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BOUNDARY VALUES OF ANALYTIC FUNCTIONS IN THE BANACH SPACE $P^t(\sigma)$ ON CRESCENTS

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1. Introduction

A simply connected domain Ω is called a crescent if it is enclosed by two Jordan curves, which intersect at a single point. We call this point the multiple boundary point. The theory of Banach spaces of analytic functions on crescents has been studied by a number of authors, but there still are many unanswered questions. Though a crescent Ω has a nice and simple boundary topologically, it does not have many of the nice properties that a Jordan domain possesses. For example, $\mathscr{P}$, the set of polynomials, is not always dense in the Hardy space $H^t(\Omega)$ (where $t \in [1, \infty)$) and this density property depends on the geometrical properties near the multiple boundary point (see, [3]). J. Akeroyd shows [3] that if Ω is bounded by two tangent circles, $\mathscr{P}$ is always dense in $H^t(\Omega)$; but this is not always true for the corresponding Bergman space $L^t_a(\Omega)$ (see [4]). We say a crescent is A-type if it is contained in D and is enclosed by ∂D and another Jordan curve whose part near the multiple boundary point coincides with the (two) sides of an angle. It is not difficult to show the polynomials are not dense in $H^t(\Omega)$ if Ω is an A-type crescent.

To introduce our results we first need some definitions. A measure σ is a harmonic measure of a simply connected domain G if $\sigma = m \circ \tilde{\varphi}^{-1}$, where m is the normalized Lebesgue measure on ∂D and $\tilde{\varphi}$ is the boundary value function of a conformal map φ of D onto G. Two harmonic measures of a domain G are boundedly equivalent. So if we say σ is the harmonic measure of G, we shall mean σ is a harmonic measure of a fixed point in G. Let $P^t(\sigma)$ be the closure of $\mathscr{P}$ in $L^t(\sigma)$. A point w is called an analytic bounded point evaluation (*abpe*) for $P^t(\sigma)$ if there exists a neighborhood U of w so that for each point λ in U there exists a function $k_\lambda \in L^q(\sigma)$ such that

$$p(\lambda) = \int p k_\lambda \, d\sigma, \; p \in \mathscr{P}, \quad \text{and} \quad \sup_{\lambda \in U} \{\|k_\lambda\|\} < \infty. \tag{1}$$

Let $\hat{f}(w) = \int f k_w \, d\sigma$ for each $f \in P^t(\sigma)$. The function $\hat{f}$ is analytic on $abpeP^t(\sigma)$, the set of *abpe*'s for $P^t(\sigma)$.

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Fix a crescent Ω with harmonic measure σ. For the sake of simplicity, we assume that $\Omega \subset D$ and $\partial D \subset \partial\Omega$. Now a fact is:

$$\text{either } abpeP^t(\sigma) = \Omega, \quad \text{or} \quad abpeP^t(\sigma) = D;$$

the former equality is equivalent to $\mathscr{P}$ being dense in $H^T(\Omega)$ (Theorem 1 of [14]). In this paper we tacitly assume that

$$abpeP^t(\sigma) = D.$$

We show (Theorem 2) that every function $f \in P^t(\sigma)$ has nontangential limits almost everywhere with respect to m on ∂D. Moreover,

$$f(\alpha) = \lim_{z \to \alpha} \hat{f}(z) \text{ a.e. } [m] \text{ on } \partial D,$$

where the limits are taken in nontangential sense. In other words, every $\hat{f}$ in $P^t(\sigma)$ has a boundary value (function) on the circle. Now a natural question is raised:

If $f \in P^t(\sigma)$ and $f|\partial D$ is bounded, is $\hat{f}(z)$ bounded?

In the classical Hardy space case, it is well known that if $f \in P^t(m)$ and $\hat{f}(z)$ has a bounded boundary value, then $\hat{f}(z)$ itself is a bounded analytic function. The measure m actually is a harmonic measure for D; it would be very reasonable for us to expect the same is true for the functions in the space $P^t(\sigma)$. Unfortunately, this is no longer the case in general.

In Section 2 we present a counter-example. In fact, we construct a domain Ω (with harmonic measure σ) and an unbounded function $h \in P^t(\sigma)$ such that $\hat{h}$ has a continuous boundary value on ∂D.

Can we have a positive answer to the question for some of these crescents Ω?

In Section 3, we give an affirmative answer if Ω is an A-type crescent.

Lastly, let μ be a finite positive measure with compact support in the plane and let S_μ be the operator defined by $S_\mu(f) = zf$ for each $f \in P^2(\mu)$. As an application of the last result, we prove that if τ is a positive measure carried by D and if Ω is an A-type crescent, then $S_{\sigma+\tau}$ and S_σ are similar if and only if τ is a Carleson measure on D.

2. A counter-example

The proof of the following lemma is elementary.

