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THE FRÖLICHER SPECTRAL SEQUENCE FOR COMPACT NILMANIFOLDS

BY

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1. Introduction

In 1955 A. Frölicher [Fr] defined a spectral sequence $\{E_r\}$ for any complex manifold M. We call it the *Frölicher spectral sequence*, but it is sometimes known as the Hodge-deRham spectral sequence. Nowadays the construction of this spectral sequence is standard, once one notes that the complex differential forms on M form a differential graded $\mathbf{C}$-module.

For a compact manifold with positive definite Kähler metric Frölicher observed that $E_1 \cong E_\infty$; Kodaira [Kod] proved that the same conclusion holds for any compact complex surface. The Iwasawa manifold I_3 (defined as the quotient of the complex Heisenberg group by the Gaussian integers) has a nonclosed holomorphic 1-form, and so $E_1^{1,0}(I_3) \not\cong E_2^{1,0}(I_3)$. Nevertheless $E_2(I_3) \cong E_\infty(I_3)$; more generally it follows from a result of Sakane (see Theorem 9 for a proof) that $E_2(M)) \cong E_\infty(M)$ for a compact complex parallelizable nilmanifold M. (A well-known result of Wang [Wa] asserts that any compact complex parallelizable manifold is the quotient of a complex Lie group by a discrete subgroup.)

In spite of the fact that Frölicher's paper has been in existence for more than 30 years, until recently no examples of complex manifolds for which $E_2 \not\cong E_\infty$ seem to have been known (see [GH, page 444]). In our note [CFG] we found compact complex manifolds of complex dimension at least 4 for which $E_2 \not\cong E_3 \cong E_\infty$. Since our examples are compact nilmanifolds, they are never simply connected. H. Pittie [Pi] has found some compact simply connected examples, the simplest of which is *Spin*(9). All of Pittie's complex manifolds must have much larger dimensions than ours.

The principal fact that led us to our examples is the observation that there are many compact nilmanifolds which possess complex structures but are not complex parallelizable. Such a manifold M is real parallelizable, however, and moreover both the deRham operator d and the Dolbeault operator $\bar{\partial}$ have explicit descriptions in terms of a canonical parallelization. Although the calculations become complicated when the dimension of M is large, it is

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possible in principle to compute the deRham cohomology, the Dolbeault cohomology and as we show below, the Frölicher spectral sequence of M.

The problem we begin to tackle in the present paper is to find for any r a compact complex manifold for which $E_r \not\cong E_\infty$. In [CFG] we did it for $r = 2$ and in the present paper we do it for $r = 3$.

The Frölicher spectral sequence of a compact complex manifold can be a very complicated object. In Section 2 we write down some useful properties of the Frölicher spectral sequence; some of these are well known general properties of spectral sequences that are especially simple for the Frölicher spectral sequence, but others are peculiar to compact complex manifolds. Fortunately to produce our examples we need only to calculate the terms $E_r^{p,0}$. It turns out that for any compact complex manifold M of complex dimension n one has

$$E_1^{n,0}(M) \cong E_\infty^{n,0}(M) \tag{1}$$

(see Section 2 for a proof). On the other hand for $r \neq s$ frequently $E_r^{p,0} \neq E_s^{p,0}$ when $1 \leq p \leq n-1$. Since the proof of (1) makes use of the assumption that M is compact, it is a special feature of the Frölicher spectral sequence.

A fact from the general theory of spectral sequences is that $E_r^{p,q} \cong E_\infty^{p,q}$ whenever $r > \max(p, q+1)$. So for an example with $E_r^{p,0} \not\cong E_\infty^{p,0}$ we must have $n > p \geq r$.

We wish to thank P. Gilkey for some useful discussions.

2. The Frölicher spectral sequence

Let us recall briefly the definition of the Frölicher spectral sequence associated to a complex manifold M of complex dimension n. Denote by $\Lambda^{r,s}$ the complex differential forms of type (r, s), and let $d = \partial + \bar{\partial}$ be the usual decomposition of the exterior differential. We put

$$F^p\Lambda = \left\{\varphi = \sum \varphi_{r,s} \in \Lambda \,|\, \varphi_{r,s} = 0 \text{ for } r < p\right\},$$

where $\Lambda = \Sigma\Lambda^{r,s}$ and $\varphi_{r,s}$ denotes the component of φ in $\Lambda^{r,s}$. Then $\{F^p\Lambda\}$ is a decreasing filtration of the complex forms on M such that $d(F^p\Lambda) \subset F^p\Lambda$. The condition "$\varphi_{r,s} = 0$ for $r < p$" can be read as "φ has dz-degree $\geq p$".

