kuranishi theory

In this section we show that the functors Y_L of §1 and $\mathrm{Hom}(\mathscr{O}_{\mathscr{K}_L} \cdot)$ of §2 are naturally isomorphic. We abuse notation and define a functor $\mathscr{K}_L\colon \mathscr{A} \to \mathrm{Sets}$ by

$$\mathscr{K}_L(A) = \mathrm{Hom}(\mathscr{O}_{\mathscr{K}_L}, A)$$

We now digress in order to recall the main construction (due to Deligne) from [GM1]. We do this in order to relate the functors $\mathscr{K}_L$ and Y_L of this paper with the functor $\mathrm{Iso}\,\mathscr{C}(L, \cdot)$ of [GM1].

In §2 of [GM1] we associated to a differential graded Lie algebra L a functor $A \to \mathscr{C}(L, A)$ from the category $\mathscr{A}$ of Artin local $\mathbf{k}$-algebras $(A, \mathfrak{m})$ to the category of transformation groupoids. We recall that a transformation groupoid is given by a small category $\mathscr{C}$ with a set of objects X and a group $\mathscr{G}$ acting on X such that for any two $\alpha, \beta \in X$, morphisms $\mathrm{Hom}(\alpha, \beta)$ correspond to $g \in \mathscr{G}$ such that $g\alpha = \beta$. Two groupoids determine the same deformation theory if they are equivalent *as categories*. The groupoid $\mathscr{C}(L, A)$ is then determined by the formulas

$$\mathscr{G} = G(A) = \exp(L^0 \otimes \mathfrak{m})$$

(the multiplication is the Baker–Campbell–Hausdorff multiplication) and

$$X = X(A) = \left\{ \omega \in L^1 \otimes \mathfrak{m} \colon d\omega + \tfrac{1}{2}[\omega, \omega] = 0 \right\}.$$

$\mathscr{G}$ acts on X according to the gauge transformation law ρ given by

$$\rho(\exp \lambda)\,\omega = \exp(\operatorname{ad} \lambda)\,\omega + \frac{1 - \exp(\operatorname{ad} \lambda)}{\operatorname{ad} \lambda}(d\lambda).$$

Then the elements of $X(A)$ are the objects and the elements of $G(A)$ the morphisms of the category $\mathscr{C}(L, A)$. We let $\operatorname{Iso}\mathscr{C}(L, A)$ denote the isomorphism classes of objects in $\mathscr{C}(L, A)$. We obtain an induced functor $\mathscr{A} \to$ Sets given by $A \to \operatorname{Iso}\mathscr{C}(L, A)$. The invariance of these functors under quasi-isomorphism is stated and proved in [GM1], Theorem 2.4. A quasi-isomorphism $f\colon L_1 \to L_2$ induces on equivalence of categories $\mathscr{C}(L_1, A) \to \mathscr{C}(L_2, A)$ which is natural in A.

The isomorphism between $\mathscr{X}_L$ and Y_L will now be established by formal Kuranishi theory. If A is an Artin local $\mathbf{k}$-algebra with maximal ideal $\mathfrak{m} \subset A$, then $G(A)$ is the nilpotent Lie group $\exp(\mathfrak{g} \otimes \mathfrak{m})$. For any ideal $\mathscr{T} \subset \mathfrak{m}$ we denote by $\pi\colon A \to A/\mathscr{T}$ the quotient projection. We will also denote by π the projection associated to any base change by the map $A \to A/\mathscr{T}$. We choose a splitting of $\mathbf{k}$-vector spaces $\sigma\colon A/\mathscr{T} \to A$. Since it is a polynomial map, the Kuranishi map $F\colon L^1 \to L^1$ induces a map $L^1 \otimes \mathfrak{m} \to L^1 \otimes \mathfrak{m}$ which we also denote by F. If $\mathscr{T}$ is as above we have a commutative diagram

$$\begin{array}{ccc} L^1 \otimes \mathfrak{m} & \xrightarrow{F} & L^1 \otimes \mathfrak{m} \\ \downarrow & & \downarrow \\ L^1 \otimes \mathfrak{m}/\mathscr{T} & \xrightarrow{F} & L^1 \otimes \mathfrak{m}/\mathscr{T}. \end{array}$$

3.1. LEMMA. *$F\colon L^1 \otimes \mathfrak{m} \to L^1 \otimes \mathfrak{m}$ is bijective.*

Proof. Artinian induction on the ring A. If $A = \mathbf{k}$ then $\mathfrak{m} = 0$ and there is nothing to prove. Suppose that $\mathscr{T} \subset A$ is an ideal such that $\mathscr{T} \cdot \mathfrak{m} = 0$ and that

$$F\colon L^1 \otimes \mathfrak{m}/\mathscr{T} \to L^1 \otimes \mathfrak{m}/\mathscr{T}$$

is bijective. Let $\sigma\colon L \otimes \mathfrak{m}/\mathscr{T} \to L \otimes \mathfrak{m}$ be the linear map induced by $\sigma\colon \mathfrak{m}/\mathscr{T} \to \mathfrak{m}$.

F is injective. Suppose that $\xi_1, \xi_2 \in L^1 \otimes \mathfrak{m}$ satisfy $F(\xi_1) = F(\xi_2)$. Then $F(\pi\xi_1) = F(\pi\xi_2)$ and the induction hypothesis implies $\pi\xi_1 = \pi\xi_2$, i.e., $\xi_1 - \xi_2 \in L^1 \otimes \mathscr{T}$. But $\mathscr{T} \cdot \mathfrak{m} = 0$ so that $[\xi_1 - \xi_2, \xi_1 - \xi_2] = \xi_1 - \xi_2, \xi_2] = 0$ whence

$$\begin{aligned} F(\xi_1) - F(\xi_2) &= (\xi_1 - \xi_2) + \delta[\xi_1 - \xi_2, \xi_2] + \tfrac{1}{2}\delta[\xi_1 - \xi_2, \xi_1 - \xi_2] \\ &= \xi_1 - \xi_2. \end{aligned}$$

Thus $\xi_1 = \xi_2$ is desired.

F is surjective. Let $\eta \in L^1 \otimes \mathfrak{m}$. By the induction hypothesis there exists $\bar{\xi} \in L^1 \otimes \mathfrak{m}/\mathscr{T}$ such that $F(\bar{\xi}) = \pi(\eta)$. Let $\eta' = \eta - F(\sigma(\bar{\xi}))$. Then $\pi(\eta') = 0$ so $\eta' \in L^1 \otimes \mathscr{T}$. Let $\xi = \sigma(\bar{\xi}) + \eta'$. It follows that

$$F(\xi) = F\big(\sigma(\bar{\xi})\big) + \eta' + \delta\big[\sigma(\bar{\xi}), \eta'\big] + \tfrac{1}{2}\delta[\eta', \eta'] = F\big(\sigma(\bar{\xi})\big) + \eta' = \eta$$

since $\delta[\sigma(\bar{\xi}), \eta'] + \frac{1}{2}\delta[\eta', \eta'] \in L \otimes (\mathfrak{m} \cdot \mathscr{T}) = 0$. ∎

3.2. *Remark.* The inverse F^{-1} of the formal Kuranishi map F: $L^1 \otimes \mathfrak{m} \to L^1 \otimes \mathfrak{m}$ carries $L^1 \otimes \mathfrak{m}$ into itself and is a polynomial map, due to the nilpotence of $\mathfrak{m}$. Indeed F^{-1} is given by a power series, which truncates to a polynomial of degree $n-1$, if $\mathfrak{m}^n = 0$.

We now set up infinitesimal product decompositions which we will need in this section and the next one. We do not use the "diffeomorphism action" of $\mathscr{G}$ but the much simpler action ρ. This is permissible because we are only using it to set up coordinates and not to define an equivalence relation on Obj $\mathscr{C}(L, A)$.

We observe that the product decompositions below do not depend on the fact that L is analytic but only on Hodge decompositions of L^0, L^1 and L^2.