LEMMA 1. *Let a and b be two positive numbers with $b > a$. Let $G_{a,b}$ denote the crescent enclosed by the circles*

$$\left\{z: \left|z - \left(1 - \frac{1}{a}\right)\right| = \frac{1}{a}\right\} \text{ and } \left\{z: \left|z - \left(1 - \frac{1}{b}\right)\right| = \frac{1}{b}\right\}.$$

Then

$$f_{a,b} = \frac{-i \exp\left(-\dfrac{a\pi i}{b-a}\right)\exp\left(-\dfrac{2\pi i}{b-a}\dfrac{1}{z-1}\right) - 1}{-i \exp\left(-\dfrac{a\pi i}{b-a}\right)\exp\left(-\dfrac{2\pi i}{b-a}\dfrac{1}{z-1}\right) + 1}$$

is a conformal map of $G_{a,b}$ onto D.

The next result is well known and it can be proved by applying a famous theorem of F. and M. Riesz [12, p. 70].

LEMMA 2. *With the notions above, let $x = f_{ab}^{-1}(0)$ and let s be arclength measure on ∂G_{ab}; then*

$$d\omega_x = \frac{1}{2\pi}|f'_{ab}|\, ds.$$

Let W_1 be the crescent enclosed by circles $C_0 = \partial D$ and $C_1 = \{z: |z - \frac{2}{3}| = \frac{1}{3}\}$; let W_2 be the crescent enclosed by C_1 and $C_2 = \{z: |z - \frac{4}{5}| = \frac{1}{5}\}$; let W_3 be the crescent enclosed by C_2 and $C_3 = \{z: |z - \frac{6}{7}| = \frac{1}{7}\}$; and let W_0 be the crescent enclosed by ∂D and C_3.

Now if we connect $z = 1$ and $z = i/100$ by a segment l, then l separates W_2 into two parts. We use V to denote the part completely contained in the upper plane. Set

$$U = W_0 \cap \{z: \operatorname{Im} z > 0\}$$

and set

$$G = U - \overline{V}.$$

G is a crescent and we use ω to denote its harmonic measure. Besides the point $z = 1$, ∂V has two other singular points (which are the intersection points of l with C_1 and C_2), and ∂G has four more singular points (which are the intersection points of l with C_2 and C_2 and the intersection points of the real axis with C_0 and C_3). For some technical reasons we, in addition, modify ∂G and ∂V slightly at a small neighborhood of each of those 'bad' points so that G and V have smooth boundaries except at $z = 1$. With these notations, now we have:

LEMMA 3. *Let h_j be the restriction of*

$$\frac{\exp\left(\frac{-\pi i}{z-1}\right)+1}{\exp\left(\frac{-\pi i}{z-1}\right)-1}$$

to W_j, $j = 1,3$, and let h_2 be the restriction of

$$\frac{\exp\left(\frac{-\pi i}{z-1}\right)-1}{\exp\left(\frac{-\pi i}{z-1}\right)+1}$$

to W_2. Then h_j maps W_j conformally onto D for each $j = 1,2,3$. Moreover, if s is arclength measure on ∂G and τ is the harmonic measure of V, then ω is boundedly equivalent to the measure

$$(*) \qquad \frac{1}{|z-1|^2}\exp\left(-\frac{|\mathrm{Im}\, z|}{|z-1|^2}\pi\right)s$$

and τ is boundedly equivalent to the restriction of $(*)$ to ∂V.

Proof. Using Lemma 1 with $a = 1$ and $b = 3$, we see $h_1 = f_{1,3}$ maps W_1 conformally onto D. Similarly, with $a = 3$ and $b = 5$ for W_2, we see $f_{3,5} = h_2$. Setting $a = 5$ and $b = 7$ for W_3 in Lemma 3 yields the desired result for $f_{5,7} = h_3$. This proves the first part of the lemma.

Now an easy computation gives

$$h_j' = \left(\frac{\exp\left(\frac{-\pi i}{z-1}\right)+1}{\exp\left(\frac{-\pi i}{z-1}\right)-1}\right)' = -\frac{\frac{2\pi i}{(z-1)^2}\exp\left(\frac{-\pi i}{z-1}\right)}{\left(\exp\left(\frac{-\pi i}{z-1}\right)-1\right)^2} \quad \text{for } j = 1,3$$

and

$$h_2' = \left(\frac{\exp\left(\frac{-\pi i}{z-1}\right) - 1}{\exp\left(\frac{-\pi i}{z-1}\right) + 1} \right)' = \frac{\frac{2\pi i}{(z-1)^2} \exp\left(\frac{-\pi i}{z-1}\right)}{\left(\exp\left(\frac{-\pi i}{z-1}\right) + 1\right)^2}.$$

