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THE ESSENTIAL NORM OF A COMPOSITION OPERATOR ON A PLANAR DOMAIN

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ABSTRACT. We generalize to finitely connected planar domains the result of Joel Shapiro which gives a formula for the essential norm of a composition operator. In the process, we define and give some properties of a generalization of the Nevanlinna counting function and prove generalizations of the Littlewood inequality, the Littlewood-Paley identity, and change of variable formulas, as well.

1. Introduction

Let Ω be a domain in the plane. For $1 \leq p < \infty$, the Hardy space $H^p = H^p(\Omega)$ is defined to be those analytic functions f on Ω for which the subharmonic function $|f(z)|^p$ has a harmonic majorant. Once we specify a base point $t_0 \in \Omega$, we define the norm of f to be the p^{th} root of the value at t_0 of the (unique) least harmonic majorant of $|f|^p$. A different choice of the base point gives an equivalent norm on H^p; this is an application of Harnack's inequality. The Hardy space H^∞ is the space of bounded analytic functions on Ω with the supremum norm. For more on the Hardy spaces, see [6], [1].

An analytic function φ that maps Ω into itself determines a composition operator C_φ on H^p given by

$$C_\varphi f = f \circ \varphi. \tag{1}$$

C_φ is a bounded operator on H^p. One simple way to see this is to note that if u_f is the least harmonic majorant of $|f|^p$, then $u_f \circ \varphi$ is an harmonic majorant of $|f \circ \varphi|^p$ and so

$$\|f \circ \varphi\|^p \leq u_f(\varphi(t_0)) \leq K u_f(t_0)$$

where K is a constant that, again by Harnack's inequality, depends only on the domain Ω, and the points t_0 and $\varphi(t_0)$.

In this paper we are concerned with H^p on a domain Ω that is finitely-connected; that is, has only a finite number of complementary components. In this setting, it is known [2] that C_φ is compact on some H^p, $1 \leq p < \infty$, if and only if it is compact on all H^p. We therefore concentrate on C_φ acting on H^2. The main result of this paper is an extension of the theorem of Joel Shapiro [7] on the essential norm—distance to the set of compact operators—of the composition operator C_φ that he proved when

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Ω is the unit disk. To understand the statement of Shapiro's theorem we must first define the Nevanlinna counting function of φ.

Definition 1.1. Let Δ be the open unit disk and suppose that φ is an analytic function mapping Δ into itself. The Nevanlinna counting function for φ is

$$N_\varphi(w) = \sum_{\varphi(z)=w} -\log|z| \qquad \text{for} \quad w \neq \varphi(0)\,. \tag{2}$$

With this background, we can state Joel Shapiro's result.

THEOREM 1.2. *Suppose that φ is an analytic function that maps Δ into Δ with $\varphi(0) = 0$. Let $\|C_\varphi\|_e$ denote the essential norm of C_φ as an operator on H^2. Then*

$$\left\|C_\varphi\right\|_e^2 = \limsup_{|w|\to 1^-} \left[\frac{N_\varphi(w)}{-\log|w|}\right].$$

In particular, C_φ is compact on H^2 if and only if

$$\lim_{|w|\to 1^-} \frac{N_\varphi(w)}{-\log|w|} = 0.$$

The development of this paper follows the arguments of Shapiro in [7] closely, altering several parts as necessary to allow for the change in setting.

2. Background

Let D be a domain in the plane whose universal covering surface is the open unit disc Δ and let Π be the covering map. The *Poincaré metric* for D is defined at $\zeta = \Pi(z) \in D$ by

$$\lambda_D(\zeta) = |\Pi'(z)|(1 - |z|^2).$$

It is shown in [3, p. 44] that the value of $\lambda_D(\zeta)$ is independent of the particular choice of $z \in \Delta$ with $\Pi(z) = \zeta$.

If D is regular for the Dirichlet problem, we denote the Green's function for D with pole at $p \in D$ by $g_D(z; p)$. The domain D is omitted unless confusion is possible.

In this paper we shall generally be concerned with a planar domain Ω whose complement consists of a finite number of disjoint non-trivial continua. Such a domain is conformally equivalent to one whose boundary consists of a finite number of disjoint analytic simple closed curves; indeed, it is conformally equivalent to a domain whose boundary components are circles. Since the conformal mapping gives an isometry of the corresponding Hardy spaces, we may assume, and shall do so, that the components $\Gamma_0, \ldots, \Gamma_p$ of Γ are circles, with Γ_0 the boundary of the unbounded component of the complement of Ω. We let ω_{t_0} denote the harmonic measure on Γ for the (fixed) base point t_0. It is standard [2] that each H^2 function f on Ω has boundary values almost everywhere on Γ, that these boundary values lie in $L^2(\Gamma, \omega_{t_0})$, and that the correspondence of f to its boundary values is an isometry of H^2 onto a closed subspace of $L^2(\Gamma, \omega_{t_0})$. We let Ω_j be the region outside Γ_j, $j = 1, \ldots, p$, including the point at ∞ and Ω_0 be the region inside Γ_0. Each of the regions Ω_j is conformally equivalent to the unit disk Δ via a linear fractional transformation. When we write $H^2(\Omega_j)$ for the Hardy space for this region, we shall always assume that the norm is taken with respect to the base point t_0.

2.1. *Factorization of H^p functions.* There is a factorization of functions in $H^p(\Omega)$, developed in [8], that parallels that for H^p functions on the unit disc. Here we give a summary; additional details may be found in [1, Section 4.7].