3.3. LEMMA. *Let $\mathfrak{h} \subset \mathfrak{g}$ be an inclusion of Lie algebras and $\mathfrak{s}$ a linear complement to $\mathfrak{h}$ in $\mathfrak{g}$. Let $S(A) = \exp(\mathfrak{s} \otimes \mathfrak{m})$. Then the natural map*

$$\nu\colon S(A) \times H(A) \to G(A)$$

is a bijection which is natural in A.

Proof. Artinian induction. If $A = \mathbf{k}$ there is nothing to prove. Suppose that $\mathscr{T} \subset A$ is an ideal with $\mathscr{T} \cdot \mathfrak{m} = 0$ and inductively assume that the assertion that ν is surjective has been proved for the $\mathbf{k}$-algebra $A/\mathscr{T}$. Let π, σ be as above. Let $g \in G(A)$ be given. By induction there exist $h \in H(A/\mathscr{T})$ and $s \in S(A/\mathscr{T})$ such that

$$sh = \pi(g).$$

Hence $\sigma(s)\sigma(h) = g\iota$ with $\iota \in \exp(\mathfrak{g} \otimes \mathscr{T})$. But $\exp(\mathfrak{g} \otimes \mathscr{T})$ is abelian and isomorphic to $\mathfrak{g} \otimes \mathscr{T}$. Consequently, we may write

$$\iota = h_1 s_1 \quad \text{with } s_1 \in \exp \mathfrak{s} \otimes \mathscr{T},\ h_1 \in \exp \mathfrak{h} \otimes \mathscr{T}.$$

We obtain

$$\sigma(s) s_1^{-1} \sigma(h) h_1^{-1} = g.$$

Now $\sigma(s)$ and s_1^{-1} commute; consequently $\log(\sigma(s)s_1^{-1}) = \log \sigma(s) - \log s_1 \in \mathfrak{s}$ and $\sigma(s)s_1^{-1} \in S(A)$.

We now prove ν is injective. Suppose there exist $h \in H(A)$, $s_1, s_2 \in S(A)$ with

$$s_1 h = s_2.$$

Applying π we find $\pi(s_1)\pi(h) = \pi(s_2)$ and by induction $\pi(h) = e$ and $h \in \exp \mathfrak{h} \otimes \mathscr{T}$. Hence h and s_1 commute and

$$\log h + \log s_1 = \log s_2.$$

Consequently, $\log h = 0$. ■

3.4. REMARK. In our application of the lemma, we take $\mathfrak{g} = L^0$ and $\mathfrak{h} = H^0(L)$. We take $\mathfrak{s} = \mathscr{A}^0(L)$.

3.5. LEMMA. *Let $V^1 \subset L^1$ be a subspace such that*

$$L^1 = V^1 \oplus dL^0.$$

Then the action of $\exp(L^0 \otimes \mathfrak{m})$ *on $L^1 \otimes \mathfrak{m}$ by ρ induces a bijection*

$$\mu\colon S(A) \times V^1 \otimes \mathfrak{m} \to L^1 \otimes \mathfrak{m}, \qquad \mu(s, v) = \rho(s)v.$$

Proof. Artinian induction. If $A = \mathbf{k}$ there is nothing to prove. Suppose that $\mathscr{T} \subset A$ is an ideal with $\mathscr{T}\mathfrak{m} = 0$ and inductively assume that the assertion has been proved for the $\mathbf{k}$-algebra $A/\mathscr{T}$. We first show that μ is surjective.

Let $\omega \in L^1 \otimes \mathfrak{m}$. By induction there exist $\bar{s} \in S(A/\mathscr{T})$ and $\bar{v} \in V^1 \otimes \mathfrak{m}/\mathscr{T}$ such that

$$\rho(\bar{s})\bar{v} = \pi(\omega).$$

We write $\bar{s} = \exp\lambda$ with $\lambda \in \mathscr{A}^0(L) \otimes \mathfrak{m}/\mathscr{T}$ and define

$$\omega' = \omega - \rho(\exp\sigma\lambda)\sigma(\bar{v}).$$

Then $\pi(\omega') = 0$ whence $\omega' \in L^1 \otimes \mathscr{T}$. Hence, there exist $v' \in V^1 \otimes \mathscr{T}$ and $\lambda' \in \mathscr{A}^0(L) \otimes \mathscr{T}$ such that $\omega' = v' + d\lambda'$.

We obtain

$$\omega = \rho(\exp\sigma\lambda)\sigma(\bar{v}) + v' + d\lambda'.$$

But we have

$$\rho(\exp\sigma\lambda)(\sigma(\bar{v}) + v') - \rho(\exp\sigma\lambda)\sigma(\bar{v}) = \exp(\operatorname{ad}\sigma\lambda)v' = v'.$$

We obtain

$$\omega = \rho(\exp\sigma\lambda)(\sigma(\bar{v}) + v') + d\lambda'.$$

Finally, applying Lemma 2.8 of [GM1] we have

$$\omega = \rho(\exp(\sigma\lambda - \lambda'))(\sigma(\bar{v}) + v').$$

Now $\sigma\lambda - \lambda' \in \mathscr{A}^0(L) \otimes \mathfrak{m}$ whence $\exp(\sigma\lambda - \lambda') \in S(A)$ and $\sigma(\bar{v}) + v' \in V^1 \otimes \mathfrak{m}$. We have established that μ is surjective.

We now prove that μ is injective. Suppose (s_1, v_1) and (s_2, v_2) are elements of $S(A) \times V^1 \otimes \mathfrak{m}$ such that

$$\rho(s_1)v_1 = \rho(s_2)v_2.$$

Applying π we obtain

$$\rho(\pi s_1)\pi(v_1) = \rho(\pi s_2)\pi(v_2)$$

and by induction

$$\pi(s_1) = \pi(s_2) \quad \text{and} \quad \pi(v_1) = \pi(v_2).$$

Consequently, there exists $\iota \in \mathscr{A}^0(L) \otimes \mathscr{T}$ such that

$$s_2 = \exp(\lambda_1 + \iota) = s_1 \exp \iota$$
$$v_2 = \rho(s_2)^{-1}\rho(s_1)v_1 = \rho(\exp \iota)^{-1}v_1 = v_1 + d\iota$$

whence

$$v_1 - v_2 = -d\iota.$$

But $v_1 - v_2 \in \mathscr{A}^1(L) \otimes \mathfrak{m}$ and $d\iota \in dL^0 \otimes \mathscr{T} \subset dL^0 \otimes \mathfrak{m}$. Hence $v_2 - v_1 = d\iota = 0$ and $v_1 = v_2$. But $\iota \in \mathscr{A}^0(L) \otimes \mathscr{T}$ so $d\iota = 0$ implies $\iota = 0$ and $s_1 = s_2$. ∎

3.6. *Remark.* In our application of this lemma we will take $V^1 = \mathscr{H}^1 + \mathscr{A}^1(L)$.

3.7. COROLLARY. *The map μ induces a bijection (natural in A)*

$$\mu\colon S(A) \times Y_L(A) \to \operatorname{Obj} \mathscr{C}(L, A).$$

Proof. Suppose $\omega \in \operatorname{Obj} \mathscr{C}(L, A)$. Then we may write $\omega = \rho(s)v$ as above. But then $v = \rho(s)^{-1}\omega$ so $v \in \operatorname{Obj} \mathscr{C}(L, A) \cap V^1 \otimes \mathfrak{m} = Y_L(A)$. ∎

We are now ready to carry over the Kuranishi map to the infinitesimal case, with infinitesimal parameters taken from an Artin local **k**-algebra A. Let $Y'(A)$ be the subset of $\mathscr{H}^1 \otimes \mathfrak{m}$ defined by

$$Y'(A) = \left\{\eta \in \mathscr{H}^1 \otimes \mathfrak{m} : \mathbf{H}(F^{-1}\eta, F^{-1}\eta) = 0\right\}.$$