So we see that

$$|h_j'|s \text{ and } \left(\frac{1}{|z-1|^2} \exp\left(- \frac{|\operatorname{Im} z|}{|z-1|^2} \pi \right) \right) s \text{ are boundedly equivalent.}$$

But ω and σ_i are boundedly equivalent on W_j, $j = 1, 3$, so using Lemma 2, we conclude that

$$\omega \text{ and } \left(\frac{1}{|z-1|^2} \exp\left(- \frac{|\operatorname{Im} z|}{|z-1|^2} \pi \right) \right) s \text{ are boundedly equivalent.}$$

Similarly, τ and the restriction of

$$\left(\frac{1}{|z-1|^2} \exp\left(- \frac{|\operatorname{Im} z|}{|z-1|^2} \pi \right) \right) s$$

to ∂V are boundedly equivalent. □

LEMMA 4. *With above notations, $abpeP^t(\omega) = U$ for all $t \in [1, \infty)$.*

Proof. Let $a \in G$ and let φ be a conformal map of D onto G that sends 0 to a. Set $\omega_a = m \circ \tilde{\varphi}$. It is well known that

$$p(a) = \int p \, d\omega_a \quad \text{for each } p \in \mathscr{P}.$$

So it follows by Hölder's inequality that

$$|p(a)| \le \left\| \frac{d\omega_a}{d\omega} \right\|_\infty \left\{ \int |p|^t \, d\omega \right\}^{1/t} \quad \text{for each } p \in \mathscr{P}.$$

Thus, $a \in bpeP^t(\omega)$, so we conclude

$$G \subset bpeP^t(\omega).$$

From Lemma 3 we see that τ and $\omega|\partial V$ are boundedly equivalent; it follows that"

$$V \subset bpeP^t(\tau) \subset bpeP^t(\omega|\partial V) \subset bpeP^t(\omega).$$

Consequently,

$$G \cup V \subset bpeP^t(\omega).$$

An appeal to Harnack's inequality (see [1]), one can easily show that

$$bpeP^t(\omega) = abpeP^t(\omega).$$

Thus

$$G \cup V \subset abpeP^t(\omega).$$

Next we show that $abpeP^t(\omega)$ is connected. This is equivalent to show that $P^t(\omega)$ contains no non-trivial characteristic function (see [18]). Let $\Delta \subset \partial G$ such that $\chi_\Delta \in P^t(\omega)$. So there exists $\{p_n\} \subset \mathscr{P}$ such that $\int |\chi_\Delta - p_n|^n \, dw \to 0$. Recall the Hardy space $H^t(G)$ consists of all analytic functions f such that $|f|^t$ has a harmonic majorant on G. For $f \in H^t(G)$, the norm can be defined as $\|f\| = u(a)^{1/t}$, where u is the least harmonic majorant of f. As a sequence in $H^t(G)$, our given sequence $\{p_n\}$ converges to a function, say x, in $H^t(G)$. Since $\chi_\Delta^2 = \chi_\Delta$, it follows that $x^2 = x$. Since G is connected, this implies that either $x = 1$ or $x = 0$. Hence, $\chi_\Delta = 0$ or $\chi_\Delta = 1$.

Let $W = abpeP^t(\omega)$. If $W \neq U$, then W is a slit simply connected domain whose boundary is contained in the union of two Jordan curves. This implies that no conformal map from D onto W is almost one-to-one on ∂W with respect to m. On the other hand, according to Thomson's theorem ([18], we have an isometrical isomorphism map from $H^\infty(W)$ to $P^t(\omega) \cap L^\infty(\omega)$. Using Theorem 94 of [13, Miller-Olin-Thomson], we conclude that W must be nicely connected. This is a contradiction; hence, $W = U$. So the proof is complete. □

LEMMA 5. *If*

$$g(z) = (z-1)^{2/t} \exp\left(\frac{\pi}{t}\frac{z+1}{1-z}\right)$$

then $g \in P^t(\omega)$.

Proof. We first show that $g \in L^t(\omega)$. Let $z = x + iy$. By Lemma 3 there is a positive constant c such that

$$\begin{aligned}
\int |g(z)|^t \, d\omega &\le c \int |g(z)|^t \frac{1}{|z-1|^2} \exp\left(- \frac{\operatorname{Im} z}{|z-1|^2} \pi \right) ds \\
&\le c \int \left\{ |(z-1)|^{2/t} \exp\left(\frac{\pi}{t} \frac{1-x^2-y^2}{|1-z|^2} \right) \right\}^t \\
&\quad \times \frac{1}{|z-1|^2} \exp\left(- \frac{\operatorname{Im} z}{|z-1|^2} \pi \right) ds \\
&\le c \int \exp\left(\pi \frac{1-x^2-y^2}{|1-z|^2} \right) \exp\left(\frac{-|y|}{|1-z|^2} \pi \right) ds \\
&\le c \int \exp\left(\pi \frac{1-x^2-y^2}{|1-z|^2} \right) ds.
\end{aligned}$$

