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INTERMEDIATE RINGS BETWEEN A LOCAL DOMAIN AND ITS COMPLETION

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ABSTRACT. We consider the structure of certain intermediate domains between a local Noetherian domain R and an ideal-adic completion R^* of R that arise as the intersection of R^* with a field containing R. In the case where the intersection domain A can be expressed as a directed union of localized polynomial extension rings of R, the computation of A is easier. We examine conditions for this to happen. We also present examples to motivate and illustrate the concepts considered.

1. Introduction

Summary. Suppose $(R, \mathbf{m})$ is an excellent normal local domain with field of fractions K and $\mathbf{m}$-adic completion $\widehat{R}$. In this paper we consider the structure of an intermediate ring A between R and $\widehat{R}$ of the form $A := K(\tau_1, \tau_2, \dots \tau_s) \cap \widehat{R}$, where $s \in \mathbb{N}$ and $\tau_1, \tau_2, \dots, \tau_s \in \widehat{\mathbf{m}}$ are certain algebraically independent elements over K. This construction follows a tradition begun by Nagata in the 1950's. The intermediate intersection rings provide interesting examples of Noetherian and non-Noetherian, excellent and non-excellent rings. If the intersection ring A can be expressed as a directed union of localized polynomial extension rings of R the computation of A is easier. We say $\tau_1, \tau_2, \dots, \tau_s$ are "limit-intersecting" if A is such a directed union. Two stronger forms of the limit-intersecting condition are useful for constructing examples and for determining if A is Noetherian and excellent. We give criteria for $\tau_1, \tau_2, \dots, \tau_s$ to have these properties. We close with several concrete examples inspired by the construction.

Background. Over the past forty years, a fruitful source of examples of local Noetherian integral domains D has been domains of the form $D := \mathcal{Q}(S) \cap (\widehat{S}/\mathbf{a})$, for certain intermediate local rings $(S, \mathbf{n})$ between R and $\widehat{R}$. Here $\widehat{S}$ denotes the $\mathbf{n}$-adic completion of S, $\mathcal{Q}(S)$ is the fraction field of S, and $\mathbf{a}$ is an ideal of $\widehat{S}$ such that the associated primes of $\mathbf{a}$ are in the generic formal fiber of S. Using this construction, examples D can be produced containing a coefficient field k such that D has finite

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transcendence degree over k, but nevertheless D is non-excellent and sometimes even non-Noetherian. Since in our setting $\widehat{S}/\mathbf{a} \cong \widehat{R}$, we often use a simpler expression for the intersection D, namely $D := L \cap \widehat{R}$, where L is an intermediate field between K and the fraction field of $\widehat{R}$.[1] The following diagram displays the containments among the rings we are discussing:

$$\begin{array}{ccccccc}
K & \xrightarrow{\subseteq} & \mathcal{Q}(S) & = & L & \xrightarrow{\subseteq} & \mathcal{Q}(\widehat{R}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
R & \xrightarrow{\subseteq} & S & \xrightarrow{\subseteq} & D := L \cap (\widehat{S}/\mathbf{a}) = L \cap \widehat{R} & \xrightarrow{\subseteq} & \widehat{R} \cong \widehat{S}/\mathbf{a}.
\end{array}$$

A description of D as a directed union of localized polynomial rings over R is often used in the construction of the examples mentioned above, such as that of [N, (E7.1), p. 210]. If such a realization of D as a directed union exists, it is easier to compute than the description of D as an intersection. In our setting there is a natural sequence of nested localized polynomial rings B_n over R having a directed union B which is contained in D and has the same fraction field as D. The classical method for demonstrating that the intersection $D = L \cap \widehat{R}$ is Noetherian has been to show that B is a Noetherian domain having completion $\widehat{R}$, and therefore that $B = L \cap \widehat{R} = D$.

In this paper we continue an investigation begun in [HRW1] but we modify the focus and approach. In that article, we found non-trivial examples of ideals $\mathbf{a}$ such that the construction described above fails to produce a new ring; that is, where $D := \mathcal{Q}(S) \cap \widehat{S}/\mathbf{a} = S$, or using the expression $D := L \cap \widehat{R}$ for the intersection, the case where D is a localized polynomial ring over R. Our primary goal here is to obtain interesting Noetherian rings, but we expand our working setting to Krull domains because such an intersection domain D may be a birational extension of S which is not Noetherian. Another modification in this paper is that we consider completions with respect to a principal ideal; this is because in most examples of new Noetherian domains produced using the $D = \mathcal{Q}(S) \cap (\widehat{S}/\mathbf{a})$ construction, the ideal $\mathbf{a}$ is extended from a completion of R with respect to a principal ideal. Finally, we analyze the intersection $D := \mathcal{Q}(S) \cap (\widehat{S}/\mathbf{a})$ (or $L \cap \widehat{R}$) more systematically here than in [HRW1]: we focus on conditions in order that the intersection be a directed union of localized polynomial rings over R.