Thus (Λ, d) is a filtered differential graded module over $\mathbf{C}$. The Frölicher spectral sequence of M is then the spectral sequence associated with (Λ, d) in the standard way (see [McC, page 33–36] or Frölicher's original paper).

More precisely let

$$\begin{aligned} Z_r^{p,q} &= F^p\Lambda^{p+q} \cap d^{-1}(F^{p+r}\Lambda^{p+q+1}), \\ B_r^{p,q} &= F^p\Lambda^{p+q} \cap d(F^{p-r}\Lambda^{p+q-1}), \\ Z_\infty^{p,q} &= F^p\Lambda^{p+q} \cap d^{-1}(0), \\ B_\infty^{p,q} &= F^p\Lambda^{p+q} \cap d(\Lambda^{p+q-1}). \end{aligned}$$

Thus we have

$$\cdots \subseteq B_{r-1}^{p,q} \subseteq B_r^{p,q} \subseteq \cdots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \cdots \subseteq Z_r^{p,q} \subseteq Z_{r-1}^{p,q} \subseteq \cdots$$

for all r. The terms $E_r^{p,q}$ in the Frölicher spectral sequence are then defined by

$$E_r^{p,q} = \frac{Z_r^{p,q}}{Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}}; \tag{2}$$

also we put

$$E_\infty^{p,q} = \frac{Z_\infty^{p,q}}{Z_\infty^{p+1,q-1} + B_\infty^{p,q}}. \tag{3}$$

Insight into the meaning of the Frölicher spectral sequence for a general complex manifold can be gained by doing the case $q = 0$. It turns out that some forms of type $(p, 0)$ are more exact than others; we can use the Frölicher spectral sequence to measure this exactness. Let us compare the $E_\infty^{p,0}$ with the $E_r^{p,0}$.

LEMMA 1. *We have*

$$E_1^{p,0} = \{\alpha \in \Lambda^{p,0} | \bar\partial\alpha = 0\}, \tag{4}$$

$$E_r^{p,0} = \frac{\{\alpha \in \Lambda^{p,0} | d\alpha = 0\}}{\{\alpha \in \Lambda^{p,0} | \alpha = d(\beta_{p-1,0} + \beta_{p-2,1} + \cdots + \beta_{p-r+1,r-2})\}} \tag{5}$$

for $r \geq 2$, *and*

$$E_r^{p,0} \cong E_\infty^{p,0} \cong \frac{\{\alpha \in \Lambda^{p,0} | d\alpha = 0\}}{\{\alpha \in \Lambda^{p,0} | \alpha = d\sigma \text{ for some } \sigma \in \Lambda\}} \tag{6}$$

for $r \geq p + 1$.

Proof. The proof of Lemma 1 is an easy consequence of the definitions. For example for $r \geq 2$ we have

$$\begin{aligned} Z_r^{p,0} &= F^p\Lambda^p \cap d^{-1}(F^{p+r}\Lambda^{p+1}) \\ &= F^p\Lambda^p \cap d^{-1}(0) = Z_\infty^{p,0}, \\ Z_{r-1}^{p+1,-1} &\subseteq F^{p+1}\Lambda^p = \{0\}, \\ B_r^{p,0} &= F^p\Lambda^p \cap d(F^{p-r}\Lambda^{p-1}), \end{aligned}$$

and so

$$E_r^{p,0} = \frac{Z_\infty^{p,0}}{F^p\Lambda^p \cap d(F^{p-r}\Lambda^{p-1})},$$

which is another way of writing (5). Equations (4) and (6) are proved similarly.

COROLLARY 2. *There is a sequence of epimorphisms*

$$E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_r^{p,0} \twoheadrightarrow \cdots.$$

Corollary 2 is a well-known fact about spectral sequences; because of Lemma 1 the proof of Corollary 2 using the Frölicher spectral sequence is particularly transparent.

The Frölicher spectral sequence is a first quadrant spectral sequence; Lemma 1 and Corollary 2 give special information about its lower edge. There is corresponding information about the vertical edge.