It is easy to verify that

$$\exp\left(\pi \frac{1-x^2-y^2}{|1-z|^2} \right)$$

is constant on each circle C_i, $i = 1, 2, 3$. So we conclude that $g \in L^t(\omega)$. Now set

$$g_n(z) = (z-1)^{2/t} \exp\left(\frac{\pi}{t} \frac{z+1}{1-z-i/n} \right).$$

Then g_n is in the function algebra

$$A(U) = \{ f \colon f \text{ is analytic on } U \text{ and is continuous on } \overline{U} \}$$

and hence

$$g_n \in P^t(\omega) \quad \text{for each } n.$$

Moreover, since

$$|z-1| < \left| z - \left(1 - \frac{i}{n}\right) \right| \quad \text{for all } z \in G,$$

we have

$$\begin{aligned}
|g_n(z)| &= \left|(z-1)^{2/t}\right|\left|\exp\left(\frac{\pi}{t}\frac{z+1}{1-z-i/n}\right)\right| \\
&= \left|(z-1)\right|^{2/t}\exp\left(\frac{\pi}{t}\frac{1-x^2-y^2-y/n}{|1-z-i/n|^2}\right) \\
&\leq \left|(z-1)\right|^{2/t}\exp\left(\frac{\pi}{t}\frac{1-x^2-y^2}{|1-z-i/n|^2}\right) \\
&\leq \left|(z-1)\right|^{2/t}\exp\left(\frac{\pi}{z}\frac{1-x^2-y^2}{|1-z|^2}\right) \\
&= |g(z)|.
\end{aligned}$$

Apparently, $g_n \to g$ pointwise. It follows by Lebesgue dominated convergence theorem that $g \in P^t(\omega)$. □

THEOREM 1. *There is a crescent Ω with harmonic measure σ and there is an unbounded function $h \in P^t(\sigma)$ such that*

$$abpeP^t(\sigma) = D \text{ and } h|\partial D \text{ is continuous}$$

Proof. Choose the region G and the function g as in Lemma 5. Let φ be a conformal map of D onto U, where U is the region as in Lemma 4. Since U is a Jordan domain, it follows by a theorem of Carathéodory that φ can be extended to be a homeomorphism from $\overline{D}$ onto $\overline{U}$ [19, p. 353]. For the sake of simplicity, we still denote this homeomorphism by φ. Let $\Omega = \varphi^{-1}(G)$ and let σ be its harmonic measure. Now we claim that

$$\varphi|\partial\Omega \in P^t(\sigma) \text{ and } \varphi^{-1}|\partial G \in P^t(\omega).$$

The proofs for them are very similar and we only prove the first one. To show that $\varphi|\partial\Omega \in P^t(\sigma)$, it suffices to prove that $\varphi|\partial\Omega \in P^\infty(\sigma)$. Since $\varphi \in P^\infty(m)$, there is a sequence of polynomials $\{p_n\}$ such that it weak-star converges to φ. That is,

$$\int_{\partial D}(\varphi - p_n)f\,dm \to 0 \quad \text{for each } f \in L^1(m).$$

Since $\Omega \subset D$, it is a well-known fact that the measure $\sigma|\partial D$ is absolutely continuous with respect to m. It now follows that

$$\int_{\partial D} (\varphi - p_n) f \, d\sigma \to 0 \quad \text{for each } f \in L^1(\sigma).$$

Also, the weak-star convergence implies that $\{p_n\}$ is uniformly bounded on D and it pointwise converges to φ on D. Using the Bounded Convergence Theorem, we conclude

$$\int_{\sigma\Omega\cap D} (\varphi - p_{fn}) f \, d\sigma \to 0 \quad \text{for each } f \in L^1(\sigma).$$

Therefore, we have

$$\int_{\partial\Omega} (\varphi - p_n) f \, d\sigma \to 0 \quad \text{for each } f \in L^1(\sigma).$$

That is, $\{p_n\}$ weak-star converges to φ; and hence $\varphi \in P^\infty(\sigma)$. The claim is proved.

The restriction of φ to Ω is a conformal map of Ω onto G. So $\sigma \circ \varphi^{-1}$ is a harmonic measure of G. Thus, $\sigma \circ \varphi^{-1}$ is boundedly equivalent to ω. Let g be the function as in Lemma 5. There is a sequence of polynomials $\{q_n\}$ such that

$$q_n \to g \quad \text{in } P^t(\omega).$$

Let $h = g \circ \varphi$. We have

$$\int_{\partial\Omega} |h - q_n \circ \varphi|^t \, d\sigma = \int_{\partial G} |g - q_n|^t \, d\left(\sigma \circ \varphi^{-1}\right) \to 0.$$