Suppose $\tau_1, \ldots, \tau_s \in \widehat{\mathbf{m}}$ are algebraically independent over K. Then $A := K(\tau_1, \ldots, \tau_s) \cap \widehat{R}$ is a quasilocal Krull domain that *dominates*[2] R and is dominated by $\widehat{R}$. Thus A, as a subring of $\widehat{R}$ and an extension ring of R, is a special type of *intermediate ring* between R and $\widehat{R}$ with fraction field $K(\tau_1, \ldots, \tau_s)$. Indeed, if $\mathbf{a}$ denotes the kernel of the canonical map to $\widehat{R}$ from the completion $\widehat{A}$ of A, then $A = \mathcal{Q}(A) \cap (\widehat{A}/\mathbf{a})$ has the form described above. The ring A *birationally dominates*[3]

[1] See the introduction to [HRW1] for more details.

[2] That is, the maximal ideal of A intersects R in $\mathbf{m}$.

[3] That is, A dominates B_0 and A is contained in the fraction field of B_0.

the localized polynomial ring $B_0 = R[\tau_1, \dots, \tau_s]_{(\mathbf{m},\tau_1,\dots,\tau_s)}$. In the present paper we explore the nature of the birational domination of A over B_0.

Many of the concepts from our earlier work are useful in this study. In [HRW1], the elements $\tau_1, \dots, \tau_s \in \widehat{\mathbf{m}}$ are defined to be *idealwise independent* over R if $A = B_0$. Here, with the assumption that each τ_i is in the completion of R with respect to a principal ideal (and the τ_i are algebraically independent over K), we investigate conditions in order that A can be realized as a directed union of localized polynomial rings over R; that is, $A = B$, where $B := \varinjlim_{n\in\mathbb{N}} B_n$, and, for each n,

$$B_n := R[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{m},\tau_{1n},\dots,\tau_{sn})} \subseteq B_{n+1},$$

where $\tau_{1n}, \dots, \tau_{sn} \in \widehat{\mathbf{m}}$ are series formed by the endpieces of the τ_i (see (2.3)). Essentially what we require in the present paper is that the conditions of [HRW1] be satisfied off the proper closed subset of Spec(R) defined by a principal ideal. This leads to the analysis of "limit-intersecting" independence properties for elements $\tau_1, \dots, \tau_s \in \widehat{\mathbf{m}}$ which are algebraically independent over K; these properties are analogs to types of "idealwise independence" over R defined in [HRW1]. As we show in §6, these modified independence conditions enable us to produce concrete examples illustrating the concepts.

Outline. We start in §2 with a motivating example and a description of the rings A and B for particular elements σ and τ of a power series ring in two variables over a field. In §3, we give some background material from [HRW1], including some definitions, terminology, and related results. Section 4 contains a description of the intermediate rings B_n, whereas §5 gives the new definitions associated with the elements τ_i of $\widehat{\mathbf{m}}$ and their basic properties. In §6 we display concrete examples of idealwise independent elements in the sense of [HRW1].

2. A motivating example

If $\sigma, \tau \in \widehat{R}$ are algebraically independent over R, then $R[t_1, t_2]$, the polynomial ring in two variables over R, can be identified with a subring of $\widehat{R}$ by means of an R-algebra isomorphism mapping $t_1 \to \sigma$ and $t_2 \to \tau$. The structure of the quasilocal domain $A = K(\sigma, \tau) \cap \widehat{R}$ depends on the residual behavior of σ and τ with respect to certain prime ideals of $\widehat{R}$. The following example illustrates this and introduces techniques that are further developed in later sections.