LEMMA 3. *For $r \geq 1$ we have*

$$E_r^{0,q} = \frac{\Lambda^q \cap d^{-1}(F^r\Lambda^{q+1})}{F^1\Lambda^q \cap d^{-1}(F^r\Lambda^{q+1}) + d(\Lambda^{q-1})}. \tag{7}$$

Proof. We have

$$\begin{aligned} E_r^{0,q} &= \frac{Z_r^{0,q}}{Z_{r-1}^{1,q-1} + B_{r-1}^{0,q}} \\ &= \frac{F^0\Lambda^q \cap d^{-1}(F^r\Lambda^{q+1})}{F^1\Lambda^q \cap d^{-1}(F^r\Lambda^{q+1}) + F^0\Lambda^q \cap d(F^{1-r}\Lambda^{q-1})} \\ &= \frac{\Lambda^q \cap d^{-1}(F^r\Lambda^{q+1})}{F^1\Lambda^q \cap d^{-1}(F^r\Lambda^{q+1}) + F^0\Lambda^q \cap d(\Lambda^{q-1})}. \end{aligned}$$

COROLLARY 4. *There is a sequence of injective homomorphisms*

$$\cdots \hookrightarrow E_{r+1}^{0,q} \hookrightarrow E_r^{0,q} \hookrightarrow \cdots \hookrightarrow E_2^{0,q} \hookrightarrow E_1^{0,q}.$$

In the next lemma we make essential use of the assumption that M is a compact complex manifold.

LEMMA 5. *Let M be a compact complex manifold of complex dimension n. Then for all r,*

$$E_r^{n,0}(M) \cong E_\infty^{n,0}(M) \cong \{\alpha \in \Lambda^{n,0} | d\alpha = 0\}. \tag{8}$$

Proof. We use the fact that there is a positive definite Hermitian inner product (,) on $\Lambda^{n,0}$ defined by

$$(\vartheta, \varphi) = (\sqrt{-1})^{n^2} \int_M \vartheta \wedge \bar{\varphi}. \tag{9}$$

Let $r \geq 2$. In view of Lemma 1 it suffices to prove that any exact form in $\Lambda^{n,0}$ vanishes. In fact if $\beta \in \Lambda$ is such that $d\beta \in \Lambda^{n,0}$ then

$$0 \leq (d\beta, d\beta) = (\sqrt{-1})^{n^2} \int_M d\beta \wedge \overline{d\beta} = (\sqrt{-1})^{n^2} \int_M d(\beta \wedge \overline{d\beta}) = 0$$

by Stokes' theorem. Since (,) is positive definite, we get $d\beta = 0$ in the case that $r \geq 2$. On the other hand any holomorphic n-form is closed so that (8) also holds when $r = 1$.

3. Lie groups with left invariant complex structures

Our examples are all constructed using real nilpotent Lie algebras with complex structures. Let G be a real Lie group. Instead of describing the Lie algebra $\mathfrak{g}$ of G in terms of its bracket we use the exterior differential on the dual space $\mathfrak{g}^*$. The two are equivalent because of the formula

$$d\alpha(X,Y) = -\alpha([X,Y]),$$

where α is any 1-form on $\mathfrak{g} \otimes \mathbf{C}$ and $X, Y \in \mathfrak{g} \otimes \mathbf{C}$.

Now suppose that G has an almost complex structure J. We choose a $\mathbf{C}$ basis $\{\omega_1, \ldots, \omega_n\}$ for the complex forms on $\mathfrak{g} \otimes \mathbf{C}$; then

$\{\omega_1, \dots, \omega_n, \overline{\omega}_1, \dots, \overline{\omega}_n\}$ is a real basis of $\mathfrak{g}^*$. The structure equations (equivalent to the bracket) have the form

$$(10)\qquad d\omega_i = \sum_{j<k} A_{ijk}\omega_j \wedge \omega_k + \sum_{j,k} B_{ijk}\omega_j \wedge \overline{\omega}_k + \sum_{j<k} C_{ijk}\overline{\omega}_j \wedge \overline{\omega}_k,$$

for $1 \le i \le n$. In general such an almost complex structure on G will not be integrable because of the presence of $(0, 2)$-forms on the right hand side of (10). However there is a simple criterion for a Lie group to have a left invariant complex structure:

THEOREM 6. *Let G be a Lie group with a left invariant almost complex structure. Then the structure equations for the Lie algebra $\mathfrak{g}$ of G have the form*

$$(11)\qquad d\omega_i = \sum_{j<k} A_{ijk}\omega_j \wedge \omega_k + \sum_{j,k} B_{ijk}\omega_j \wedge \overline{\omega}_k \quad (1 \le i \le n).$$

Conversely the structure equations (11) *define a Lie group G with left invariant complex structure. Hence quotients of G have complex structures.*

Proof. Defining an almost complex structure J on G is equivalent to saying that $\{\omega_1, \dots, \omega_n\}$ form a basis for the $(1, 0)$ forms on G. Since on a complex manifold the differential of a $(1, 0)$-form must be the sum of forms of types $(2, 0)$ and $(1, 1)$, we see that a necessary condition for an almost complex structure J be integrable is that all the coefficients C_{ijk} in (10) vanish. Moreover this condition is also sufficient. (See for example [KN, volume 2, Theorem 2.8].)

COROLLARY 7. *The structure equations for the Lie algebra $\mathfrak{g}$ of a nilpotent Lie group G with a left invariant almost complex structure have the form*

$$(12)\qquad d\omega_i = \sum_{j<k<i} A_{ijk}\omega_j \wedge \omega_k + \sum_{j,k<i} B_{ijk}\omega_j \wedge \overline{\omega}_k \quad (1 \le i \le n).$$

Conversely the structure equations (12) *define a nilpotent Lie group G with left invariant complex structure.*

Proof. The condition that G be nilpotent is equivalent to the condition that each $d\omega_i$ is a linear combination of wedge products of the ω_j's and their conjugates with $j < i$. Combining this fact with Corollary 7 we get the required result.

COROLLARY 8. *Let G be a nilpotent Lie group for which the coefficients A_{ijk} and B_{ijk} in structure equations* (12) *are integers. Then there is a discrete subgroup Γ of G such that the quotient G/Γ is a compact complex manifold that is real parallelizable.*

Proof. A well-known result of Mal'čev [Ma] implies that a nilpotent Lie group G has compact quotients provided there exists a basis of the 1-forms such that the coefficients in the structure equations are integers. It is easy to see that the left invariant forms descend to the quotient, and consequently give rise to a real parallelization.

If G has a left invariant almost complex structure $\tilde{J}$, it also descends to an almost complex structure J on the quotient. Moreover J is integrable if and only if $\tilde{J}$ is.

The complex parallelizable manifolds of Wang [Wa] are the quotients of Lie groups for which in (11) the B_{ijk}'s vanish and the A_{ijk}'s are holomorphic. However there are many other complex manifolds that are real parallelizable but not complex parallelizable. The simplest such example (see [FGG]) is the Kodaira-Thurston manifold. It is a quotient of the nilpotent Lie group of real dimension 4 with left invariant complex structure whose structure equations are

$$d\varphi = 0, \quad d\psi = \varphi \wedge \overline{\varphi}.$$

Our examples of complex manifolds for which $E_2 \neq E_3$ can be viewed as simultaneous generalizations of the complex parallelizable manifolds and the Kodaira-Thurston manifold.

4. The Iwasawa manifold and complex parallelizable manifolds

In order to make subsequent examples clearer we first compute that portion of Frölicher spectral sequence for the Iwasawa manifold I_3 that we need. Explicitly $I_3 = G/\Gamma$ where G is the group of complex matrices of the form

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

and Γ is the subgroup of G consisting of those matrices whose entries are Gaussian integers. The functions z_1, z_2, z_3 in (13) are natural complex coordinates on G; moreover it is easy to check that $\omega_1, \omega_2, \omega_3, \overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3$ form a basis for the left invariant 1-forms on G where

$$\omega_1 = dz_1, \qquad \omega_2 = dz_2, \qquad \omega_3 = dz_3 - z_1\, dz_2.$$

We have

$$d\omega_1 = 0, \qquad d\omega_2 = 0, \qquad d\omega_3 = -\omega_1 \wedge \omega_2;$$

thus $\bar{\partial}\omega_3 = 0$ but ω_3 is not $\bar{\partial}$-exact. It follows that

$$E_1^{1,0} \cong \{\omega_1, \omega_2, \omega_3\},$$
$$E_2^{1,0} \cong \{\omega_1, \omega_2\} \cong E_\infty^{1,0}.$$

These calculations are a special case of a much more general result.