Since $\varphi \in P^t(\sigma) \cap L^\infty(\sigma)$, and since $P^t(\sigma) \cap L^\infty(\sigma)$ is a Banach algebra, it follows that

$$q_n \circ \varphi \in P^t(\sigma) \cap L^\infty(\sigma).$$

Consequently,

$$h \in P^t(\sigma).$$

Now we want to show that

$$abpe P^t(\sigma) = D.$$

Let σ be a conformal map of Ω onto D. In light of Theorem 1 of [14], the fact $abpe P^t(\sigma) = D$ is equivalent to the fact that α is not in $P^t(\sigma)$. To prove the latter, we argue by contradicting; suppose there is a sequence of polynomials $\{p_n\}$ such that

$$\int |p_n - \alpha|^t \, d\sigma \to 0.$$

Note that

$$\int_{\partial G} |\alpha \circ \varphi^{-1} - p_n \circ \varphi^{-1}|^t \, d(\sigma \circ \varphi^{-1}) = \int_{\partial\Omega} |\alpha - p_n|^t \, d\sigma \to 0.$$

Also note that $p_n \circ \varphi^{-1} \in P^t(\omega)$ for each n. It follows that

$$\alpha \circ \varphi^{-1} \in p^t(\omega).$$

On the other hand, the restriction of $\alpha \circ \varphi^{-1}$ to G is also a conformal map of G onto D. Using Theorem 1 of [14] again, we conclude

$$abpeP^t(\omega) = G \neq U,$$

a contradiction to Lemma 4, Hence $abpeP^t(\sigma) = D$.

Finally, one verifies that

$$g(z) = (z - 1)^{2/t} \exp\left(\frac{\pi}{t} \frac{z + 1}{1 - z}\right)$$

is continuous on ∂D and g is unbounded on G. Since φ is a homeomorphism from $\overline{D}$ onto $\overline{U}$, we see that $h = g \circ \varphi$ is the desired function. The proof is complete. □

3. On A-type crescents

The following theorem says that $\hat{f}$ has boundary values on ∂D for each f in $P^t(\sigma)$.

THEOREM 2. *Let Ω be a crescent with harmonic measure σ. If $abpeP^t(\sigma) = D$, then $\hat{f}$ has nontangential limits almost everywhere with respect to m on ∂D for every $f \in P^t(\tau)$. Moreover,*

$$f(\sigma) = \lim_{z \to \sigma} \hat{f}(z) \text{ nontangentially a.e. } [m] \text{ on } \partial D.$$

Remark. The hypothesis $abpeP^t(\sigma) = D$ implies that Ω is a crescent which has ∂D as its outer boundary. Also note that we do not require Ω to be an A-type crescent (see the definition of A-type crescent at the beginning of this article). Theorem 2 works for all crescent Ω with $abpeP^t(\sigma) = D$.

Proof. Suppose that $f \in P^t(\sigma)$. Choose $p_n \in \mathscr{P}$ such that

$$\int |p_n - f|^t \, d\sigma \to 0.$$

Let φ be a conformal map of D onto Ω and let $\tilde{\varphi}$ denote the boundary value function on ∂D. Note

$$\int |p_n \circ \tilde{\varphi} - f \circ \tilde{\varphi}|^t \, dm \to 0.$$

Since $p_n \circ \tilde{\varphi} \in P^t(m)$ for each n, we get

$$f \circ \tilde{\varphi} \in P^t(m).$$

Now we claim that

$$\hat{f} \circ \varphi(z) = \widehat{f \circ \tilde{\varphi}}(z) \quad \text{for all } z \in D.$$

Let $L = \tilde{\varphi}^{-1}(\partial\Omega - \partial D)$ (note, ∂D is a part of $\partial\Omega$). It follows that L is an open arc on the unit circle ∂D (this can be proved using Carathéodory's Theorem [19, p. 353]). Since $\hat{f}(z)$ is continuous at every point of $\partial\Omega - \partial D$, it follows that for every $e^{ix} \in L$,

$$\begin{aligned}\lim_{z \to e^{ix}} \hat{f} \circ \varphi(z) &= \hat{f}\Big(\lim_{z \to e^{ix}} \varphi(z)\Big) \\ &= \hat{f}\big(\tilde{\varphi}(e^{ix})\big) \\ &= f\big(\tilde{\varphi}(e^{ix})\big),\end{aligned}$$

where all limits are taken nontangentially. On the other hand, if we take the nontangential limit

$$\lim_{z \to e^{ix}} \widehat{f \circ \tilde{\varphi}}(z) = (f \circ \tilde{\varphi})\big(e^{ix}\big) = f\big(\tilde{\varphi}(e^{ix})\big)$$

for almost every $x \in [-\pi, \pi]$. Consequently, the nontangential limit

$$\lim_{z \to e^{ix}} \Big[\hat{f} \circ \varphi(z) - \widehat{f \circ \tilde{\varphi}}(z)\Big] = 0$$

for almost every point e^{ix} on the arc L. Applying Lusin-Privaloof's theorem [19, p 320], we conclude

$$\hat{f} \circ \varphi(z) = \widehat{f \circ \tilde{\varphi}}(z) \quad \text{for } z \in D.$$

The claim is proved.