2.1. *Example.* Let k be a field, let x and y be indeterminates over k, and let

$$\sigma := \sum_{i=0}^{\infty} a_i x^i \in k[[x]] \quad \text{and} \quad \tau := \sum_{i=0}^{\infty} b_i y^i \in k[[y]]$$

be formal power series that are algebraically independent over the fields $k(x)$ and $k(y)$, respectively. Consider the integral domain

$$A := k(x, y, \sigma, \tau) \cap k[[x, y]].$$

Using an interesting result of Valabrega in [V], it is easy to show:

2.2. PROPOSITION. *With the notation of* (2.1), *A is a two-dimensional regular local domain with maximal ideal $(x, y)A$ and completion $\widehat{A} = k[[x, y]]$.*

Proof. The ring $C = k(x, \sigma) \cap k[[x]]$ is a rank-one discrete valuation domain with completion $k[[x]]$, and the field $k(x, y, \sigma, \tau) = L$ is an intermediate field between the fields of fractions of the rings $C[y]$ and $C[[y]]$. Hence by [V, Proposition 3], $A = L \cap C[[y]]$ is a regular local domain with completion $k[[x, y]]$. □

In order to give a more explicit description of A, we use the last parts or the *endpieces* of the power series σ and τ. Since in later sections of this article endpieces of other power series are used, we describe endpiece power series in general here.

2.3. *Endpiece Notation.* Let $(T, \mathbf{n})$ be a quasilocal domain such that the $\mathbf{n}$-adic completion $\widehat{T}$ is a normal domain and let $0 \neq z \in \mathbf{n}$. Let T^* be the z-adic completion of T. For $\gamma \in T^*$, write

$$\gamma := \sum_{i=0}^{\infty} c_i z^i, \text{ where } c_i \in T.$$

Then for each $n \in \mathbb{N}$, we define γ_n, the n^{th} *endpiece of* γ with respect to z:

$$\gamma_n := \sum_{i=n+1}^{\infty} c_i z^{i-n}.$$

For each $n \in \mathbb{N}$, we have the relations

$$\gamma_n = c_n z + \gamma_{n+1} z. \tag{2.3.1}$$

Returning to Example 2.1, we describe A using σ_n and τ_n, for $n \in \mathbb{N}$, the endpiece series with respect to x and y, respectively, as described in (2.3). We define

$$C_n := k[x, \sigma_n]_{(x,\sigma_n)}, \qquad D_n := k[y, \tau_n]_{(y,\tau_n)} \qquad \text{and} \qquad B_n := k[x, y, \sigma_n, \tau_n]_{(x,y,\sigma_n,\tau_n)}.$$

These rings are all dominated by $k[[x, y]]$, and the relations (2.3.1) imply the inclusions $C_n \subseteq C_{n+1}$, $D_n \subseteq D_{n+1}$, and $B_n \subseteq B_{n+1}$. Moreover, for each of these inclusions we have also birational domination of the larger ring over the smaller. It is easy to see that the rank-one discrete valuation domains C and D given below can also be described as the direct limits shown:

$$C := k(x, \sigma) \cap k[[x]] = \varinjlim(C_n) = \cup_{n=1}^{\infty} C_n;$$

$$D := k(y, \tau) \cap k[[y]] = \varinjlim(D_n) = \cup_{n=1}^{\infty} D_n.$$

We define

$$B := \varinjlim(B_n) = \cup_{n=1}^{\infty} B_n.$$

Thus B is the directed union of a chain of four-dimensional regular local domains that are essentially finitely generated over k. We show below that the dimension of B is either two or three. We have

$$xB \cap B_n = (x, \sigma_n)B_n \quad \text{and} \quad yB \cap B_n = (y, \tau_n)B_n,$$

where $(x, \sigma_n)B_n$ and $(y, \tau_n)B_n$ are height-two prime ideals of the 4-dimensional regular local domain B_n. Therefore the unique maximal ideal of B is $(x, y)B$. Also, $B_{(xB)}$ and $B_{(yB)}$ are rank-one discrete valuation domains, since each is the contraction to the field $k(x, y, \sigma, \tau)$ of the (x)-adic or the (y)-adic valuations of $k[[x, y]]$. Moreover, B is birationally dominated by the two-dimensional regular local domain $A = k(x, y, \sigma, \tau) \cap k[[x, y]]$.

To summarize and elaborate, we have the following.

2.4. THEOREM. *With notation as above, B is a quasilocal Krull domain with maximal ideal $\mathbf{n} = (x, y)B$, the dimension of B is either 2 or 3, and B is Hausdorff in the topology defined by the powers of $\mathbf{n}$. The $\mathbf{n}$-adic completion $\widehat{B}$ of B is canonically isomorphic to $k[[x, y]]$. Depending on the choice of σ and τ it may or may not be that B is Noetherian, and the following statements are equivalent:*

(1) *B is Noetherian.*
(2) $\dim(B) = 2$.
(3) $B = A$.
(4) *Every finitely generated ideal of B is closed in the $\mathbf{n}$-adic topology on B.*

In particular there exist certain values for σ and τ such that $B \neq A$ and other values such that $B = A$.