The following theorem is the infinitesimal analogue of Theorem 3 and has a similar proof. The main point is that the statement of Lemma 2.5 can be replaced by

$$F^{-1}(Y'(A)) \subset Y_L(A).$$

3.8. THEOREM. *F maps $Y_L(A)$ bijectively onto $Y'(A)$.*

Now Let $\mathcal{O}_{\mathcal{K}_L}$ be the analytic local ring of the Kuranishi space at 0. We may realize $\mathcal{O}_{\mathcal{K}_L}$ concretely as follows. Choose a basis $\{\eta_1, \ldots, \eta_l\}$ for $\mathcal{H}^1$ and let $\{x_1, \ldots, x_l\}$ be the dual basis. Then $\mathcal{O}_{\mathcal{H}^1}$, the analytic local ring of the linear subspace $\mathcal{H}^1$ at 0, is the ring

$$\mathbf{R}\{x_1, \ldots, x_l\}$$

of convergent power series in $x_1, \ldots, x_n$ and $\mathcal{O}_{\mathcal{K}_L}$ is the quotient of $\mathbf{R}\{x_1, \ldots, x_l\}$ by the ideal generated by the components (relative to a basis of $\mathcal{H}^2$) of the $\mathcal{H}^2$-valued analytic equation

$$\mathbf{H}\left(\left[F^{-1}\left(\sum_{i=1}^{l} x_i\eta_i\right), F^{-1}\left(\sum_{i=1}^{l} x_i\eta_i\right)\right]\right) = 0.$$

Since $\mathcal{O}_{\mathcal{K}_L}$ is a quotient of $\mathcal{O}_{\mathcal{H}^1}$ there is an embedding $\iota\colon \mathcal{K}(A) \hookrightarrow \mathcal{H}^1(A)$, where $\mathcal{H}^1(A)$ denotes the set of $\mathbf{k}$-algebra homomorphisms $\mathcal{O}_{\mathcal{H}^1} \to A$ and $\mathcal{K}(A)$ denotes the set of $\mathbf{k}$-algebra homomorphisms $\mathcal{O}_{\mathcal{K}_L} \to A$. Since a $\mathbf{k}$-algebra homomorphism $\mathcal{O}_{\mathcal{H}^1} \to A$ is determined by its values on the generators $x_1, \ldots, x_l$, which can be arbitrary elements of $\mathfrak{m}$, it follows that there is a canonical isomorphism

$$e\colon \mathcal{H}^1(A) \to \mathcal{H}^1 \otimes \mathfrak{m}.$$

The composition $e \circ \iota$ maps $\mathcal{K}(A)$ isomorphically onto $Y'(A) \subset \mathcal{H}^1 \otimes \mathfrak{m}$. There is a commutative diagram

$$\begin{tikzcd} \mathcal{H}^1(A) \arrow[r, "e"] & \mathcal{H}^1 \otimes \mathfrak{m} \\ \mathcal{K}(A) \arrow[u, "\iota"'] \arrow[r] & Y'(A) \arrow[u] \end{tikzcd}$$

in which the horizontal arrows are isomorphisms of functors.

We obtain the following theorems since the germ $(\mathcal{K}_L, 0)$ clearly represents $\mathcal{K}(A)$.

3.9. THEOREM. *The functor $A \to Y_L(A)$ is represented by the analytic germ $(\mathcal{K}_L, 0)$.*

3.10. COROLLARY. *R_L is the completion of the analytic local ring $\mathcal{O}_{\mathcal{K}_L}$.*

3.11. THEOREM. *If $H^0(L) = 0$, the functor $A \to \operatorname{Iso}\mathcal{C}(L, A)$ is represented by the analytic germ $(\mathcal{K}_L, 0)$.*

Proof. In this case Corollary 3.7 states that

$$\mu\colon G(A) \times Y_L(A) \to \operatorname{Obj} \mathscr{C}(L, A)$$

is a natural bijection. Thus $Y_L(A)$ is a cross-section to the orbits of $G(A)$ and the theorem follows.

3.12. COROLLARY. *If $H^0(L_1) = 0$ and $f\colon L_1 \to L_2$ is a quasi-isomorphism then the analytic germs $(\mathscr{K}_{L_1}, 0)$ and $(\mathscr{K}_{L_2}, 0)$ are analytically isomorphic.*

Proof. The corollary follows immediately from Theorems 2.4 and 3.1 of [GM1].

4. The invariance of the Kuranishi space under quasi-isomorphism

We now assume we are given two differential graded Lie algebras (L_1, d_1) and (L_2, d_2) with finite-dimensional first cohomology and that we have chosen complements $C^1(L_i)$, $i = 1, 2$, as in §1. We choose (non-topological) splittings

$$C^1(L_i) = \mathscr{A}^1(L_i) \oplus \mathscr{H}^1(L_i), \qquad i = 1, 2,$$

and then further splittings of L_i^0, L_i^1 and L_i^1, to get (non-topological) Hodge decompositions analogous to those of §2. We then obtain product decompositions of $\operatorname{Obj} \mathscr{C}(L_i, A)$ as in §3.

Now assume we are given a homomorphsm $f\colon L_1 \to L_2$ such that the induced maps on cohomology $H^i(f)\colon H^i(L_1) \to H^i(L_2)$ satisfy:

(i) $H^1(f)$ is an isomorphism;

(ii) $H^2(f)$ is an injection.

Our goal in this section is to prove the following theorem relating the complete local **k**-algebras R_{L_1} and R_{L_2}. We abbreviate the functors Y_{L_1} and Y_{L_2} of §1 to Y_1 and Y_2.

4.1. THEOREM. *If $f\colon L_1 \to L_2$ satisfies (i) and (ii) then R_{L_1} and R_{L_2} are isomorphic.*

Proof. By Corollary 3.7 we have a diagram

$$\begin{tikzcd} S_1(A) \times Y_1(A) \arrow[r, "\mu_1"] & \operatorname{Obj} \mathscr{C}(L_1, A) \arrow[d, "f"] \\ S_2(A) \times Y_2(A) \arrow[r, "\mu_2"] & \operatorname{Obj} \mathscr{C}(L_2, A). \end{tikzcd}$$

We let $j\colon Y_1(A) \to S_1(A) \times Y_1(A)$ be the map

$$j(y) = (e, y)$$

and $p\colon S_2(A) \times Y_2(A) \to Y_2(A)$ be the map

$$p(s, y) = y \quad \text{for} \quad i = 1,2.$$

We define a map $\phi\colon Y_1(A) \to Y_2(A)$ by

$$\phi(y) = p(f(j(y))).$$

Our theorem will follow if we can prove ϕ is a bijection. This we accomplish in the next two lemmas.

4.2. LEMMA. *ϕ is injective.*

Proof. Artinian induction. Suppose y_1, y_1' satisfy $\phi(y_1) = \phi(y_1') = y_2$. Then there exist s, $s' \in S_2(A)$ such that

$$f(y_1) = \rho(s)y_2, \quad f(y_1') = \rho(s')y_2.$$

Now $\phi(\pi y_1) = \phi(\pi y_1')$ whence by induction $\pi y_1 = \pi y_1'$ and consequently $f(\pi y_1) = f(\pi y_1') = \eta$. Applying π to the equations above we obtain

$$\eta = \rho(\pi s)\pi(y_2) = \rho(\pi s')\pi(y_2)$$

whence $s' = s \exp \lambda$ with $\lambda \in L_2^0 \otimes \mathscr{T}$. Also $y_1' = y_1 + \iota$ with $\iota \in \mathscr{H}^1(L) \otimes \mathscr{T}$. We obtain

$$f(y_1 + \iota) = \rho(s)\rho(\exp \lambda)y_2$$

whence

$$f(y_1) + f(\iota) = \rho(s)y_2 + d\lambda$$

and

$$f(\iota) = -d\lambda.$$

But $\iota \in \mathscr{H}^1$ and $H^1(f)$ is injective. Hence $f(\iota)$ exact implies $\iota = 0$. We obtain $y_1' = y_1$. ∎

4.3. *Remark.* In the case that L_1 and L_2 are analytic this lemma may be proved in a more conceptual fashion as follows. Let A be the dual numbers and observe that in this case we have

$$\mathscr{K}_{L_1}(A) = Y_1(A) = \mathscr{H}^1(L_1) \otimes \mathfrak{m}$$
$$\mathscr{K}_{L_2}(A) = Y_2(A) = \mathscr{H}^1(L_2) \otimes \mathfrak{m}$$
$$\phi = \mathscr{H}_2 \circ H^1(f) \circ \mathscr{H}_1^{-1}.$$

Thus ϕ is an isomorphism in this case. Now we have the general result that a map of analytic germs $f\colon (X, x_0) \to (Y, y_0)$ that is injective on Zariski tangent spaces induces an injective map $f\colon X(A) \to Y(A)$ for all Artin local **k**-algebras A. Indeed, suppose $f(a_1) = f(a_2)$. We may assume that X and Y are embedded in their Zariski tangent spaces and f extends to these ambient Zariski tangent spaces. By Artinian induction we have $a_2 = a_1 + \iota$ with $\iota \in T_{x_o}(X) \otimes \mathscr{T}$. We obtain

$$f(a_1) = f(a_2) = f(a_1 + \iota) = f(a_1) + df(a_1) \cdot \iota$$

whence

$$df(a_1) \cdot \iota = 0.$$

But $df(a_1) \cdot \iota = df(0) \cdot \iota$ whence $\iota = 0$ and $a_1 = a_2$.