Now let $J = \tilde{\varphi}^{-1}(\partial D)$ (again, ∂D is a part of the boundary of Ω). Again applying Carathéodory's Theorem we see J is an arc on the unit circle. Since both ∂D and J are smooth, φ^{-1} preserves angles at almost every point of ∂D. Therefore, for almost every $\beta \in \partial D$ we have

$$\begin{aligned}
\lim_{w \to \beta} \hat{f}(w) &= \lim_{z \to \tilde{\varphi}^{-1}(\beta)} \hat{f}(\varphi(z)) \\
&= \lim_{z \to \tilde{\varphi}^{-1}(\beta)} \hat{f} \circ \varphi(z) \\
&= \lim_{z \to \tilde{\varphi}^{-1}(\beta)} \widehat{f \circ \tilde{\varphi}}(z) \\
&= f \circ \tilde{\varphi}(z)\big(\tilde{\varphi}^{-1}(\beta)\big) \\
&= f(\beta)
\end{aligned}$$

where all limits are nontangential. The proof is complete. □

The next lemma can be found in [3]; a proof can also be given using Lemma 2 (one may also consult Ahlfors [1, p. 236]). We state it her for reader's convenience.

LEMMA 6. *Let G be a Jordan domain such that ∂G is smooth except at one point a. Suppose that ∂G forms an angle α at a with $\alpha < \pi$. Then the harmonic measure for G and the measure $(|z - a|^{\frac{\pi}{\alpha}-1})s$ are boundedly equivalent, where s is arclength on ∂G.*

The work in [3] is the inspiration of the next lemma as well as our work on A-type crescents.

LEMMA 7. *Let Ω be an A-type crescent and let σ be its harmonic measure. Then*

$$abpe P^t(\sigma) = D.$$

Proof. Let γ be the Jordan curve such that $\gamma \cup \partial D = \partial\Omega$. Without loss of generality we may assume $z = 1$ is the multiple boundary point of $\partial\Omega$. Since Ω is an A-type domain, there exist two segments $l_1 \subset \gamma$ and $l_2 \subset \gamma$ such that l_1 and l_2 together form an angle at $z = 1$ (note, by definition a crescent is enclosed by *two* Jordan curves, so γ has only one intersection point with ∂D). Clearly l_i also forms an angle α_i with the vertical line $\operatorname{Re} z = 1$, for each $i = 1, 2$. We may assume that $\alpha_1 < \alpha_2 < \pi$ (note, $\alpha_i > 0$ by the definition of an A-type crescent). Using the above lemma we can find constants $c > 0$ and $r > 0$ such that

$$d\sigma|\partial D \geq c|z - 1|^r \, dm.$$

This implies that $\log(d\sigma/dm) \in L^1(m)$. Now a simple application of Szegö's Theorem [9, p. 136] shows that $abpeP^t(\sigma) = D$. So the conclusion follows by the above measure inequality. □

The next theorem is a maximum principle type result for functions in $P^t(\sigma)$ on A-type crescents.

THEOREM 3. *Let Ω be an A-type crescent with harmonic measure σ. If $f \in P^t(\sigma)$ and $f|\partial D \in L^\infty(\sigma)$, then $f \in P^t(\sigma) \cap L^\infty(\sigma)$.*

Proof. Suppose that $f \in P^t(\sigma)$ and $|\partial D \in L^\infty(\sigma)$. So there exists $\{p_n\} \subset \mathscr{P}$ such that

$$\int_{\partial\Omega} |p_n - f|^t \, d\sigma \to 0 \text{ as } n \to 0.$$

We may assume that $z = 1$ is the multiple boundary point. As in the proof of the previous lemma, there exist two constants $c > 0$ and $r > 0$ such that

$$d\sigma|\partial D \geq c|z - 1|^r \, dm.$$

Hence

$$\int |p_n - f|^t |z - 1|^r \, dm \to 0.$$

We express the last limit as

$$\int_{\partial D} \left| p_n(z - 1)^{r/t} - f_0(z - 1)^{r/t} \right|^t dm \to 0,$$

where $f_0 = f|\partial D$. Set $u = f_0(z - 1)^{r/t}$. Then

$$u = f_0(z - 1)^{r/t} \in P^t(m).$$

Since $(z-1)^{r/t}$ is bounded, it follows from the hypothesis that

$$u \in L^\infty(m) \cap P^t(m) = P^\infty(m).$$

Now if we let $v = f_0^{t/2r}$, then

$$v = u^{t/2r}(z-1)^{-1/2}.$$

One can directly check that

$$\sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{1-re^{i\theta}}\right|^{1/2} d\theta\right\} < \infty,$$