Proof. We have already observed that B is a quasilocal domain with maximal ideal $\mathbf{n} = (x, y)B$. Since B is dominated by $k[[x, y]]$, B is Hausdorff in the topology defined by the powers of $\mathbf{n}$. Since $\mathbf{n}$ is finitely generated, $\widehat{B}$ is Noetherian [N, (31.7)]. Therefore $\widehat{B}$ is a 2-dimensional regular local domain that canonically surjects onto $k[[x, y]]$. This canonical surjection must have kernel (0), so we have $\widehat{B} = k[[x, y]]$.

To see that B is a Krull domain, observe that if $\mathbf{q}$ is a height-one prime of B_n, then $\mathbf{q}$ is contained in the union $(x, \sigma_n)B_n \cup (y, \tau_n)B_n$ if and only if $\mathbf{q} \subseteq xB \cup yB$, and if $\mathbf{q}$ is not contained in $xB \cup yB$, then $B_{n+1} \subseteq (B_n)_{\mathbf{q}}$. It follows that if $\mathbf{q}$ is not contained in $xB \cup yB$, then $B \subseteq (B_n)_{\mathbf{q}}$. Moreover, the canonical map $\mathrm{Spec}(B_{n+1}) \to \mathrm{Spec}(B_n)$ restricts to a biregular correspondence of the height-one primes of B_{n+1} not contained in $xB \cup yB$ with the height-one primes of B_n not contained in $xB \cup yB$. It follows that if U_n is the multiplicative system $U_n = B_n - ((xB \cap B_n) \cup (yB \cap B_n))$, then

$$(B_0)_{U_0} = \cdots = (B_n)_{U_n} = \cdots = B[1/xy].$$

Since xB and yB are principal height-one prime ideals of B, we have $x^i(B)_{xB} \cap B = x^iB$ and $y^j(B)_{yB} \cap B = y^jB$ for all positive integers i, j, so $B = B[1/xy] \cap (B)_{xB} \cap (B)_{yB}$. In particular, it follows that B is a Krull domain.

Since the maximal ideal of B is finitely generated, it follows from Nishimura [Ni, Theorem, page 397] that B is Noetherian if $\dim(B) = 2$. On the other hand, since $\mathbf{n} = (x, y)B$, it is clear that if B is Noetherian, then B is a 2-dimensional regular local domain with completion $k[[x, y]]$. Since the completion of a local Noetherian ring is a faithfully flat extension, if B is Noetherian we have $B = k(x, y, \sigma, \tau) \cap k[[x, y]]$ and hence $B = A$. That B satisfies the condition in statement (4) if and only if B is Noetherian follows from [N, (31.8), page 110]. Since B is a birational extension of the 3-dimensional Noetherian domain $C[y, \tau]$, the dimension of B is at most 3.

To see that B can be strictly smaller than $A := k(x, y, \sigma, \tau) \cap k[[x, y]]$, observe that if $\tau = \sigma(y)$, that is, if $a_i = b_i$ for all $i \in \mathbb{N}$ (for example, $\sigma = e^x - 1, \tau = e^y - 1$) then $(\sigma - \tau)/(x - y)$ is in A. But the description given above for B as an intersection of DVR's shows that if $\mathbf{q}$ is the height-one prime $(x - y)B_n$, then $B \subseteq (B_n)_{\mathbf{q}}$. while $(\sigma - \tau)/(x - y) \notin (B_n)_{\mathbf{q}}$. Therefore $(\sigma - \tau)/(x - y) \notin B$, so $B < A$. This shows the existence of a three-dimensional quasilocal Krull domain B having a two-generated maximal ideal such that B birationally dominates a three-dimensional regular local domain.

To complete the proof of (2.4) it remains to show the existence of σ and τ for which $B = A$. We establish in Example 6.10 that $\sigma = e^x - 1$ and $\tau = e^{e^y} - 1$ have this property. $\square$

3. Background material

We review the main definitions and relevant results from [HRW1]. The flatness conditions ((3.1), (3.2) and (3.4)) are used in the limit-intersecting independence definitions of §5.