THEOREM 9. *Let M be a compact complex parallelizable nilmanifold. Then for all p and q*

$$E_2^{p,q}(M) \cong E_\infty^{p,q}(M). \tag{14}$$

Proof. Let $M = G/\Gamma$ where G is a simply connected complex nilpotent Lie group and Γ is a uniform subgroup. Let $\mathfrak{g}$ denote the Lie algebra and J the canonical complex structure of G. Then

$$\mathfrak{g} \otimes \mathbf{C} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$$

where

$$\mathfrak{g}^\pm = \{X \in \mathfrak{g} | JX = \pm\sqrt{-1}X\}.$$

Sakane [Sa] has shown that

$$H_{\bar{\partial}}^{p,q}(M) \cong H^q(\mathfrak{g}^-) \otimes \Lambda^p(\mathfrak{g}^+)^*, \tag{15}$$

where $H^q(\mathfrak{g}^-)$ denotes the Lie algebra cohomology of $\mathfrak{g}^-$. From (15) we have

$$E_2^{p,q} \cong H^q(\mathfrak{g}^-) \otimes H^p(\mathfrak{g}^+).$$

On the other hand it follows from Nomizu's theorem [No] that

$$H^r(M) \cong H^r(\mathfrak{g} \otimes \mathbf{C}). \tag{16}$$

Since M is complex parallelizable, we have

$$[\mathfrak{g}^+, \mathfrak{g}^-] = 0. \tag{17}$$

From (15), (16) and (17) we get (14).

5. A nilmanifold with $E_2 \not\cong E_3 \cong E_\infty$

Since $E_2 \cong E_\infty$ for any compact complex parallelizable nilmanifold, we must search for more complicated nilmanifolds to achieve $E_2 \not\cong E_3$. We give an example in complex dimension 4 simpler than that of [CFG]; it will be a starting point for constructing examples with $E_n \not\cong E_{n+1}$ for $n \geq 3$.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of complex dimension 4 with left invariant complex structure given by the equations

$$
(18) \qquad \begin{aligned}
d\omega_1 &= 0, \\
d\omega_2 &= 0, \\
d\varphi_1 &= \omega_1 \wedge (\omega_2 + \overline{\omega}_2), \\
d\varphi_2 &= \overline{\omega}_1 \wedge \omega_2.
\end{aligned}
$$

By Corollary 7 these structure equations determine a nilpotent Lie group G with a left invariant complex structure, and by Corollary 8 we get a compact nilmanifold with a complex structure.

It will be convenient to introduce an abbreviated notation for wedge products. We write

$$
\omega_{ij} = \omega_i \wedge \omega_j, \qquad \omega_{i\bar{j}} = \omega_i \wedge \overline{\omega}_j, \qquad \omega_{\bar{i}\bar{j}} = \overline{\omega}_i \wedge \overline{\omega}_j, \quad \omega_{ijk} = \omega_i \wedge \omega_j \wedge \omega_k,
$$

and so forth.

To show that $E_2 \not\cong E_3$ we consider the differential form $\gamma = \omega_{12}$ of type $(2, 0)$. Then γ is closed, hence $\bar{\partial}$-closed, so that it represents a nontrivial class in $E_1^{2,0}$. Moreover by Lemma 1 we see that γ also defines a nontrivial class in $E_2^{2,0}$ because

$$
\gamma \notin d(\Lambda^{1,0}) = \{\omega_{12} + \omega_{1\bar{2}}, \omega_{\bar{1}2}\}.
$$

However using Lemma 1 again, we see that γ defines the zero class in $E_3^{2,0}$ because

$$
\gamma = d(\varphi_1 - \overline{\varphi}_2) \in d(\Lambda^{1,0} \oplus \Lambda^{0,1}).
$$

Thus $E_2^{2,0} \not\cong E_3^{2,0}$.