We conclude the proof of the theorem by the following lemma.

4.4. LEMMA. *ϕ is surjective.*

Proof. Artinian induction. Let $y_2 \in Y_2(A)$. By induction there exists $y_1 \in Y_1(A/\mathscr{T})$ and $s_2 \in S_2(A/\mathscr{T})$ such that

$$f(y_1) = \rho(s_2) \cdot \pi(y_2).$$

We claim there exists $\tilde{y}_1 \in Y_1(A)$ with $\pi(\tilde{y}_1) = y_1$. Let $o_2(y_1)$ be the obstruction to lifting y_1 [GM1, §2.7]. We have

$$H^2 f(o_2(y_1)) = o_2(f(y_1)) = o_2(\rho(s_2)\pi(y_2)) = H^2(s_2)o_2(\pi(y_2)) = 0$$

and since $H^2(f)$ is injective we have $o_2(y_1) = 0$. Hence there exists $\omega_1 \in \operatorname{Obj}\mathscr{C}(L_1, A)$ with $\pi(\omega_1) = y_1$. By Lemma 3.5 there exists $\omega_1' \in Y_1(A)$ and $s_1 \in S_1(A)$ such that

$$\omega_1 = \rho(s_1)\omega_1'.$$

Applying π we find $y_1 = \rho(\pi(s_1))y_1'$ where $\pi(\omega_1') = y_1' \in Y_1(A/\mathscr{T})$ and $\pi(s_1) \in S_1(A/\mathscr{T})$. Hence by Lemma 3.3 we have $\rho(\pi(s_1)) = e$ and $y_1 = y_1'$. Hence $\pi(\omega_1') = y_1$. We put $\tilde{y}_1 = \omega_1'$ and the claim is proved.

We now consider $\eta = f(\tilde{y}_1) - \rho(\sigma(s_2))y_2$. Here if $s_2 = \exp\lambda_2$ then $\sigma(s_2) = \exp\sigma(\lambda_2)$ whence $\sigma(s_2) \in S_2(A)$. Then $\pi(\eta) = f(y_1) - \rho(s_2)\pi(y_2) = 0$ whence $\eta \in L_2^1 \otimes \mathscr{T}$. But $f(\tilde{y}_1)$ and $\rho(\sigma(s_2))y_2$ are objects so $d\eta = 0$. We may then write

$$\eta = h + d\lambda_1$$

with $h \in \mathscr{H}^1(L_2) \otimes \mathscr{T}$ and $\lambda_1 \in \mathscr{A}^0(L_2) \otimes \mathscr{T}$. We obtain

$$f(\tilde{y}_1) = \rho(\sigma(s_2))y_2 + h + d\lambda_1.$$

Since f is surjective on first cohomology there exist $\iota \in \mathscr{H}^1(L_1) \otimes \mathscr{T}$ and $\lambda_2 \in \mathscr{A}^0(L_2) \otimes \mathscr{T}$ such that

$$f(\iota) = h + d\lambda_2.$$

We obtain

$$f(\tilde{y}_1 - \iota) = \rho(\sigma(s_2))y_2 + d(\lambda_1 - \lambda_2).$$

We now observe that $\tilde{y} = \tilde{y}_1 - \iota \in Y_1(A)$. Also if we put $s = \exp(\lambda_2 - \lambda_1) \cdot \sigma(s_2)$ then by Lemma 2.8 of [GM1],

$$\rho(s)y_2 = \rho(\exp(\lambda_2 - \lambda_1))\rho(\sigma(s_2))y_2 = \rho(\sigma(s_2))y_2 + d(\lambda_1 - \lambda_2).$$

Also $\log s = \lambda_2 - \lambda_1 + \log \sigma(s_2)$ since $\exp(\lambda_2 - \lambda_1)$ and $\sigma(s_2)$ commute. Hence $\log s \in \mathscr{A}^0(L_2) \otimes \mathfrak{m}$ and $s \in S_2(A)$. We obtain $f(\tilde{y}) = \rho(s)y_2$ and consequently $\phi(\tilde{y}) = y_2$. ■

4.5. *Remark.* The surjectivity of ϕ for general A does not follow from the surjectivity of ϕ for the case $A = \mathbf{k}[\varepsilon]$ as is easily seen from the example of the inclusion map of the coordinate axes into the plane. The extra input here is that one can lift points from $A/\mathscr{T}$ to A provided one can lift the image points. As we have seen this follows from the obstruction theory of [GM1]. In order to make this precise we make the following definition.

4.6. DEFINITION. A map of germs $f\colon (X, x_0) \to (Y, y_0)$ is smooth at x_0 if the natural map

$$X(A) \to X(A/\mathscr{T}) \times_{Y(A/\mathscr{T})} Y(A)$$

is a surjection for any A and $\mathscr{T}$ as above.

We have just seen that ϕ is smooth. Then in the case that L_1 and L_2 are analytic, Lemma 4.4 follows from the next elementary lemma.

4.7. LEMMA. *Suppose $f\colon (X, x_0) \to (Y, y_0)$ is smooth at X_0. Then $f\colon X(A) \to Y(A)$ is surjective for all A.*

Proof. Artinian induction. Putting $A = \mathbf{k}[\varepsilon]$ and $\mathscr{T} = (\varepsilon)$ we see that $df(x_0)$ is surjective. Let $y \in Y(A)$ be given. Then by induction we can find $\bar{x} \in X(A/\mathscr{T})$ such that $f(\bar{x}) = \pi(y)$. Since $\pi(y)$ lifts to y we can find x in $X(A)$ lifting $\bar{x}$. Consequently $f(x) = y + \iota_2$ with $\iota_2 \in T_{y_0}(Y) \otimes \mathscr{T}$. We choose

ι_1 such that $df(x_0) \cdot \iota_1 = \iota_2$. Then

$$f(x + \iota_1 \cdot) = f(x) + df(x)\iota_1 = f(x) + df(x_0)\iota_1 = y. \qquad \blacksquare$$

Theorem 4.1 has the following immediate consequence.

4.8. THEOREM. *Suppose (L_1, d_1) and (L_2, d_2) are analytic differential graded Lie algebras which are quasi-isomorphic as differential graded Lie algebras. Then the analytic germs $(\mathscr{K}_{L_1}, 0)$ and $(\mathscr{K}_{L_2}, 0)$ are analytically isomorphic.*

Proof. By Corollary 3.11 and Theorem 4.1 $\mathscr{O}_{\mathscr{K}_{L_1}}$ and $\mathscr{O}_{\mathscr{K}_{L_2}}$ have isomorphic completions R_{L_1} and R_{L_2}. By [A] they are isomorphic. $\blacksquare$

5. Formality and quadratic cones

5.1. DEFINITION. Two analytic differential graded Lie algebras (L, d) and $(\bar{L}, \bar{d})$ are quasi-isomorphic if there exists a sequence of homomorphisms of differential graded Lie algebras

$$L = L_0 \to L_1 \leftarrow L_2 \to \cdots \leftarrow L_{m-1} \to L_m = \bar{L}$$

such that each homomorphism induces an isomorphism on cohomology.