3.1. *Definition.* Let $\phi: S \longrightarrow T$ be an injective morphism of commutative rings and let $k \in \mathbf{N}$ be an integer with $1 \leq k \leq d = \dim(T)$ where d is an integer or $d = \infty$. Then ϕ is called *locally flat in height k*, abbreviated LF_k, if, for every prime ideal Q of T with $\mathrm{ht}(Q) \leq k$, the induced morphism on the localizations $\phi_Q: S_{Q \cap S} \longrightarrow T_Q$ is faithfully flat.

3.2. *Definition.* Let $S \hookrightarrow T$ be an extension of Krull domains. We say that T is a *height-one preserving* extension of S if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.

The height-one preserving property is crucial for our work, and so it is fortunate that it holds in the situations we consider. In particular the following result, which extends [HRW1, (2.7)] by eliminating a Noetherian hypothesis, shows that the height-one preserving property holds within completions.

3.3. PROPOSITION. *Suppose $(\widehat{C}, \hat{\mathbf{n}})$ is a complete normal local Noetherian domain that dominates a quasilocal Krull domain $(D, \mathbf{m})$. Assume the injection $D \longrightarrow \widehat{C}$ is height-one preserving, and suppose $\tau \in \hat{\mathbf{n}}$ is algebraically independent over the fraction field L of D. Let $S = D[\tau]_{(\mathbf{m},\tau)}$. Then the local inclusion morphism φ: $S \longrightarrow \widehat{C}$ is height-one preserving.*

Proof. Let P be a height-one prime ideal of S.

Case (i). If $\operatorname{ht}(P \cap D) = 1$, then $P = (P \cap D)S$. Since $D \longrightarrow \widehat{C}$ is height-one preserving, $(P \cap D)\widehat{C} \subseteq Q$, for some height-one prime ideal Q of $\widehat{C}$. Then $P\widehat{C} = (P \cap D)S\widehat{C} \subseteq Q$ as desired.

Case (ii). Suppose $P \cap D = (0)$. Let U denote the multiplicative set of nonzero elements of D. Let t be an indeterminate over D and let $S_1 = D[t]_{(\mathbf{m},t)}$. Consider the following commutative diagram where the map from S_1 to S is the D-algebra isomorphism taking t to τ and λ is the natural extension to $\widehat{C}[[t]]$.

$$\begin{tikzcd}
 & U^{-1}S_1 \arrow[r, "\subseteq"] & U^{-1}\widehat{C}[t]_{(\hat{\mathbf{n}},t)} & \\
 & \arrow[u, "\cup"] & \arrow[u, "\cup"] & \\
D \arrow[r, "\subseteq"] \arrow[d, "="] & S_1 = D[t]_{(\mathbf{m},t)} \arrow[r, "\subseteq"] \arrow[d, "\cong"] & \widehat{C}[t]_{(\hat{\mathbf{n}},t)} \arrow[r, "\subseteq"] & \widehat{C}[[t]] \arrow[d, "\lambda"] \\
D \arrow[r, "\subseteq"] & S = D[\tau]_{(\mathbf{m},\tau)} \arrow[rr, "\varphi"] & & \widehat{C}.
\end{tikzcd}$$

Under the above isomorphism of S with S_1, P correponds to a height-one prime ideal P_0 of S_1 such that $P_0 \cap D = (0)$. Thus P_0 is contracted from the localization $U^{-1}S_1$. Since $U^{-1}S_1$ is a localization of the polynomial ring $L[t]$, it is a principal ideal domain. Hence P_0 is contained in a proper principal ideal of $U^{-1}S_1$. Therefore P_0 is contained in a proper principal ideal of $U^{-1}\widehat{C}[t]_{(\hat{\mathbf{n}},t)}$, and hence in a height-one prime ideal of $\widehat{C}[t]_{(\hat{\mathbf{n}},t)}$. Now $\widehat{C}[t]_{(\hat{\mathbf{n}},t)} \longrightarrow \widehat{C}[[t]]$ is faithfully flat because $\widehat{C}$ is Noetherian; thus $P_0\widehat{C}[[t]]$ is contained in a height-one prime ideal of $\widehat{C}[[t]]$. Since $\widehat{C}$ is catenary and $\ker(\lambda) = (t - \tau)$ is principal, $P_0 + \ker(\lambda) = P + \ker(\lambda)$ has height two in $\widehat{C}[[t]]$. It follows that $P\widehat{C}$ is contained in a height-one prime of $\widehat{C}$. □

Next we review the concept of weak flatness defined in [HRW1].

3.4. *Definition.* Let $S \hookrightarrow T$ be an extension of Krull domains. We say that T is *weakly flat* over S if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.