Let $\mathcal{G}$ be the group of linear fractional transformations of Δ onto itself that leave the covering map Π invariant: $\Pi \circ \tau = \Pi, \tau \in \mathcal{G}$. An analytic function h on Δ is *modulus automorphic* if for each $\tau \in \mathcal{G}$ there is a unimodular constant $c = c(\tau)$ such that $h \circ \tau = ch$. Each modulus automorphic function h corresponds to a function f on Ω by $h(z) = f(\Pi(z)), z \in \Delta$. The modulus of f is single-valued, but f itself has unimodular periods in the sense that analytic continuation of a function element $(f, \mathcal{O})$ along any curve γ in Ω leads to the function element $(cf, \mathcal{O})$, where c is a unimodular constant that depends only on the homotopy class of γ. The class of such multiple-valued analytic functions with single-valued modulus whose p^{th} power has a harmonic majorant will be denoted by $MH^p(\Omega)$.

A *Blaschke product* B is an element of $MH^\infty(\Omega)$ with

$$\log|B(z)| = -\sum_k g_\Omega(z; w_k), \qquad \sum_k g_\Omega(w_k; t_0) < \infty.$$

If there are only a finite number of zeros, then the second condition is automatically satisfied.

A *singular inner function* S is an element of MH^∞ with

$$\log|S(z)| = -\int_\Gamma P(s; z) d\nu(s)$$

where ν is a non-negative Borel measure on Γ that is singular with respect to harmonic measure ω_{t_0} and $P(\cdot; z)$ is the Poisson kernel for $z \in \Omega$.

An *outer function* in MH^p is an element F of MH^p of the form

$$\log|F(z)| = \int_\Gamma u(s) P(s; z) d\omega_{t_0}(s)$$

where $u \in L^1(\Gamma, \omega_{t_0})$ and $e^u \in L^p(\Gamma, \omega_{t_0})$.

The basic theorem on factorization is this.

THEOREM 2.1. *Each function $f \in MH^p(\Omega)$ has a factorization as*

$$f = BSF$$

where B is a Blaschke product, S is a singular inner function, and F is an outer function in $MH^p(\Omega)$. The factors are unique up to multiplication by unimodular constant. Even if f is single-valued, the factors need not be.

2.2. *The Nevanlinna counting function.* Our first goal is to generalize the Nevanlinna counting function to the domain Ω and understand some of its properties.

Definition 2.2. Let $\varphi \colon \Omega \to \Omega$ be an analytic function. The Nevanlinna counting function for φ, $N_\varphi(w)$ for $w \in \Omega \setminus \{\varphi(t_0)\}$, is

$$N_\varphi(w) = \sum_{\varphi(z)=w} g_\Omega(z; t_0).$$

Note that this reduces to the counting function defined previously if Ω is the unit disk Δ and $t_0 = 0$.

For the Nevanlinna counting function on the unit disk, there is the classical theorem of Littlewood [4]:

THEOREM 2.3. *Let ψ be a holomorphic self-map of the unit disk Δ. Then*

$$N_\psi(w) \le \log \left| \frac{1 - \overline{\varphi(0)} w}{\varphi(0) - w} \right|, \quad w \in \Delta \setminus \{\psi(0)\} \tag{3}$$

with equality holding for quasi-every w (i.e., all w except those in a set of capacity zero) exactly when ψ is inner.

If $\psi(0) = 0$, then (3) reduces to

$$N_\psi(w) \le -\log |w|$$

which is an improvement of the Schwarz inequality.

For counting functions on Ω, we have the following generalization of Littlewood's inequality:

THEOREM 2.4. *Let $\varphi \colon \Omega \to \Omega$ be analytic and fix the point t_0. Then*

$$N_\varphi(w) = \sum_{\varphi(z)=w} g(z; t_0) \le g(w; t_0) \text{ for all } w \in \Omega \setminus \{t_0\},$$

with equality holding (for quasi-every w) exactly when $\varphi(\Gamma) \subset \Gamma$, by which we will mean that the boundary values of φ on Γ lie in Γ almost everywhere (with respect to ω_{t_0}).

Proof. Let $g(z; w)$ be the Green's function for Ω with pole at w. The function $g(\varphi(z); w)$ is harmonic on Ω except at the collection of isolated points where $\varphi(z) = w$; at such a point, $g(\varphi(z); w)$ has a logarithmic pole. Let $^*g(\varphi(z); w)$ be the (multiple-valued) harmonic conjugate of $g(\varphi(z); w)$ on $\Omega \setminus \{\varphi(z) = w\}$ and set

$$Q_w(z) = e^{-g(\varphi(z); w) - i^*g(\varphi(z); w)}.$$

Q_w lies in MH^∞; indeed, its modulus is bounded by one. Using Theorem 2.1, we factor Q_w in $H^2(\Omega)$ as $Q_w(z) = B_w(z) S_w(z) F_w(z)$ where the factors are a Blaschke product, a singular inner function, and an outer function, respectively. We then get

$$\begin{aligned} -\log|Q_w(t_0)| &= g(\varphi(t_0); w) \\ &= g(t_0; w) \\ &= -\log|B_w(t_0)| - \log|S_w(t_0)| - \log|F_w(t_0)|. \end{aligned}$$

The function $Q_w(z)$ has zeros exactly where $\varphi(z) = w$, so we have

$$-\log|B_w(t_0)| = \sum_{\varphi(z)=w} g(t_0; z) = N_\varphi(w).$$

Thus we see that

$$g(w; t_0) = N_\varphi(w) - \log|S_w(t_0)| - \log|F_w(t_0)|,$$

so

$$g(w; t_0) \geq N_\varphi(w).$$

We have equality when both $\log|S_w(t_0)|$ and $\log|F_w(t_0)|$ are zero, which happens when both S_w and F_w are unimodular constants. If $\varphi(\Gamma) \subset \Gamma$, then we will have $|Q_w| = 1$ almost everywhere on Γ and thus $F_w \equiv 1$. By the extension of Frostman's theorem, Theorem 2.6, which is proved below since Q_w is a Blaschke product composed with φ, it has trivial singular factor for quasi-every w in Ω. □

The well-known theorem of Frostman, for functions on the unit disk, can be stated as follows:

THEOREM 2.5. *Let ψ be an inner function on Δ. Then for $|w| < 1$, the function*

$$q_w(z) = \frac{\psi(z) - w}{1 - \overline{w}\psi(z)} \tag{4}$$

is a Blaschke product except possibly for a set of w in Δ of logarithmic capacity zero.