It follows from Theorem 4.8 that quasi-isomorphic analytic differential graded Lie algebras have isomorphic Kuranishi spaces.

5.2. DEFINITION. An analytic differential graded Lie algebra is formal if it is quasi-isomorphic to its cohomology (with zero differential).

The Kuranishi space of an analytic differential graded Lie algebra L with zero differential is the quadratic cone

$$\mathscr{K}_L = \{\eta \in L^1 | [\eta, \eta] = 0\}$$

and we obtain:

5.3. THEOREM. *Suppose (L, d) is a formal analytic differential graded Lie algebra. Then the Kuranishi space of L is isomorphic to the quadratic cone $\mathscr{Q} \subset H^1(L)$ given by*

$$\{\eta \in H^1(L) | [\eta, \eta] = 0\}.$$

5.4. COROLLARY. *Suppose that L is formal and the cup-product on $H^1(L)$ is zero. Then $\mathscr{K}_L \cong H^1(L)$.*

Remark. This corollary will allow us to prove certain deformation spaces are smooth manifolds even though $H^2(L) \neq 0$.

In order to apply the above theorem and its corollary we need to be able to prove that certain differential graded Lie algebras are formal. We give two examples.

Flat bundles over Kähler manifolds.

Let $P \to M$ be a principal G-bundle with a flat connection ω_0 where G is a compact Lie group. Let $\operatorname{ad} P = P \times_G \mathfrak{g}$ denote the $\mathfrak{g}$-bundle associated to P by the adjoint representation. Its complexification $\operatorname{ad} P \otimes \mathbf{C}$ is a flat complex vector bundle with a parallel Riemannian metric (since G is compact). Let $\mathscr{A}^p(M; \operatorname{ad} P)$ denote the vector space of $\operatorname{ad} P$-valued exterior p-forms on M. Let

$$D\colon \mathscr{A}^p(M; \operatorname{ad} P) \to \mathscr{A}^{p+1}(M; \operatorname{ad} P)$$

denote the covariant exterior differential operator corresponding to the flat connection on $\operatorname{ad} P$. Then

$$\Big(\bigoplus_{p \geq 0} \mathscr{A}^p(M; \operatorname{ad} P), D\Big)$$

is a differential graded Lie algebra and $\mathscr{K}$ is the versal deformation space for isomorphism classes of flat connections on P near ω_0. We obtain:

THEOREM. *Let M be a compact Kähler manifold. Then $(\mathscr{A}^*(M; \operatorname{ad} P), D)$ is formal.*

COROLLARY. *$\mathscr{K}$ is isomorphic to a quadratic cone.*

Remark. With a little more work, we deduce that the algebraic variety $\operatorname{Hom}(\pi_1(M), G)$ has quadratic singularities. The hypothesis that G be compact can be considerably weakened: using Corlette's existence theorem for harmonic metrics, Simpson [Si] has proved formality of $\mathscr{A}^*(M; \operatorname{ad} P), D)$ under the hypothesis that the monodromy representation of P is reductive.

Deformations of complex structures.

Let M be a complex manifold of complex dimension n. Then the Kuranishi space of the differential graded Lie algebra

$$\Big(\bigoplus_{q \geq 0} \mathscr{A}^{0,q}(M, T^{1,0}M), \bar\partial\Big)$$

is the versal deformation space for deformations of complex structures on M.

Suppose that M is a compact Kähler manifold which admits a nowhere vanishing holomorphic $(n, 0)$-form ω.

THEOREM. (1) $(\oplus_{q\geq 0}\mathscr{A}^{0,q}(M, T^{1,0}M), \bar\partial)$ *is formal.*
(2) *The cup-product in* H^1 *is zero.*

All the work involved here was done by Tian [Ti] and Todorov [To]. We are merely paraphrasing their proofs, which we now sketch. There is a vector bundle isomorphism

$$\phi\colon T^{1,0}M \to \Lambda^{n-1}T^{1,0}$$

given by

$$\phi(X) = \iota_X(\omega)$$

where ι_X denotes interior multiplication by X. There is thus an induced structure of a differential graded Lie algebra on

$$\left(\bigoplus_{q\geq 0}\mathscr{A}^{n-1,q}(M), \bar\partial\right).$$

The induced bracket is rather mysterious, but Tian and Todorov showed the following:

TIAN-TODOROV LEMMA. *The bracket of a* ∂*-closed element of* $\mathscr{A}^{n-1,q'}$ *and a* ∂*-closed element of* $\mathscr{A}^{n-1,q}$ *is* ∂*-exact.*

The theorem follows immediately from this lemma. Indeed let $\mathfrak{Z}$ be the subspace of $\oplus_{q\geq 0}\mathscr{A}^{n-1,q}(M)$ consisting of ∂-closed forms. By the Tian-Todorov lemma, $\mathfrak{Z}$ is a sub-algebra and its inclusion is a quasi-isomorphism. Let $\mathfrak{B}\subset\mathfrak{Z}$ be the subspace of ∂-exact forms. Again by the Tian-Todorov lemma, $\mathfrak{B}$ is an ideal in $\mathfrak{Z}$ and the projection $\mathfrak{Z}\to\mathfrak{Z}/\mathfrak{B}$ is a quasi-isomorphism. Thus there is a quasi-isomorphism (where $\mathfrak{Z}$ and $\mathfrak{Z}/\mathfrak{B}$ are given the induced differential $\bar\partial$)

$$\bigoplus_{q\geq 0}\mathscr{A}^{n-1,q}(M) \leftarrow \mathfrak{Z} \to \mathfrak{Z}/\mathfrak{B}.$$

LEMMA. *The differential on* $\mathfrak{Z}/\mathfrak{B}$ *induced by* $\bar\partial$ *is zero.*

Proof. Let $\omega\in\mathfrak{Z}$ and $\omega = \bar\partial\tau$ with $\tau\in\mathfrak{Z}$. Then ω is ∂-closed and $\bar\partial$-exact. By the principle of two types [DGMS], there exists η with

$$\omega = \partial\bar\partial\eta.$$

But then ω has zero class in $\mathfrak{Z}/\mathfrak{B}$ and the lemma is proved. ∎

Thus the differential graded Lie algebra $\oplus_{q\geq 0}\mathscr{A}^{n-1,q}(M)$ is formal.

To establish the vanishing of the cup product on H^1 we need only observe that if $\omega\in H^1$ is harmonic, then ω is ∂-closed and consequently the bracket

$[\omega, \omega]$ is ∂-exact by the Tian-Todorov lemma. Since $[\omega, \omega]$ is $\bar{\partial}$-closed, the principle of two types implies that there exists η such that $[\omega, \omega] = \bar{\partial}\partial\eta$ and the class of $[\omega, \omega]$ is zero in H^1. The proof of the theorem is complete.

6. The Kuranishi space of a compact complex parallelizable nilmanifold

Let N be a simply-connected complex nilpotent Lie group of complex dimension n with Lie algebra $\mathfrak{n}$ (we will regard $\mathfrak{n}$ as the Lie algebra of left-invariant vector fields on N). Suppose N admits a cocompact lattice Γ. In this section we will apply our general theory, to determine the Kuranishi space $\mathscr{K}$ parametrizing the versal family of deformations of the complex structure on $M = \Gamma \backslash N$ inherited from that of N. We will also describe the versal family. Let $\bar{\mathfrak{n}}$ denote the algebra obtained by replacing the complex structure on $\mathfrak{n}$ by its conjugate. We recall that $\mathfrak{n}$ is said to have a real structure if $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are isomorphic as complex Lie algebras. The existence of such an isomorphism is equivalent to the existence of a basis for $\mathfrak{n}$ relative to which all structure constants are real. We now state the main theorem of this section.