3.5. PROPOSITION [HRW1, (2.10), (2.14)]. *Let ϕ: $S \hookrightarrow T$ be an extension of Krull domains and let F denote the fraction field of S.*

(1) *Suppose $PT \neq T$ for every height-one prime ideal P of S. Then $S \hookrightarrow T$ is weakly flat $\iff$ $S = F \cap T$.*
(2) *If $S \hookrightarrow T$ is weakly flat, then ϕ is height-one preserving and, moreover, for every height-one prime ideal P of S with $PT \neq T$, there is a height-one prime ideal Q of T with $Q \cap S = P$.*

3.6 *Remark.* The height-one preserving condition does *not* imply weakly flat. To see this, consider a domain $(D, \mathbf{m})$, as in (3.3), such that $\dim(\widehat{C} \otimes_D \mathcal{Q}(D)) = 0$, where $(\widehat{C}, \hat{\mathbf{n}})$, τ, and S are as in the statement of (3.3), and so the local inclusion morphism φ: $S \longrightarrow \widehat{C}$ is height-one preserving. (For example, take $\widehat{C} = k[[x, y]]$ and $D = k[[x]][y]_{(x,y)}$.) There exists a height-one prime ideal P of S such that $P \cap D = 0$; then $P\widehat{C} \neq \widehat{C}$. Since φ is height-one preserving, there exists a height-one prime ideal $\widehat{Q}$ of $\widehat{C}$ such that $P\widehat{C} \subseteq \widehat{Q}$. Also $\dim(\widehat{C} \otimes_D \mathcal{Q}(D)) = 0$ implies $\widehat{Q} \cap D \neq 0$. We have $P \subseteq \widehat{Q} \cap S$ and $P \cap D = 0$. It follows that P is strictly smaller than $\widehat{Q} \cap S$, so $\widehat{Q} \cap S$ has height greater than one and so the extension φ is not weakly flat.

4. Intersections and directed unions

In general the intersection of a normal Noetherian domain with a subfield of its field of fractions is a Krull domain, but is not Noetherian. The Krull domain B in the motivating example (2.1)–(2.4) (in the case where $B \neq A$) illustrates that a directed union of normal Noetherian domains may be a non-Noetherian Krull domain. Thus, in order to apply an iterative procedure in §5, we consider a quasilocal Krull domain $(T, \mathbf{n})$ which is not assumed to be Noetherian, but is assumed to have a Noetherian completion. To distinguish from the earlier Noetherian hypothesis on R, we let T denote the base domain.

As we mention in the introduction, completions with respect to principal ideals are used in our constructions.

4.1. *Setting and notation.* Let $(T, \mathbf{n})$ be a quasilocal Krull domain with fraction field F. Assume there exists a nonzero element $y \in \mathbf{n}$ such that the y-adic completion $\widehat{(T, (y))} := (T^*, \mathbf{n}^*)$ of T is an analytically normal local Noetherian domain. It then follows that the $\mathbf{n}$-adic completion $\widehat{T}$ of T is also a normal local domain, since the $\mathbf{n}$-adic completion of T is the same as the $\mathbf{n}^*$-adic completion of T^*. Since T^* is

Noetherian, if F^* denotes the field of fractions of T^*, then $T^* = \widehat{T} \cap F^*$. Therefore $F \cap T^* = F \cap \widehat{T}$. Let d denote the dimension of the Noetherian domain T^*. It follows that d is also the dimension of $\widehat{T}$.[4]

(1) Assume that $T = F \cap T^* = F \cap \widehat{T}$, or equivalently by (3.5.1), that T^* and $\widehat{T}$ are weakly flat over T.
(2) Let $\widehat{T}_y := \widehat{T}[1/y]$, the localization of $\widehat{T}$ at the powers of y, and similarly, let $T^*_y := T^*[1/y]$. The domains $\widehat{T}_y$ and T^*_y are of dimension $d-1$.
(3) Let $\tau_1, \dots, \tau_s \in \mathbf{n}^*$ be algebraically independent over F.
(4) For each i with $1 \le i \le s$, we have an expansion $\tau_i := \Sigma_{j=1}^{\infty} c_{ij} y^j$ where $c_{ij} \in T$.
(5) For each $n \in \mathbb{N}$ and each i, $1 \le i \le s$, we define the n^{th}-endpiece of τ_i with respect to y as in (2.3), so that

$$\tau_{in} := \Sigma_{j=n+1}^{\infty} c_{ij} y^{j-n}, \qquad \tau_{in} = y\tau_{i,n+1} + c_{in} y.$$