For our generalization, we will prove the following theorem and associated lemma, which are suggested in [1, Ch. 5, Exercise 2, 3]:

THEOREM 2.6. *Let φ be an analytic function on Ω with $\varphi(\Gamma) \subset \Gamma$, and $B_w(z) = \exp\{-g(z;w) - i^*g(z;w)\}$ be the Blaschke product on Ω with zero at w. Then the function*

$$Q_w(z) = B_w(\varphi(z))$$

is a Blaschke product (on Ω), except possibly for a set of w in Ω of logarithmic capacity zero.

Proof. For Π, the universal covering map from Δ onto Ω (with $\Pi(0) = t_0$), we have the pull-back map ψ: $\Delta \to \Delta$ which satisfies $\varphi \circ \Pi = \Pi \circ \psi$. It is easy to see that if $\varphi(\Gamma) \subset \Gamma$, then ψ must be inner. Define

$$E = \left\{ w \in \Delta : \frac{\psi(z) - w}{1 - \overline{w}\psi(z)} \text{ has a nontrivial singular factor} \right\}.$$

By Theorem 2.5 above, E has logarithmic capacity zero, thus so does $\Pi(E)$ (in Ω). We write

$$\begin{aligned} Q_w \circ \Pi &= B_w \circ \varphi \circ \Pi \\ &= B_w \circ \Pi \circ \psi. \end{aligned}$$

By Lemma 2.7, below, $B_w \circ \Pi$ is a Blaschke product on Δ, with zeros at those points z with $\Pi(z) = w$. The function $B_w \circ \Pi \circ \psi$ is thus a (constant times a) product of terms of the form (4). If w is not in $\Pi(E)$, then each of these terms is a Blaschke product. Thus $B_w \circ \Pi \circ \psi = B_w \circ \varphi \circ \Pi$ is a Blaschke product, and, again by Lemma 2.7, $B_w \circ \varphi = Q_w$ is a Blaschke product on Ω. □

LEMMA 2.7. *The analytic function B on Ω is a Blaschke product if and only if $B \circ \Pi$ is a Blaschke product on Δ.*

Proof. If B is a Blaschke product on Ω, we can write

$$B(z) = e^{-\Sigma g(z;z_j) - i^*(\Sigma g(z;z_j))}$$

for some sequence $\{z_j\}$ with the property that $\sum_1^\infty g(\zeta; z_j) < \infty$ for each $\zeta \in \Omega$. $B \circ \Pi$ is easily seen to be an inner function on Δ, so we can write

$$B \circ \Pi = bS, \tag{5}$$

where b is a Blaschke product on Δ and S is a singular inner function. The Blaschke product b has a zero at any z with $\Pi(z) = z_j$ for some j. We now see that

$$\begin{aligned} -\log|B \circ \Pi(0)| &= -\log|B(t_0)| \\ &= \sum_j g(t_0; z_j) \\ &= \sum_j \sum_{\Pi(z)=z_j} \log\frac{1}{|z|} \\ &= -\log|b(0)|. \end{aligned}$$

The third line above comes from the fact [5, VII.5] that we can write the Green's function for Ω in terms of Green's functions on the unit disk,

$$g(w; t_0) = \sum_{\Pi(a)=w} \log \frac{1}{|a|}.$$

But (5) gives us $-\log|B \circ \Pi(0)| = -\log|b(0)| - \log|S(0)|$, so $|S(0)| = 1$, and thus $S \equiv 1$; i.e., $B \circ \Pi = b$ is a Blaschke product in Δ.

Now assume $B \circ \Pi$ is a Blaschke product on Δ. It is easy to see that B must be an inner function on Ω, so it has the factorization in $H^\infty(\Omega)$,

$$B = bS$$

where b is a Blaschke product on Ω and S is a singular inner function on Ω (i.e., S has boundary values of modulus 1 a.e., and has no zeros on Ω). We then have

$$B \circ \Pi = (b \circ \Pi)(S \circ \Pi),$$

and we can easily see that $S \circ \Pi$ is a function on Δ which has no zeros and has boundary values of 1 a.e., so $S \circ \Pi$ is a singular inner function. But $B \circ \Pi$ is a Blaschke product, so has only trivial singular inner factor; i.e., $S \circ \Pi$ is trivial, so S must be trivial, and B must be a Blaschke product. □

2.3. *The sub-mean-value property.* We will need the following property for the counting function on Ω:

THEOREM 2.8. *Let h be an analytic function on a domain U. Suppose that D is an open disk in $U \backslash h^{-1}(t_0)$ with center at a and that $h(D) \subset \Omega$. Then*

$$N_\varphi(h(a)) \le \frac{1}{A(D)} \int_D N_\varphi(h(w))\,dA(w) \tag{6}$$

where N_φ is the counting function for Ω and A is area measure.