6.1. THEOREM. *The analytic germ* $(\mathscr{K}, 0)$ *is isomorphic to the germ* $(\mathrm{Hom}_{\mathrm{alg}}(\bar{\mathfrak{n}}, \mathfrak{n}), 0)$, *the germ at* 0 *of the affine variety of complex Lie algebra homomorphisms from* $\bar{\mathfrak{n}}$ *to* $\mathfrak{n}$. *If* $\mathfrak{n}$ *has a real structure this latter variety is isomorphic to the variety* $\mathrm{End}_{\mathrm{alg}}(\mathfrak{n})$ *of Lie algebra endomorphisms of* $\mathfrak{n}$.

We now prove the theorem.

Let $L_2 = \oplus_{q=0}^{n} \mathscr{A}^{0,q}(M, T^{1,0})$ and let L_1 be the subalgebra consisting of the projections to M of the left-invariant elements of $\oplus_{q=0}^{n} \mathscr{A}^{0,q}(N, T^{1,0})$. We have vector space isomorphisms

$$L_1^q = \mathrm{Hom}_{\mathbf{C}}(\Lambda^q \bar{\mathfrak{n}}, \mathfrak{n}), \quad q = 0, 1, \dots, n.$$

We will henceforth regard $\mathfrak{n}$ as the space of left-invariant complex vector fields on N of type $(1,0)$ and $\bar{\mathfrak{n}}$ as the left-invariant fields of type $(0,1)$. We define a differential $\bar{D}$ on L_1 for $\omega \in \mathrm{Hom}_{\mathbf{C}}(\Lambda^p \bar{\mathfrak{n}}, \mathfrak{n})$ and $X_1, \dots, X_{p+1} \in \mathfrak{n}'$ by

$$\bar{D}\omega(X_1, \dots, X_{p+1}) = \sum_{i<j} (-1)^{ij} \omega\big([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{p+1}\big)$$

and a bracket $[\ ,\ ]' \colon L_1^p \otimes L_1^q \to L_1^{p+q}$ by

$$[\omega, \eta]' = \frac{1}{p!q!} \sum_{\sigma} \varepsilon(\sigma) \big[\omega(X_{\sigma(1)}, \dots, X_{\sigma(p)}), \eta(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)})\big].$$

6.2. Lemma. *The differential $\overline{D}$ and the bracket $[\ ,\ ]'$ are the ones induced on L_1 by the inclusion $j\colon L_1 \to L_2$. Consequently $j\colon (L_1, [\ ,\ ]', \overline{D}) \to (L_2, [\ ,\ ], \bar\partial)$ is a homomorphism of differential graded Lie algebras.*

Proof. We first prove the statement for the differential. Let $P\colon T(M) \otimes \mathbf{C} \to T^{1,0}(M)$ be the projection given by

$$P = \tfrac{1}{2}[I - iJ].$$

We then have the well-known formula for $\bar\partial\omega$ with $\omega \in L_2^q$:

$$\begin{aligned}\bar\partial\omega(X_1, \ldots, X_{q+1}) &= P\left(\sum_{i=1}^{q+1} (-1)^{i-1}\left[X_i, \omega\left(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1}\right)\right]\right)\\ &\quad + \sum_{i<j} (-1)^{i+j}\omega\left(\left[X_i, X_j\right], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1}\right).\end{aligned}$$

But if $\omega \in L_1^q$ and $X_1, \ldots, X_{q+1}$ are in $\bar{\mathfrak{n}}$ then $\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1}) \in \mathfrak{n}$ and the terms in the first summand are all zero since the elements of $\mathfrak{n}$ and the elements of $\bar{\mathfrak{n}}$ commute. We obtain the required formula for the differential.

To compute the induced bracket we use the following formula [Ni] for the bracket of two elements $\omega \in L_2^p$ and $\eta \in L_2^q$ evaluated on arbitrary smooth vector fields of type $(0,1)$:

$$\begin{aligned}[\omega, \eta](X_1, \ldots, X_{p+q}) &= \sum_\sigma \varepsilon(\sigma)\left[\omega\left(X_{\sigma(1)}, \ldots, X_{\sigma(p)}\right), \eta\left(X_{\sigma(p+1)}, \ldots, X_{\sigma(p)}\right)\right]\\ &\quad + \frac{(-1)^{pq+q+1}}{(p-1)!q!}\sum_\sigma \varepsilon(\sigma)\,\omega\left(\left[X_{\sigma(1)}, \eta\left(X_{\sigma(2)}, \ldots, X_{\sigma(q+1)}\right)\right],\right.\\ &\qquad\qquad\qquad\qquad \left. X_{\sigma(q+1)}, \ldots, X_{\sigma(pq)}\right)\\ &\quad + \frac{(-1)^{p}}{p!(q-1)!}\sum_\sigma \varepsilon(\sigma)\,\eta\left(\left[X_{\sigma(1)}, \omega\left(X_{\sigma(2)}, \ldots, X_{\sigma(p+1)}\right)\right],\right.\\ &\qquad\qquad\qquad\qquad \left. X_{\sigma(p+2)}, \ldots, X_{\sigma(p+q)}\right).\end{aligned}$$

If the X_i's are now assumed to be left-invariant and ω and η are in L_1 then the second and third terms contain brackets of elements of $\bar{\mathfrak{n}}$ (namely $X_{\sigma(1)}$) with elements of $\mathfrak{n}$ (the values of ω or η) and consequently are zero. ∎

We now establish that j is a quasi-isomorphism.

6.3. THEOREM. *The inclusion j: $L_1 \to L_2$ induces an isomorphism of cohomology.*

We begin by reducing the theorem to a result about the usual Dolbeault cohomology groups. The natural map α: $\mathscr{A}^{0,q}(M) \otimes \mathfrak{n} \to L_2^q$ given (on decomposables) by

$$\alpha(\omega \otimes X)(v_1, \dots, v_q) = \omega(v_1, \dots, v_q) X$$

is an isomorphism of vector spaces (but not of Lie algebras). We see that under the isomorphism α the differential $\bar{\partial} \otimes 1$ goes to $\bar{\partial}$. We obtain a commutative diagram of complexes:

$$\begin{array}{ccc} \left(\bigoplus_{q=0}^{n} \mathscr{A}^{0,q}(M) \otimes \mathfrak{n}, \bar{\partial} \otimes 1 \right) & \xrightarrow{\alpha} & (L_2, \bar{\partial}) \\ \uparrow & & \uparrow j \\ (\Lambda^{0,q}(\bar{\mathfrak{n}})^* \otimes \mathfrak{n}, \bar{\partial} \otimes 1) & \xrightarrow{\mathrm{id}} & (L_1, \bar{D}). \end{array}$$

The vertical arrows are the inclusions of the projections to M of left-invariant forms on N.

We see then that the theorem will follow from the corresponding result for the Dolbeault algebra of scalar-valued differential forms of type $(0, q)$ for $0 \leq q \leq n$. This result is proved in Lemma 6.5 and is based on an unpublished version proved by Richard Penney. Our proof is a modification of his more analytical one. We will use the term "complex torus" to denote a compact connected complex Lie group—necessarily the quotient of $\mathbf{C}^m$ by a cocompact lattice. Jim Carrell has pointed out that there are holomorphic vector fields on compact complex manifolds M (e.g., the Iwasawa manifold) acting non-trivially on $H^0(M, \Omega^1)$. Thus some extra "compactness" condition on G is necessary.