(6) For each $n \in \mathbb{N}$, we define $B_n := T[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{n}, \tau_{1n}, \dots, \tau_{sn})}$. In view of (5), we have $B_n \subseteq B_{n+1}$ and B_{n+1} dominates B_n for each n. We define

$$B := \varinjlim_{n \in \mathbb{N}} B_n = \bigcup_{n=1}^{\infty} B_n, \qquad \text{and} \qquad A := F(\tau_1, \dots, \tau_s) \cap \widehat{T}.$$

Thus, B and A are quasilocal domains and A birationally dominates B. We are especially interested in conditions which imply that $B = A$.
(7) Let A^* denote the y-adic completion $(A, (y))^\wedge$ of A and B^* the y-adic completion of B.

4.2 *Remark.* The motivating example (2.1)–(2.4) with $T := B \neq A$ (from the notation of (2.1) shows that $T \longrightarrow T^*_y$ can satisfy the other conditions of (4.1) but not satisfy the assumption (4.1.1); that is, such an extension is always height-one preserving (by (3.3)) but not in general weakly flat.

4.3 PROPOSITION. *The definitions of B and B_n are independent of representations for $\tau_1, \dots, \tau_s$ as power series in y with coefficients in T.*

Proof. For $1 \le i \le s$, assume that τ_i and $\omega_i = \tau_i$ have representations

$$\tau_i = \sum_{j=1}^{\infty} a_{ij} y^j \quad \text{and} \quad \omega_i = \sum_{j=1}^{\infty} b_{ij} y^j,$$

[4]If T is Noetherian, then $\dim(T) = d$. However, without the hypothesis that T is Noetherian, it is unclear whether T has dimension d.

where each $a_{ij}, b_{ij} \in T$. We define the n^{th}-endpieces τ_{in} and ω_{in} as in (2.3):

$$\tau_{in} = \sum_{j=n+1}^{\infty} a_{ij} y^{j-n} \quad \text{and} \quad \omega_{in} = \sum_{j=n+1}^{\infty} b_{ij} y^{j-n}.$$

Then we have

$$\tau_i = \Sigma_{j=1}^{\infty} a_{ij} y^j = \Sigma_{j=1}^{n} a_{ij} y^j + y^n \tau_{in} = \Sigma_{j=1}^{\infty} b_{ij} y^j = \Sigma_{j=1}^{n} b_{ij} y^j + y^n \omega_{in} = \omega_i.$$

Therefore, for $1 \leq i \leq s$ and each positive integer n,

$$y^n \tau_{in} - y^n \omega_{in} = \Sigma_{j=1}^{n} b_{ij} y^j - \Sigma_{j=1}^{n} a_{ij} y^j,$$

and so

$$\tau_{in} - \omega_{in} = \frac{\Sigma_{j=1}^{n} (b_{ij} - a_{ij}) y^j}{y^n}.$$

Since $\Sigma_{j=1}^{n} (b_{ij} - a_{ij}) y^j \in T$ is divisible by y^n in T^* and $T = F \cap T^*$, it follows that y^n divides $\Sigma_{j=1}^{n} (b_{ij} - a_{ij}) y^j$ in T. Therefore $\tau_{in} - \omega_{in} \in T$. It follows that B_n and $B = \cup_{n=1}^{\infty} B_n$ are independent of the representation of the τ_i. □

4.4 THEOREM. *Assume the setting and notation of* (4.1). *Then the intermediate rings B_n, B and A have the following properties*:

(1) $yA = yT^* \cap A$ *and* $yB = yA \cap B = yT^* \cap B$. *More generally, for every* $t \in \mathbb{N}$, *we have* $y^t A = y^t T^* \cap A$ *and* $y^t B = y^t A \cap B = y^t T^* \cap B$.
(2) $T/y^t T = B/y^t B = A/y^t A = T^*/y^t T^*$ *for each positive integer* t.
(3) *Every ideal of* T, B *or* A *that contains* y *is finitely generated by elements of* T. *In particular, the maximal ideal* $\mathbf{n}$ *of* T *is finitely generated, and the maximal ideals of* B *and* A *are* $\mathbf{n}B$ *and* $\mathbf{n}A$.
(4) *For every* $n \in \mathbb{N}$: $yB \cap B_n = (y, \tau_{1n}, \dots, \tau_{sn})B_n$, *an ideal of* B_n *of height* $s+1$.
(5) *If* $P \in \mathit{Spec}(A)$ *is minimal over* yA *and* $Q = P \cap B$, $W = P \cap T$, *then* $T_W \subseteq B_Q = A_P$, *and all three localizations are DVRs.*
(6) *For every* $n \in \mathbb{N}$, $B[1/y]$ *is a localization of* B_n, *i.e., for each* $n \in \mathbb{N}$, *there exists a multiplicatively closed subset* S_n *of* B_n *such that* $B[1/y] = S_n^{-1} B_n$.
(7) $B = B[1/y] \cap B_{\mathbf{q}_1} \cap \cdots \cap B_{\mathbf{q}_r}$, *where* $\mathbf{q}_1, \dots, \mathbf{q}_r$ *are the prime ideals of* B *minimal over* yB.