Proof. This sub-mean-value property follows from the version proved in [7], since we can express our counting function on Ω as a counting function on the unit disk:

$$N_\varphi(w) = \sum_{\varphi(z)=w} g(z; t_0).$$

As we did earlier, we write the Green's function of Ω in terms of the Green's function on Δ to get

$$N_\varphi(w) = \sum_{\varphi(z)=w} g(z; t_0)$$

$$= \sum_{\varphi(z)=w} \sum_{\Pi(a)=z} \log \frac{1}{|a|}$$
$$= \sum_{\varphi\circ\Pi(a)=w} \log \frac{1}{|a|}$$
$$= N_{\varphi\circ\Pi}(w),$$

for Π the universal covering map of the unit disk onto Ω which maps 0 to t_0. In this last line, $N_{\varphi\circ\Pi}(w)$ is the counting function on the unit disk. It is shown in [7] that $N_{\varphi\circ\Pi}(h(w))$ has the required sub-mean-value property, so $N_{\varphi}(h(w))$ has the same property. □

2.4. *The Littlewood-Paley identity.* For functions in $H^2(\Delta)$, we have the Littlewood-Paley identity [7]:

THEOREM 2.9. *For functions* $f \in H^2(\Delta)$,

$$\|f\|^2_{H^2(\Delta)} = \frac{1}{2\pi}\int_T \left|f(e^{i\theta})\right|^2 d\theta = \frac{2}{\pi}\int_\Delta \left|f'(z)\right|^2 \log(1/|z|)dA(z) + |f(0)|^2 .$$

The corresponding theorem on Ω is:

THEOREM 2.10. *For functions* $f \in H^2(\Omega)$,

$$\|f\|^2_{H^2(\Omega)} = \int_\Gamma |f|^2\, d\omega_{t_0} = \frac{2}{\pi}\int_\Omega \left|f'(z)\right|^2 g(z;t_0)dA + |f(t_0)|^2,$$

where ω_{t_0} *is harmonic measure on* Γ *for* t_0.

Proof. Let r be a small positive number and let $\Omega_r = \Omega \setminus \{z : |z-t_0| \le r\}$. The boundary of Ω_r is $\Gamma_r = \Gamma \cup \{z : |z-t_0| = r\}$. We begin with Green's formula:

$$\int_{\Gamma_r} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds = \int_{\Omega_r} (u\Delta v - v\Delta u)\, dA.$$

We take $u = |f|^2$ and $v = g(\cdot;t_0)$. We have

$$d\omega_{t_0} = \frac{-1}{2\pi}\frac{\partial v}{\partial n} ds \quad \text{and} \quad \Delta u = 4|f'|^2.$$

On the circle $|z-t_0| = r$, the normal derivative of v is the radial derivative and equals $1/r$ plus a bounded term. This gives a term on the left-hand side of $2\pi|f(t_0)|^2$ as $r \to 0$. On the other hand, v itself is $\log r$ plus a bounded term and so the other term from the left-hand side goes to zero as $r \to 0$. On the right-hand side, the Laplacian of v is identically zero on Ω_r and v itself is $\log s$, $0 < s \le r$ plus a bounded term on the circle $|z-t_0| = s$. Thus, the right-hand side approaches $-4\int_\Omega |f'(z)|^2 dA$ as $r \to 0$. Rearrangement gives the conclusion. □

2.5. *Change of variable formulas.* We also have the change of variable formula for the disk (see [7]):

THEOREM 2.11. *For any positive, measurable function F on Δ, and analytic self-map ψ of Δ,*

$$\int_\Delta F(\psi(z)) \left|\psi'(z)\right|^2 \log(1/|z|)dA(z) = \int_\Delta FN_\psi dA.$$

We will need the corresponding theorem on the domain Ω.

THEOREM 2.12. *For any positive, measurable function F on Ω, and analytic self-map φ of Ω,*

$$\int_\Omega F(\varphi(z)) \left|\varphi'(z)\right|^2 g(z; t_0)dA(z) = \int_\Omega F(z)N_\varphi(z)dA(z).$$

Proof. The proof follows closely the one in [7]. Since φ is a local homeomorphism on the open set Ω' formed by deleting from Ω the zeros of φ', there exists a countable collection $\{R_j\}$ of disjoint open regions in Ω' the union of whose closures is Ω, and such that φ is one-to-one on each R_j. Let ψ_j denote the inverse of the restriction of φ to R_j, so that ψ_j is a one-to-one map taking $\varphi\left(R_j\right)$ back onto R_j. By the usual change of variable formula applied on R_j, with $z = \psi_j(w)$,

$$\int_{R_j} F(\varphi(z)) \left|\varphi'(z)\right|^2 g(z; t_0)dA = \int_{\varphi(R_j)} F(w)g\left(\psi_j(w); t_0\right) dA(w).$$

Thus, if χ_j denotes the characteristic function of the set $\varphi\left(R_j\right)$,

$$\int_\Omega (F \circ \varphi) \left|\varphi'\right| g(\cdot; t_0)dA = \int_\Omega F(w) \left\{\sum_j \chi_j(w)g\left(\psi_j(w); t_0\right)\right\} dA(w).$$

This is the desired formula, since the term in curly braces on the right side of the equation above is $N_\varphi(w)$. $\square$

We will also need the following version of this change of variable formula.

COROLLARY 2.13. *For each f analytic on Ω,*

$$\|f \circ \varphi\|^2_{H^2(\Omega)} = \frac{2}{\pi} \int_\Omega \left|f'\right|^2 N_\varphi dA + |f(\varphi(t_0))|^2 .$$

Proof. The generalized form of the Littlewood–Paley identity applied to $f \circ \varphi$ yields

$$\begin{aligned} \|f \circ \varphi\|^2_{H^2(\Omega)} &= \frac{2}{\pi} \int \left|(f \circ \varphi)'(z)\right|^2 g(z; t_0)\, dA + |f(\varphi(t_0))|^2 \\ &= \frac{2}{\pi} \int \left|f' \circ \varphi\right|^2 \left|\varphi'\right|^2 g(z; t_0)\, dA + |f(\varphi(t_0))|^2 \end{aligned}$$

(by the chain rule). An application of the change of variable formula, with $F = \left|f'\right|^2$, completes the proof. □

2.6. *A basis for* $H^2(\Omega)$.