It is also possible to compute the $E_r^{0,1}$ using Lemma 3. We find that

$$
\begin{aligned}
&E_1^{0,1} \cong \{\overline{\omega}_1, \overline{\omega}_2, \overline{\varphi}_2\} \cong \{\overline{\omega}_1, \overline{\omega}_2, \overline{\varphi}_2 - \varphi_1\} \cong E_2^{0,1}, \\
&E_3^{0,1} \cong E_\infty^{0,1} \cong \{\overline{\omega}_1, \overline{\omega}_2\}.
\end{aligned}
$$

The equations (18) can be integrated explicitly, and thus determine complex coordinates $\{z_1, z_2, z_3, z_4\}$ on G:

$$
(19) \qquad \begin{aligned}
\omega_1 &= dz_1, \\
\omega_2 &= dz_2, \\
\varphi_1 &= dz_3 - (z_2 + \bar{z}_2)\, dz_1, \\
\varphi_2 &= dz_4 + \bar{z}_1\, dz_2.
\end{aligned}
$$

Hence G can be realized as the nilpotent group of complex matrices of the form

$$\begin{pmatrix} 1 & -\bar{a}_1 & a_4 & a_2 + \bar{a}_2 & a_3 \\ & 1 & a_2 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & a_1 \\ & & & & 1 \end{pmatrix}.$$

Then G/Γ is a compact nilmanifold that is complex with $E_1 \not\cong E_2 \not\cong E_3 \cong E_\infty$.

6. A nilmanifold with $E_3 \not\cong E_4 \cong E_\infty$

This case is only slightly more complicated than the preceding example. Let $\mathfrak{g}$ be the nilpotent Lie algebra of complex dimension 6 defined by the structure equations

$$\begin{aligned} d\omega_1 &= 0, \\ d\omega_2 &= 0, \\ d\omega_3 &= 0, \\ d\varphi_1 &= \omega_1 \wedge (\omega_2 + \bar{\omega}_2), \\ d\varphi_2 &= \bar{\omega}_1 \wedge \omega_2, \\ d\varphi_3 &= \omega_1 \wedge (\varphi_1 + \bar{\omega}_3). \end{aligned} \tag{20}$$

To prove that the Frölicher spectral sequence of M satisfies $E_3 \not\cong E_4$ we consider the differential form $\gamma = \omega_1 \wedge \omega_2 \wedge \varphi_1$ of type $(3,0)$. We have

$$d(\omega_2 \wedge \varphi_3) = \gamma + \omega_{12\bar{3}} \quad \text{and} \quad \omega_{12} = d(\varphi_1 - \bar{\varphi}_2).$$

Furthermore

$$\omega_{12\bar{3}} = d(\varphi_1 \wedge \bar{\omega}_3 - \bar{\varphi}_2 \wedge \bar{\omega}_3) \in d(\Lambda^{1,1} \oplus \Lambda^{0,2}) \subset d(\Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}),$$

but a direct computation shows $\omega_{12\bar{3}} \notin d(\Lambda^{2,0} \oplus \Lambda^{1,1})$. Consequently

$$\begin{aligned} \gamma &= d(\omega_2 \wedge \varphi_3) - \omega_{12\bar{3}} \\ &= d(\omega_2 \wedge \varphi_3 - \varphi_1 \wedge \bar{\omega}_3 + \bar{\varphi}_2 \wedge \bar{\omega}_3) \\ &\in d(\Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}). \end{aligned}$$

Therefore by Lemma 1, γ defines a nonzero class in $E_3^{3,0}$, but γ defines the zero class in $E_4^{3,0}$.

An explicit realization of the compact complex manifold M corresponding to (20) can be obtained in the same way as we obtained the compact manifold corresponding to (18). In fact, integrating the equations (20) we determine complex coordinates $\{z_1, z_2, z_3, z_4, z_5, z_6\}$ on G such that

$$\begin{aligned}\omega_1 &= dz_1,\\ \omega_2 &= dz_2,\\ \omega_3 &= dz_3,\\ \varphi_1 &= dz_4 - (z_2 + \bar{z}_2)\, dz_1,\\ \varphi_2 &= dz_5 + \bar{z}_1\, dz_2,\\ \varphi_3 &= dz_6 - (z_4 + \bar{z}_3)\, dz_1.\end{aligned} \tag{21}$$

Hence G can be realized as the nilpotent group of complex matrices of the form

$$\begin{pmatrix} 1 & -\bar{a}_1 & a_5 & a_2 + \bar{a}_2 & a_4 & \bar{a}_3 & a_6 \\ & 1 & a_2 & 0 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 \\ & & & 1 & a_1 & 0 & \frac{1}{2}a_1^2 \\ & & & & 1 & 0 & a_1 \\ & & & & & 1 & a_1 \\ & & & & & & 1 \end{pmatrix}.$$

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