so $(1/(1-z))^{1/2} \in P^1(m)$ (see [8], or [10]). Obviously $u^{t/2r} \in P^\infty(m)$, it now follows that

$$v = u^{1/2r}(z-1)^{-1/2} \in P^1(m);$$

i.e.,

$$f_0 = v^{2r/t} \in P^{t/2r}(m).$$

But $f_0 \in L^\infty(m)$, so

$$f_0 \in L^\infty(m) \cap P^{t/2r}(m) = P^\infty(m).$$

Now let $\hat{f}_0$ the analytic extension of f_0 on D. We note $\hat{f}_0 \in H^\infty(D)$ and $\lim_{z\to\lambda}\hat{f}_0(\lambda)$ nontangentially a.e. $[m]$ on ∂D. Combining this fact with the proceeding theorem, we obtain

$$\lim_{z\to\lambda}\left[\hat{f}(z) - \hat{f}_0(z)\right] = 0 \quad \text{nontangentially a.e. } [m] \text{ on } \partial D.$$

Using Lusin-Pfrivaloff's theorem [19, p. 320], we conclude

$$\hat{f}(z) = \hat{f}_0(z) \quad \text{for each } z \in D.$$

Hence $\hat{f}$ is bounded on D. But $\hat{f}(z) = f(z)$ a.e. $[\sigma] z \in \partial\Omega \cap D$, so we obtain

$$f \in L^\infty(\sigma) \cap P^t(\sigma).$$

The proof is complete. □

4. An application to operator theory

A positive measure τ on D is a *Carleson measure* if there is a positive constant c such that for all $t \in [1, \infty)$,

$$\|p\|_{L^t(\tau)} \leq c\|p\|_{L^t(m)} \quad \text{for } p \in \mathscr{P}.$$

A theorem of *Carleson* [10, p. 238] shows that a measure τ on the unit disc is a *Carleson* measure if and only if there exists a positive constant A such that

$$\mu(C_h) \leq Ah$$

for each *Carleson* square

$$C_h = \left\{z = re^{it}: 1 - h \leq r < 1;\, t_0 \leq t \leq t_0 + h\right\}.$$

THEOREM 4. *Let Ω be an A-type crescent and let σ be the harmonic measure of Ω. If τ is a finite positive measure on D, then $S_{\sigma+\tau}$ and S_σ are similar if and only if τ is a Carleson measure on D.*

Proof. Suppose that $S_{\sigma+\tau}$ and S_σ are similar. Let A: $P^2(\sigma + \tau) \to P^2(\sigma)$ be an invertible operator such that

$$AS_{\tau+\sigma} = S_\sigma A.$$

For every $p \in \mathscr{P}$, one verifies that

$$A(p) = \left(A(1)\right)p \quad \text{and} \quad A^{-1}(p) = \left(A^{-1}(1)\right)p.$$

Moreover, if we let $u = A(1)$ and $v = A^{-1}(1)$, then

$$\|A^{-1}\|^{-1}\|p\|_{\sigma+\tau} \leq \|up\|_\sigma \leq \|A\|\;\|p\|_{\sigma+\tau} \tag{2}$$

and

$$\|A\|^{-1}\|p\|_\sigma \leq \|vp\|_{\sigma+\tau} \leq \|A^{-1}\|\;\|p\|_\sigma. \tag{3}$$

Replacing p by $z^n p$ in (1) and letting $n \to \infty$, we obtain (note, $|z| < 1$ on D)

$$\|up\|_{\sigma_0} \leq \|A\|\;\|p\|_{\sigma_0} \quad \text{for each } p \in \mathscr{P}, \tag{4}$$

where $\sigma_0 = \sigma | \partial D$. Now we claim that

$$u \in L^\infty(\sigma_0).$$

In fact, (3) implies that the operator M_u, defined by

$$M_u(p) = up \quad \text{for each } p \in \mathscr{P},$$

is a bounded linear operator on $P^2(\sigma_0)$. So we have

$$\begin{aligned}\int |u^n|^2 \, d\sigma_0 &= \int |M_u^n(1)|^2 \, d\sigma_0 \\ &\leq \|M_u\|^{2n} \int d\sigma_0 \\ &= \|M_u\|^{2n} \sigma_0(\partial D) \quad \text{for each } n \geq 1.\end{aligned}$$

Thus,

$$\int \left| \frac{u}{\|M_u\|} \right|^{2n} d\sigma_0(\partial D) \quad \text{for all } n.$$