THEOREM 2.14. *Let Ω be bounded by $p + 1$ disjoint circles, $\Gamma_0, \dots, \Gamma_p$, and let Ω_i, $i = 0, 1, \dots p$ be defined as at the beginning of this section. There is an orthonormal basis of $H^2(\Omega)$, say $u_0, u_1, u_2, \dots$, with this property: if $f \in H^2$ has the expansion $\sum_0^\infty c_k u_k$, then $f - \sum_0^{m(p+1)} c_k u_k$ has a zero at t_0 of order at least m, $m = 1, 2, 3, \dots$.*

Proof. Let ϕ_j be the linear fractional transformation that maps Ω_j onto the unit disk Δ, normalized so that $\phi_j(t_0) = 0$. Let $u_{jk} = \phi_j^k$; then $\{u_{jk}\}_{k=0}^\infty$ is an orthonormal basis of $H^2(\Omega_j)$, $j = 0, \dots p$ and u_{jk} vanishes to order k at t_0. Arrange the functions u_{jk} as

$$u_{00}, u_{10}, \dots, u_{p0}, u_{01}, u_{11}, \dots, u_{p1}, u_{02}, \dots$$

and renumber them as $v_0, v_1, v_2, \dots$. Now let E_n be the closed linear span of $\{v_{n+1}, v_{n+2}, \dots\}$, $n = 0, 1, 2, \dots$. Finally, let u_m be the (normalized) projection of v_m onto the orthogonal complement in H^2 of E_n.[1] We now check that these functions have the desired properties. Evidently, by its very construction, u_0 is orthogonal to $v_1, v_2, \dots$. We write $u_1 = v_1 + h_1$ where $h_1 \in E_1$. Hence,

$$0 = \langle u_0, v_1\rangle = \langle u_0, u_1\rangle - \langle u_0, h_1\rangle = \langle u_0, u_1\rangle,$$

and so u_0 is orthogonal to u_1. Next, u_2 is orthogonal to $v_3, v_4, \dots$. We write $u_2 = v_2 + h_2$ where $h_2 \in E_2$. Hence,

$$0 = \langle u_0, v_2\rangle = \langle u_0, u_2\rangle - \langle u_0, h_2\rangle = \langle u_0, u_2\rangle$$

and

$$0 = \langle u_1, v_2\rangle = \langle u_1, u_2\rangle - \langle u_1, h_2\rangle = \langle u_1, u_2\rangle$$

[1] Thanks to Todd Young for his contribution of this idea to the proof.

so that u_2 is orthogonal to both of u_0 and u_1. In a similar way we can see that the functions $u_0, u_1, u_2, \ldots$ are mutually orthogonal. Next, it is easy to establish that the linear span of $u_0, \ldots, u_n$ is the orthogonal complement of E_n, $n = 0, 1, 2, \ldots$ and so if $v \in H^2$ is orthogonal to $u_0, u_1, \ldots$, then

$$v \in \bigcap_{n=0}^{\infty} E_n.$$

and thus $v = 0$. Finally, suppose $f \in H^2$ has an orthonormal expansion $f = \sum c_k u_k$. Those u_k with $k > (p+1)m$ vanish at t_0 to order at least m and hence $f - \sum_0^{m(p+1)} c_k u_k$ has a zero at t_0 of order at least m.

PROPOSITION 2.15. *Suppose that $f \in H^2(\Omega)$ vanishes to order n at t_0. Let Π be the universal covering map from Δ to Ω. For $\zeta \in \Omega$ and $\Pi(z) = \zeta$, let λ_Ω be the Poincaré metric for Ω. Then:*

(a) $|f(\zeta)| \le \frac{|z|^n}{\sqrt{1-|z|^2}} (\lambda_\Omega(\zeta)\|\Pi\|_\infty)^{\frac{1}{2}} \|f\|_2$;

(b) $|f'(\zeta)| \le \sqrt{2}n \frac{|z|^{n-1}}{\sqrt{1-|z|^2}} (\lambda_\Omega(\zeta))^{\frac{3}{2}} \|f\|_2 \|\Pi\|_\infty^{\frac{1}{2}}$.

Proof. Let $g = f \circ \Pi$ so that g has a zero of order n at the origin. Then use the standard estimates (see [7]) in the unit disk plus the fact that $\lambda_\Omega(\zeta) = |\Pi'(z)|(1 - |z|^2) \le \|\Pi\|_{Bloch} \le \|\Pi\|_\infty$.

3. The main theorem

With the background of Section 2 in place, we are now ready to state and prove the main result of this paper.

THEOREM 3.1. *Suppose that Ω is finitely connected and that φ is an analytic function mapping Ω into itself with $\varphi(t_0) = t_0$. Let $\|C_\varphi\|_e$ denote the essential norm of C_φ, regarded as an operator on $H^2(\Omega)$. Then*

$$\left\|C_\varphi\right\|_e^2 = \limsup_{w \to \Gamma} \frac{N_\varphi(w)}{g(w; t_0)}.$$

In particular, C_φ is compact on H^2 if and only if

$$\lim_{w \to \Gamma} \frac{N_\varphi(w)}{g(w; t_0)} = 0.$$

We will prove the theorem by proving separately upper and lower bounds for the essential norm of C_φ.