6.4. LEMMA. *Let M be a compact complex manifold and G a connected complex Lie group acting on M. Assume that the action $G \times M \to M$ is holomorphic in both variables and that G acts by isometries of a hermitian metric h on M. Then G acts trivially on all the sheaf cohomology groups $H^p(M, \Omega^q)$.*

Proof. It suffices to prove the Lie algebra $\underline{\mathfrak{g}}$ of G acts trivially. We first observe that any anti-holomorphic vector field $\bar{Z}$ acts trivially on $\bar{\partial}$-cohomology. This follows immediately from the formula

$$\theta(\bar{Z}) = \iota(\bar{Z}) \circ \bar{\partial} + \bar{\partial} \circ \iota(\bar{Z})$$

which follows in turn by a comparison of types in the usual Cartan formula

$$\theta(\bar{Z}) = \iota(\bar{Z}) \circ d + d \circ \iota(\bar{Z}).$$

It remains to check that the holomorphic vector fields tangent to G act trivially on the above cohomology groups. But since G preserves the metric h, G acts by unitary transformations on $\mathscr{H}^{p,q}(M)$, the space of harmonic (for the $\bar{\partial}$-Laplacian associated to h) forms on M of type (p, q). Hence the Lie algebra $\mathfrak{g}$ acts by skew-hermitian transformations on $\mathscr{H}^{p,q}(M)$ and extends to a Lie algebra homomorphism ρ: $\mathfrak{g} \otimes \mathbf{C} \to \mathrm{End}(\mathscr{H}^{p,q}(M))$. If $X \in \mathfrak{g}$ we let $\hat{X}$ denote the corresponding vector field on M. Since the action is holomorphic in the first variable we have $J\hat{X} = \widehat{JX}$. Now consider $Z = \hat{X} - iJ\hat{X} = \hat{X} - i\widehat{JX}$. Then $\rho(Z) = \rho(\hat{X}) - i\rho(\widehat{JX})$ and the adjoint $\rho(Z)^*$ of $\rho(Z)$ is given by

$$\begin{aligned} \rho(Z)^* &= \rho(\hat{X})^* + i\rho(\widehat{JX})^* \\ &= -\rho(\hat{X}) - i\rho(\widehat{JX}) \\ &= -\rho(\bar{Z}) \end{aligned}$$

where $\bar{Z} = \hat{X} + iJ\hat{X}$. But we have seen that $\rho(\bar{Z}) = 0$ and the lemma follows.

COROLLARY. *Let M be a compact complex manifold and G a complex torus acting on M such that the action map $G \times M \to M$ is holomorphic in both variables. Then G acts trivially on all the cohomology groups $H^p(M, \Omega^q)$.*

6.5. LEMMA. *Let $A_2 = \oplus_{q=0}^n \mathscr{A}^{0,q}(M)$ be the Dolbeault algebra and let $A_1 = \oplus_{q=0}^n A_1^q$ where A_1^q is the subspace of projections to M of the left-invariant $(0, q)$-forms on N. Then the inclusion*

$$j\colon (A_1, \bar{\partial}) \to (A_2, \bar{\partial})$$

induces an isomorphism of cohomology.

Proof. By Matsushima [M], the center $Z(N)$ of N is connected and $\Gamma_1 = Z(N) \cap \Gamma$ is cocompact in $Z(M)$. Thus $T = \Gamma_1 \backslash Z(N)$ is a complex torus which acts freely on M. We obtain a holomorphic principal fibration $T \to M \to M/T$ with base a compact nilmanifold of dimension strictly less than that of M.

In order to adapt the inductive argument of Nomizu [No] to our context it suffices to show that the inclusion

$$i\colon A_2^T \to A_2$$

of the T-invariant elements A_2^T in A_2 induces an isomorphism of cohomology.

But this follows from the previous lemma. Nomizu's argument now applies. ■

By Theorem 4.1 there is an analytic equivalence from $(\mathscr{X}_{L_1}, 0)$ to $(\mathscr{X}_{L_2}, 0)$. It remains to determine $\mathscr{X}_{L_1}$. We have seen that

$$(L_1)^1 = (\bar{\mathfrak{n}})^* \otimes \mathfrak{n} = \mathrm{Hom}_{\mathbf{C}}(\bar{\mathfrak{n}}, \mathfrak{n}).$$

Since $(L_1)^1$ is finite dimensional we can replace $\mathscr{X}_1$ by Y_1. Now we have $(L_1)^0 = H^0(L_1)$ so there are no 1-boundaries. Hence the linear complement $A^1(L_1)$ is all of $(L_1)^1$ and we have

$$Y_1 = \left\{\omega \in \mathrm{Hom}_{\mathbf{C}}(\bar{\mathfrak{n}}, \mathfrak{n}) : \bar{D}\omega + \tfrac{1}{2}[\omega, \omega] = 0\right\}.$$

6.6. LEMMA. *Y_1 is complex analytically isomorphic to* $\mathrm{Hom}_{\mathrm{alg}}(\bar{\mathfrak{n}}, \mathfrak{n})$, *the affine variety of Lie algebra homomorphisms from $\bar{\mathfrak{n}}$ to $\mathfrak{n}$.*

Proof. We have $\omega \in Y_1$ if and only if

$$\bar{D}\omega + \tfrac{1}{2}[\omega, \omega] = 0.$$

But $X, Y \in \bar{\mathfrak{n}}$ we have

$$\begin{aligned} \bar{D}\omega(X, Y) &= -\omega([X, Y]), \\ [\omega, \omega](X, Y) &= 2[\omega(X), \omega(Y)]. \end{aligned}$$

Thus the above equations hold if and only if for all $X, Y \in \bar{\mathfrak{n}}$ we have

$$\omega([X, Y]) = [\omega(X), \omega(Y)]$$ ■

We complete the proof of Theorem 6.1 by observing that a real structure σ: $\mathfrak{n} \to \bar{\mathfrak{n}}$ induces an analytic equivalence

$$\sigma^* : \mathrm{Hom}_{\mathrm{alg}}(\bar{\mathfrak{n}}, \mathfrak{n}) \to \mathrm{Hom}_{\mathrm{alg}}(\mathfrak{n}, \mathfrak{n})$$

given by

$$\sigma^* T = T \circ \sigma.$$

We now consider some examples of analytic germs that can be obtained in this way.

Quadratic cones.

Let $\mathfrak{n}$ be the (Heisenberg) Lie algebra generated by $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ subject to these relations:

(i) all three-fold brackets are zero;
(ii) $[X_i, Y_i] = [X_j, Y_j]$ all i, j;
(iii) $[X_i, Y_j] = 0$, $i \neq j$;
(iv) $[X_i, X_j] = 0$;
(v) $[Y_i, Y_j] = 0$.

Then $\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{n}, \mathfrak{n})$ is clearly the homogeneous quadratic cone consisting of all elements $X_1', \ldots, X_n', Y_1', \ldots, Y_n'$ in $\mathfrak{n}^{2n}$ satisfying (ii), (iii), (iv) and (v). We thank Robert Steinberg for pointing out this example.

Cones of degree n.

Let $\mathfrak{n}$ be the Lie algebra generated by X, Y satisfying the relations:
(i) all $(n + 1)$-fold brackets are zero;
(ii) $\mathrm{ad}^{n-1}X(Y) = \mathrm{ad}^{n-1}Y(X)$.

Then $\mathrm{Hom}_{\mathrm{alg}}(\mathfrak{n}, \mathfrak{n})$ is the cone of degree n consisting of $X', Y' \in \mathfrak{n}^2$ satisfying (ii).

It is clear that by prescribing $\mathfrak{n}$ by generators and relations as above one can write down a very large number of germs that are Kuranishi spaces of complex manifolds. In particular the above examples show that the Kodaira-Spencer algebra is not always formal.

We conclude this section by describing the versal deformation (X_2, π_2, ι_2) of M (see the appendix for the definition). We first form the versal deformation of M in the space of complex structures on M coming from left-invariant complex structures on N. We will call such structures locally left-invariant. We obtain the latter deformation (X_1, π_1, ι_1) by forming the product $\mathrm{Hom}_{\mathrm{alg}}(\bar{\mathfrak{n}}, \mathfrak{n}) \times M$. We then use id: $\mathrm{Hom}_{\mathrm{alg}}(\bar{\mathfrak{n}}, \mathfrak{n}) \to \mathrm{Hom}_{\mathrm{alg}}(\bar{\mathfrak{n}}, \mathfrak{n})$ to twist the product complex structure (see the appendix). Intuitively one changes the product almost complex structure at (A, m) for $A \in \mathrm{Hom}_{\mathrm{alg}}(\bar{\mathfrak{n}}, \mathfrak{n})$ and $m \in M$ by using A to change the almost complex structure on the tangent vectors to M at m. Clearly we have a morphism of deformations

$$\phi\colon (X_1, \pi_1, \iota_1) \to (X_2, \pi_2, \iota_2).$$

But we have just seen that ϕ induces an isomorphism on the base. By Lemma 7.6 of the appendix we find that ϕ is an isomorphism. As a consequence we obtain the following theorem.