Hence

$$|u(z)| \leq \|M_u\| \text{ a.e. } [\sigma_0] \quad \text{on } \partial D,$$

which proves the claim. By Theorem 3 we conclude

$$|u| \leq \|M_u\| \quad \text{a.e. } [\sigma].$$

Now we claim:

$$v = A^{-1}(1) = \frac{1}{A(1)} = \frac{1}{u} \quad \text{a.e. } [\sigma + \tau].$$

Let $\{p_n\} \supset \mathscr{P}$ such that p_n converges to $A^{-1}(1)$ in $P^2(\sigma + \tau)$. By passing to a subsequence if necessary, we see that $p_n \to A^{-1}(1)$ a.e. $[\sigma + \tau]$. Now the continuity of A implies that

$$up_n \to 1 \quad \text{in } P^2(\sigma).$$

So there exists a subsequence $\{p_{n_i}\}$ such that up_{n_i} converges to 1 a.e. $[\sigma]$.

Since

$$D \supset abpeP^2(\sigma + \tau) \supset abpeP^2(\sigma) = D,$$

it follows that p_{n_i} converges to $\hat{v}$, the analytic extension of v on D, uniformly on compact subsets of D. But $v = \hat{v}$ a.e. on D, so $uv = 1$ a.e. $[\sigma + \tau]$. Thus $v = 1/u$ a.e. $[\sigma + \tau]$. The claim is proved. Now for $p \in \mathscr{P}$, we have

$$\begin{aligned} \|p\|_\tau &\leq \|u(vp)\|_{\sigma+\tau} \\ &\leq \|M_u\| \ \|vp\|_{+\tau} \\ &= \|M_u\| \ \|A^{-1}(p)\|_{\sigma+\tau} \\ &\leq \|M_u\| \ \|A^{-1}\| \ \|p\|_\sigma \end{aligned}$$

It follows that τ is a *Carleson* measure on D.

Conversely, assume that τ is a *Carleson* measure on D. There exists a constant $c > 0$ such that

$$\|p\|_\tau \leq c\|p\|_\sigma \quad \text{for each } p \in \mathscr{P}. \tag{5}$$

Define an operator A: $P^2(\sigma) \to P^2(\sigma + \tau)$ via $A(p) = p$ for each $p \in \mathscr{P}$. Then (4) implies that A is bounded. A is one-to-one and onto, so it follows by the Open Mapping Theorem that A is invertible. Clearly, $AS_\sigma = S_{\sigma+\tau}A$. So S_σ and $S_{\sigma+\tau}$ are similar. □

References

1. L.V. Ahlfors, *Complex analysis*, Third Ed., McGraw-Hill, New York, 1979.
2. J. Akeroyd, *Polynomial approximation in the mean with respect to harmonic measure on crescents II*, Trans. Amer. Math. Soc. **303** (1987), 193–199.
3. J. Akeroyd, D. Khavinson and H.S. Shapiro, *Remarks concerning cyclic vectors in Hardy spaces and Bergman spaces*, Michigan Math. J. **38** (1991), 191–205.
4. J.E. Brennan, *Approximation in the mean by polynomials on non-Carathéodory domains*, Ark. Mat. **15** (1977), 117–168.
5. L. Carleson, *Interpolations by bounded analytic functions and the Corona problem*, Ann. of Math. **76** (1962), 547–559.
6. W.S. Clary, *Quasi-similarity and subnormal operator*, Ph.D. thesis, University of Michigan, 1973.
7. J.B. Conway, *The Theory of Subnormal Operators*, Math. Surveys and Monographs, vol. 36, Amer. Math. Soc., Providence, R.I., 1991.
8. P.L. Duren, *Theory of H^p-spaces*, Academic Press, New York, 1970.
9. T.W. Gamelin, *Uniform algebras*, Prentice Hall, Englewood Cliffs, N.J., 1969.
10. J.B. Garnett, *Bounded analytic functions*, Academic Press, San Diego, CA, 1981.
11. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
12. P. Koosis, *Introduction to H_p spaces*, London Math. Soc. Lecture Note Series, Vol. 40, Cambridge Univ. Press, Cambridge, 1980.

13. T. Miller, R. Olin and J. Thomson, *Subnormal operators and representations of algebra of bounded analytic functions and other uniform algebras*, Mem. Amer. Math. Soc. No. 354, Amer. Math. Soc., Providence, R.I., 1986.
14. J.Z. Qiu, *Density of polynomials*, Houston J. Math., to appear.
15. ______, *Equivalence classes of subnormal operators*, J. Operator Theory, to appear.
16. ______, *Carleson measure and polynomial approximation*, to appear.
17. ______, *Polynomial approximation and Carleson measure on a general domain and equivalence classes of subnormal operators*, P.h.D. thesis, Virginia Tech, June 1993.
18. J. Thomson, *Approximation in the mean by polynomials*, Ann. of Math. **133** (1991), 477–507.
19. M. Tsuji, *Potential theory in modern function theory*, Chelsea, New York, 1975.

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