3.1. *The upper bound.* We will use the following general formula from [7] for the essential norm of a linear operator on a Hilbert space:

THEOREM 3.2. *Suppose T is a bounded linear operator on a Hilbert space H. Let $\{K_n\}$ be a sequence of compact self-adjoint operators on H, and write $R_n = I - K_n$. Suppose $\|R_n\| = 1$ for each n, and $\|R_n x\| \to 0$ for each $x \in H$. Then $\|T\|_e = \lim_n \|TR_n\|$.*

The goal now is to show that, for an analytic function $\varphi\colon \Omega \to \Omega$ which fixes the point t_0,

$$\left\|C_\varphi\right\|_e^2 \leq \limsup_{w\to\Gamma} \frac{N_\varphi(w)}{g(w;t_0)}. \tag{7}$$

We do this by applying Theorem 3.2 above with K_n the operator which takes f to the sum of the first $(p+1)n$ terms in its expansion relative to the basis we have chosen for $H^2(\Omega)$ in Theorem 2.14.

For this orthonormal basis $u_0, u_1, u_2, \ldots$ of $H^2(\Omega)$, we can write any $f \in H^2(\Omega)$ as $f = \sum_{k=0}^{\infty} c_k u_k$, and then $K_n f = \sum_{k=0}^{(p+1)n} c_k u_k$. $R_n = I - K_n$ will then be an operator with the property that $R_n f = \sum_{k=1+(p+1)n}^{\infty} c_k u_k$ has a zero of order at least n at t_0.

The operator K_n is self-adjoint and compact. Since $R_n = I - K_n$, its norm is 1, so that the hypotheses of the proposition are fulfilled, and

$$\left\|C_\varphi\right\|_e = \lim_{n\to\infty} \left\|C_\varphi R_n\right\|.$$

To estimate the right side of the above, fix a function f in the unit ball of $H^2(\Omega)$, and a positive integer n. Then by Corollary 2.13 we get

$$\left\|C_\varphi R_n f\right\|_{H^2(\Omega)}^2 = \frac{2}{\pi}\int \left|(R_n f)'\right|^2 N_\varphi dA + |R_n f(\varphi(t_0))|^2.$$

Since $\|f\|_{H^2(\Omega)} \leq 1$, the same is true of $R_n f$.

Now fix $r < 1$. Split the integral above into two parts, $\Omega_r = \Pi(r\Delta)$ (where Π is the universal covering map of Ω which maps the origin to t_0), and the other its complement in Ω, Ω_r^c. Then take the supremum of both sides of the resulting inequality over all functions f in the unit ball B of $H^2(\Omega)$. We obtain

$$\begin{aligned}\left\|C_\varphi R_n\right\|^2 \leq\ & \sup_B \frac{2}{\pi}\int_{\Omega_r} \left|(R_n f)'\right|^2 N_\varphi dA \\ & + \sup_B \frac{2}{\pi}\int_{\Omega_r^c} \left|(R_n f)'\right|^2 N_\varphi dA + |R_n f(\varphi(t_0))|^2.\end{aligned}$$

We now use the pointwise estimate for $(R_n f)'$, from Proposition 2.15 part (b). $R_n f$ has a zero of order at least n at t_0, and, for $w \in \Omega_r$, $w = \pi(z)$ for some z with $|z| < r$.

Thus we have

$$\left|(R_n f)'(w)\right| \le \sqrt{2}n \frac{|r|^{n-1}}{\sqrt{1-|r|^2}} (\lambda_\Omega(\zeta))^{\frac{3}{2}} \|f\|_2 \|\Pi\|_\infty^{\frac{1}{2}},$$

so

$$\begin{aligned}
&\frac{2}{\pi}\int_{\Omega_r} \left|(R_n f)'\right|^2 N_\varphi dA \\
&\qquad \le \left(\sqrt{2}n \frac{|r|^{n-1}}{\sqrt{1-|r|^2}} (\lambda_\Omega(\zeta))^{\frac{3}{2}} \|f\|_2 \|\Pi\|_\infty^{\frac{1}{2}}\right)^2 \frac{2}{\pi}\int_{\Omega_r} N_\varphi dA.
\end{aligned}$$

Since the right side is $\left(n|r|^{n-1}\right)^2$ multiplied by terms which are bounded (independent of n), we get

$$\frac{2}{\pi}\int_{\Omega_r} \left|(R_n f)'\right|^2 N_\varphi dA \to 0$$

as $n \to \infty$.

We can also easily see that $|R_n f(\varphi(t_0))|^2 = |R_n f(t_0)|^2 = 0$ for all $n \ge 1$. As f runs through the unit ball of $H^2(\Omega)$, $R_n f$ runs through a subset of the ball, so we can replace $R_n f$ in the remaining integral with f and only increase the right side. We use $h(w) = N_\varphi(w)/g(w;t_0)$, to obtain

$$\begin{aligned}
\lim_{n\to\infty} \left\|C_\varphi R_n\right\|^2 &\le \lim_{n\to\infty} \sup_B \frac{2}{\pi}\int_{\Omega_r^c} \left|(R_n f)'\right|^2 N_\varphi dA \\
&\le \sup_B \frac{2}{\pi}\int_{\Omega_r^c} \left|f'(w)\right|^2 \frac{N_\varphi(w)}{g(w;t_0)} g(w;t_0) dA \\
&= \sup_B \frac{2}{\pi}\int_{\Omega_r^c} \left|f'(w)\right|^2 h(w) g(w;t_0) dA \\
&\le \sup\{h(w) : w \in \Omega_r^c\} \sup_B \frac{2}{\pi}\int_{\Omega} \left|f'(w)\right|^2 g(w;t_0) dA \\
&\le \sup\{h(w) : w \in \Omega_r^c\},
\end{aligned}$$

where the last line follows from the generalized version of the Littlewood-Paley identity, Theorem 2.10. As $r \to 1$, $w \in \Omega_r^c$ means that w is the image under Π of only those z with $|z| \ge r$, so $w \to \Gamma$; thus

$$\sup\{h(w) : w \in \Omega_r^c\} \to \limsup_{w\to\Gamma} h(w),$$

giving us (7).