6.7. THEOREM. *Any complex structure on M sufficiently close to the locally bi-invariant one is isomorphic to a locally left-invariant complex structure.*

Appendix: The versal deformation of a compact complex manifold

The basic reference for this appendix is [Ku2].

7.1. DEFINITION. Let M be a compact manifold. Let (S, s_0) be the germ of a complex analytic space. A deformation of M over (S, s_0) is a triple (X, π, ι) where X is a complex analytic space, $\pi\colon X \to S$ is a proper smooth holomorphic mapping and $\iota\colon M \to X$ is an embedding which induces an isomorphism from M onto $\pi^{-1}(s_0)$.

The hypothesis that π is smooth means that every point $x \in X$ has a local product neighborhood in X with respect to π.

7.2. DEFINITION. Let (X_1, π_1, ι_1) and (X_2, π_2, ι_2) be two deformations of M over (S, s_0). Then a morphism $\phi = (\tilde{f}, f)$ from (X_1, π_1, ι_1) to (X_2, π_2, ι_2) is a diagram

$$\begin{array}{ccc} X_1' & \xrightarrow{\tilde{f}} & X_2' \\ {\scriptstyle \pi_2}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ (S_1', s_0) & \xrightarrow{f} & (S_2', s_0) \end{array}$$

such that $\tilde{f} \circ \iota_1 = \iota_2$ where S_i' is a neighborhood of s_0 in S_i and $X_i' = \pi_i^{-1}(S_i')$ for $i = 1, 2$.

If $f\colon (S, s_0) \to (T, t_0)$ is a morphism and (X, π, ι) is a deformation of M over (T, t_0) then we may define a deformation $(f^*X, f^*\pi, f^*\iota)$ in the usual way as the fiber product of f and π.

7.3. DEFINITION. A deformation (X, π, ι) of M over (T, t_0) is *complete* if given any deformation (X', π', ι') of M over (S, s_0) there exists a neighborhood S' of s_0, a morphism $f\colon (S', s_0) \to (T, t_0)$ and an isomorphism from (X', π', ι') to $(f^*X, f^*\pi, f^*\iota)$.

We call f a classifying map for the deformation (X', π', ι'). We do not require f to be unique.

7.4. DEFINITION. A complete deformation (X, π, ι) of M over T is *versal* if any two classifying maps for a deformation (X', π', ι') of M over (S, s_0) have the same derivative at s_0.

Remark. In what follows we will frequently use the result that an endomorphism of an analytic local **k**-algebra inducing an automorphism of the Zariski tangent space is an automorphism.

7.5. LEMMA. *Any two versal deformations of M are isomorphic.*

Proof. Let (X, π, ι) and (X', π', ι') be two versal deformations of M. Then we have classifying maps f and g for X and X' respectively and diagrams

$$\begin{array}{ccc} X & \xrightarrow{\tilde f} & X' \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & S', \end{array} \qquad \begin{array}{ccc} X' & \xrightarrow{\tilde g} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S. \end{array}$$

It suffices to prove that $\tilde f$ maps a neighborhood of $\pi^{-1}(s_0)$ in X isomorphically onto a neighborhood of $(\pi')^{-1}(s_0')$ in X'.

We have a diagram (with $\tilde h = \tilde f \circ \tilde g$ and $h = f \circ g$)

$$\begin{array}{ccc} X' & \xrightarrow{\tilde h} & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{h} & S'. \end{array}$$

Since $\tilde h \circ \iota = \iota$ we have $\tilde h | \pi^{-1}(s_0) = \mathrm{id}$ and $dh(s_0) = \mathrm{id}$. Thus if $x \in (\pi')^{-1}(s_0')$ the map $\tilde h$ leaves x fixed and induces a map of the analytic local ring $\mathcal{O}_{X', x}$ which induces the identity on the Zariski tangent space. By the remark above there exists a neighborhood U_x' of x in X' which is mapped isomorphically onto itself. We obtain a neighborhood U' of $\pi^{-1}(s_0)$ in X' such that $\tilde h$ carries U' onto itself and is a local isomorphism. By shrinking U' we may assume $\tilde h$ is injective (since $\tilde h$ is injective on $\pi^{-1}(s_0)$). Finally we may assume U' is the inverse image of a neighborhood of s_0' in S'.

Now consider the morphism $\tilde f$: $X \to X'$. We replace X' by U' and X by $\tilde f^{-1}(U')$. Then

$$\tilde f \circ \tilde g \circ \tilde h^{-1} = \mathrm{id}$$

so $\tilde f$ has a right inverse $\tilde k = \tilde g \circ \tilde h^{-1}$. Hence $\tilde f$ is onto. But an argument analogous to that of the paragraph above produces a neighborhood U of $\pi^{-1}(s_0)$ such that $(\tilde g \circ \tilde f)|U$ is invertible. Hence $\tilde f|U$ is injective. But clearly $\tilde f|U$ surjects onto $\tilde k^{-1}(U)$. We replace X' by $\tilde k^{-1}(U)$.

COROLLARY. *The parametrizing germs (S, s_0) and (S', s_0') for any two versal deformations of M are isomorphic.*

We observe that the method of proof of the previous lemma can be used to prove the following.

7.6. LEMMA. *Suppose $\phi = (\tilde{f}, f)$ is a morphism of deformations with f an isomorphism of parametrizing germs. Then ϕ is an isomorphism.*

Proof. Clearly the induced deformation f^*X' is isomorphic to X' and it suffices to compare X and f^*X'; that is, we may assume $S' = S$ and $f = \mathrm{id}$. We have a diagram

$$\begin{tikzcd} X \arrow[rr, "\tilde{f}"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}.$$

Let $x \in \pi^{-1}(s_0)$ and so $\tilde{f}(x) \in \pi^{-1}(s_0)$. The analytic local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X',\tilde{f}(x)}$ are isomorphic and $d\tilde{f}$ induces an isomorphism of Zariski tangent spaces. Thus by shrinking X and X' we may assume $\tilde{f}$ is a local isomorphism, in particular it is an open map. But there exists a neighborhood U of $\pi^{-1}(s_0)$ such that $f|U$ is injective. We replace X by U and X' by $f(U)$. ∎

In 1962, Kuranishi [Ku1] proved the existence of a versal deformation of any compact complex manifold. We will explain the idea of the proof of Kuranishi's theorem in order to make the connection with differential graded Lie algebras and the material of §2 of this paper.

Let L be the Kodaira-Spencer algebra and $\mathscr{F}$ be the subset of L^1 given by

$$\mathscr{F} = \left\{\eta \in L^1 \colon \bar{\partial}\eta + \tfrac{1}{2}[\eta, \eta] = 0\right\}.$$

7.7. DEFINITION. Let (S, s_0) be an analytic germ. An analytic family of elements of $\mathscr{F}$ parametrized by (S, s_0) is a map $\eta\colon S \to L^1$ such that η is a real analytic section of the pull-back of $T^{0,1}(M)^* \otimes T^{1,0}(M)$ to $M \times S$ which is complex analytic in S.

We now describe briefly how an analytic family of elements of $\mathscr{F}$ parametrized by (S, s_0) induces a deformation of M with parametrizing germ (S, s_0). Form the product analytic space $M \times S$. Let η be an analytic family as above. Twist the product structure on $M \times S$ using η via the complex Frobenius theorem with parameters as explained in [Ku2], Chapter VII. We obtain the required deformation of M which we denote $M \times_\eta S$. Now let $F^{-1}\colon \mathscr{K}_L \to Y_L$ be the Kuranishi family described in §2. Then F^{-1} is an analytic family [Ku2, p. 82]. We put $X = M \times_{F^{-1}} \mathscr{K}_L$, let π be the projection onto the second factor and $\iota\colon M \to X$ be the inclusion $\iota(x) = (x, 0)$.

7.8. THEOREM (KURANISHI). *(X, π, ι) is a versal deformation of M.*

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