3.2. *The lower bound.* We now wish to complete the proof of Theorem 3.1 by showing

$$\|C_\varphi\|_e^2 \geq \limsup_{w\to\Gamma} \frac{N_\varphi(w)}{g(w;t_0)}. \tag{8}$$

The following elementary proposition will be used.

PROPOSITION 3.3. *Suppose T is a bounded operator on a Banach space X and $\{x_n\}$ is a sequence in the unit ball of X that goes weakly to zero. Then $\|T\|_e \geq \limsup_{n\to\infty} \|Tx_n\|$.*

Proof. Let K be a compact operator. Then

$$\|T-K\| \geq \limsup \|(T-K)x_n\| = \limsup \|Tx_n\|$$

since $\|Kx_n\| \to 0$. Now take the infimum over all K to get the desired conclusion. □

We will apply Proposition 3.3 to the operator C_φ with the role of $\{x_n\}$ played by normalized reproducing kernels for the spaces $H^2(\Omega_j)$. We fix $j, 0 \leq j \leq p$ and let ϕ be the linear fractional transformation that maps Ω_j onto the unit disk Δ, with $\phi(t_0) = 0$. We then know that

$$\int_{\Gamma_j} u\circ\phi\, d\omega_j = \frac{1}{2\pi}\int_T u\, d\theta, \quad u \text{ continuous on } T,$$

where ω_j is the harmonic measure on Γ_j relative to Ω_j for the point t_0. It then follows that the reproducing kernel for a on $H^2(\Omega_j)$ is given by

$$K_a^{\Omega_j}(z) = \frac{1}{1-\overline{\phi(a)}\phi(z)}.$$

From Proposition 3.3, we see that

$$\|C_\varphi\|_e^2 \geq \limsup_{a\to\Gamma_j} \frac{\|C_\varphi K_a^{\Omega_j}\|^2}{\|K_a^{\Omega_j}\|^2}. \tag{9}$$

We may compute terms on the right above in the following way:

$$\begin{aligned}
\|C_\varphi K_a^{\Omega_j}\|^2 &= \int_\Gamma |K_a^{\Omega_j}\circ\varphi|^2 d\omega_{t_0} \\
&= \frac{2}{\pi}\int_\Omega |K_a^{\Omega_j\prime}(\varphi(z))||\varphi'(z)|^2 dA(z) + |K_a^{\Omega_j}(t_0)|^2 \\
&= \frac{2}{\pi}\int_\Omega \frac{|\phi(a)|^2}{|1-\overline{\phi(a)}\phi(\varphi(z))|^4}|\phi'(\varphi(z))|^2|\varphi'(z)|^2 dA(z) + 1
\end{aligned}$$

$$= \frac{2}{\pi} \int_{\Omega} \frac{|\phi(a)|^2}{|1 - \overline{\phi(a)}\phi(w)|^4} |\phi'(w)|^2 N_\varphi(w) dA(w) + 1$$
$$= \frac{2}{\pi} \int_U \frac{|b|^2}{|1 - \overline{b}\zeta|^4} N_\varphi(\phi^{-1}(\zeta)) dA(\zeta) + 1,$$

where $b = \phi(a)$, and $U = \phi(\Omega)$. We now make the change of variables $\zeta = \tau_b(\xi) = \frac{b-\xi}{1-\overline{b}\xi}$ or $\xi = \tau_b(\zeta)$. We then obtain

$$\|C_\varphi K_a^{\Omega_j}\|^2 \geq \frac{|b|^2}{(1 - |b|^2)^2} \frac{2}{\pi} \int_{U_b} N_\varphi(\phi^{-1}(\tau_b(\xi))) dA(\xi) \tag{10}$$

where $U_b = \tau_b(U)$.

We need to compute the norm of $K_a^{\Omega_j}$ in $H^2(\Omega)$ exactly. For simplicity, we set $b = \phi(a)$. Then

$$\begin{aligned}
\|K_a^{\Omega_j}\|^2 &= \int_\Gamma \left| \frac{1}{1 - \overline{b}\phi(z)} \right|^2 d\omega_{t_0}(z) \\
&= 1 + \frac{2}{\pi} \int_\Omega \frac{1}{|1 - \overline{b}\phi(z)|^4} |b|^2 |\phi'(z)|^2 g_\Omega(z; t_0) dA(z) \\
&= 1 + \frac{2}{\pi} \int_U \frac{1}{|1 - \overline{b}\zeta|^4} |b|^2 g_U(\zeta; 0) dA(\zeta) \qquad (\text{where } U = \phi(\Omega)) \\
&= 1 + \frac{2}{\pi} \frac{|b|^2}{(1 - |b|^2)^4} \int_{U_b} g_{U_b}(\xi; b) dA(\xi) \qquad \left(\text{using } \xi = \frac{b - \zeta}{1 - \overline{b}\zeta}\right).
\end{aligned}$$

This last integral may be computed using Theorem 2.10 with $f(z) = z$. This will give

$$\begin{aligned}
\frac{2}{\pi} \int_{U_b} g_{U_b}(\xi; b) dA(\xi) &= -|b|^2 + \int_{\partial U_b} |z|^2 d\omega_b^{U_b}(z) \\
&= -|b|^2 + \int_{\partial U} \left[\frac{|\zeta - b|}{|1 - \overline{b}\zeta|} \right]^2 d\omega_0^U(\zeta) \\
&= -|b|^2 + 1 - (1 - |b|^2) \int_S \frac{1 - |z|^2}{|1 - \overline{b}z|^2} d\omega_0^U(z) \\
&= (1 - |b|^2) \left(1 - \int_S \frac{1 - |z|^2}{|1 - \overline{b}z|^2} d\omega_0^U(z) \right)
\end{aligned}$$

where S is the union of that part of the boundary of U that lies inside Δ, so that S is p disjoint circles lying inside Δ. When this is substituted into the expression for $\|K_a^{\Omega_j}\|^2$ we obtain

$$\|K_a^{\Omega_j}\|^2 = \left(\frac{1}{1 - |b|^2} \right) \left(1 - |b|^2 \int_S \frac{1 - |z|^2}{|1 - \overline{b}z|^2} d\omega_0^U \right) \tag{11}$$

$$= \left(\frac{1}{1 - |b|^2} \right) (1 - I(b)) \tag{12}$$

where

$$I(b) = |b|^2 \int_S \frac{1-|z|^2}{|1-\bar{b}z|^2} d\omega_0^U.$$

We now put (12) together with (10) and obtain

$$\frac{\|C_\varphi K_a^{\Omega_j}\|^2}{\|K_a^{\Omega_j}\|^2} \geq \frac{1}{(1-|b|^2)(1-I(b))} \frac{2}{\pi} \int_{U_b} N_\varphi(\phi^{-1}(\tau_b(\xi)))dA(\xi). \tag{13}$$

The open set $U = \phi(\Omega)$ has the form $U = \Delta \setminus P$ where P is the union of p closed disks. As $|b| \to 1$, τ_b converges uniformly on compact subsets on Δ to a unimodular constant. Thus, $\tau_b(P)$ (as a subset of Δ) converges to the unit circle as $|b| \to 1$; in particular, if $r < 1$ is given, then U_b contains the disk $\{|\xi| \leq r\}$ when $|b|$ is near enough to 1; that is, when a is near enough to Γ_j. Thus,

$$\int_{U_b} N_\varphi(\phi^{-1}(\tau_b(\xi)))dA(\xi) \geq \int_{|\xi|\leq r} N_\varphi(\phi^{-1}(\tau_b(\xi)))dA(\xi).$$

However, by the sub-mean-value property for the counting function, Theorem 2.8, we obtain

$$\frac{2}{\pi}\int_{|\xi|\leq r} N_\varphi(\phi^{-1}(\tau_b(\xi)))dA(\xi) \geq 2N_\varphi(\phi^{-1}(\tau(0)))r^2 = 2N_\varphi(a)r^2.$$

When this and (13) are applied to (9) we obtain

$$\|C_\varphi\|_e \geq \limsup_{a\to\Gamma_j} \frac{2}{(1-|b|^2)(1-I(b))} N_\varphi(a)r^2.$$

The number r may be arbitrarily near 1 so that it may be removed from this last inequality yielding

$$\|C_\varphi\|_e \geq \limsup_{a\to\Gamma_j} \frac{2N_\varphi(a)}{(1-|b|^2)(1-I(b))}, \qquad b = \phi(a). \tag{14}$$

Next, $g_\Omega(a; t_0) = g_U(b; 0)$ and so

$$\frac{2}{(1-|b|^2)} N_\varphi(a) = \frac{N_\varphi(a)}{g_\Omega(a;t_0)} \frac{g_U(b;0)}{\frac{1}{2}\left(1-|b|^2\right)} \tag{15}$$

We now claim that

$$\lim_{b\to e^{it}} \frac{g_U(b;0)}{\frac{1}{2}\left(1-|b|^2\right)(1-I(b))} = 1, \qquad 0 \leq t \leq 2\pi. \tag{16}$$

(14), (15), and (16) imply that

$$\|C_\varphi\|_e \geq \limsup_{a\to\Gamma_j} \frac{N_\varphi(a)}{g_\Omega(a;t_0)} \tag{17}$$

which together with (7) proves the theorem.

To see that (16) holds, we first note that

$$\lim_{b\to e^{it}} \frac{g_U(b;0)}{\frac{1}{2}\left(1-|b|^2\right)} = \lim_{b\to e^{it}} \frac{g_U(b;0)}{1-|b|} = \frac{\partial g_U}{\partial n}(e^{it}) = V(e^{it})$$

where V is the function such that $V dt = 2\pi d\omega_0^U$ on the unit circle $\mathbb{T}$. Now let u be any continuous function on $\mathbb{T}$ and let $\widetilde{u}$ denote its harmonic extension to Δ via the Poisson kernel. Then

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{T}} u(t)\left(1-\int_S \frac{1-|z|^2}{|1-\overline{e^{it}}z|^2}d\omega_0^U\right)dt \\
&\qquad = \widetilde{u}(0) - \int_S \left(\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{e^{it}}z|^2}u(t)dt\right)d\omega_0^U(z) \\
&\qquad = \widetilde{u}(0) - \int_S \widetilde{u}(z)d\omega_0^U(z) \\
&\qquad = \widetilde{u}(0) - \int_{\partial U} \widetilde{u}(z)d\omega_0^U(z) + \int_{\mathbb{T}} u d\omega_0^U(z) \\
&\qquad = \frac{1}{2\pi}\int_{\mathbb{T}} uV dt.
\end{aligned}$$

This shows that

$$\begin{aligned}
V(e^{it}) &= 1-\int_S \frac{1-|z|^2}{|1-\overline{e^{it}}z|^2}d\omega_0^U \\
&= \lim_{b\to e^{it}} (1-I(b)),
\end{aligned}$$

which gives us (16), so we are done.

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