Proof. Let $K := F(\tau_1, \dots \tau_s)$, the field of fractions of A and B. Then $A = T^* \cap K$ implies $yA \subseteq yT^* \cap A$. Let $g \in yT^* \cap A \subseteq yT^* \cap K$. Then $g/y \in T^* \cap K = A \Longrightarrow g \in yA$. Since $B = \bigcup_{n=1}^{\infty} B_n$, we have $yB = \bigcup_{n=1}^{\infty} yB_n$. It is clear that $yB \subseteq yA \cap B \subseteq yT^* \cap B$. We next show $yT^* \cap B = yB$. Let

$g \in yT^* \cap B$. Then there is an $n \in \mathbf{N}$ with $g \in B_n$ and, multiplying g by a unit of B_n if necessary, we may assume that $g \in T[\tau_{1n}, \dots, \tau_{sn}]$. Write $g = r_0 + g_0$ where $g_0 \in (\tau_{1n}, \dots, \tau_{sn})T[\tau_{1n}, \dots, \tau_{sn}]$ and $r_0 \in T$. Substituting $\tau_{jn} = y\tau_{jn+1} + c_{jn}y \in yT^*$ from (4.1.5) yields that $g_0 \in yT^*$ and so $r_0 \in yT^* \cap T = yT$. Since by (4.1.5), $(\tau_{1n}, \dots, \tau_{sn})B_n \subseteq yB_{n+1}$, it follows that $g \in yB$. Now $yB = yT^* \cap B$ implies $y^2B = y(yT^* \cap B) = y^2T^* \cap yB = y^2T^* \cap B$. Similarly $y^tB = y^tT^* \cap B$ for every $t \in \mathbb{N}$.

Since $y^tT^* \cap T = y^tT$, $T/y^tT = T^*/y^tT^*$, and $T/(y^tT) \hookrightarrow B/(y^tB) \hookrightarrow A/(y^tA) \hookrightarrow T^*/y^tT^*$, the assertion in (2) follows.

Since T^* is Noetherian, the assertions of (3) follow from (2).

For (4), let $f \in yB \cap B_n$. After multiplication by a unit of B_n, we may assume that $f \in T[\tau_{1n}, \dots, \tau_{sn}]$, and hence f is of the form

$$f = \sum_{(i) \in \mathbf{N}^s} a_{(i)}\tau_{1n}^{i_1} \dots \tau_{sn}^{i_s}$$

with $a_{(i)} \in T$. Since $\tau_{jn} \in yB$, we see that $a_{(0)} \in yB \cap T \subseteq yT^* \cap T$, and we can write $a_{(0)} = y\widehat{b}$ for some element $\widehat{b} \in T^*$. This implies that $\widehat{b} \in F \cap T^* = T$; the last equality uses (4.1.1). Therefore $a_{(0)} \in yT$ and $f \in (y, \tau_{1n}, \dots, \tau_{sn})B_n$. Furthermore if $g \in (y, \tau_{1n}, \dots, \tau_{sn})B_n$, then $\tau_{in} \subseteq yB \cap B_n$, so $g \in yB \cap B_n$.

For (5), since T^* and hence A is Krull, P has height one and A_P is a DVR. Also A_P has the same fraction field as B_Q. By (2), W is a minimal prime of yT. Since T is a Krull domain, T_W is a DVR and the maximal ideal of T_W is generated by $u \in T$. Thus by (2) the maximal ideal of B_Q is generated by u and so B_Q is a DVR dominated by A_P. Therefore they must be the same DVR.

Item (6) follows from (4.1.5).

For (7), suppose $\beta \in B[1/y] \cap B_{\mathbf{q}_1} \cap \cdots \cap B_{\mathbf{q}_r}$. Now $B_{\mathbf{q}_1} \cap \cdots \cap B_{\mathbf{q}_r} = (B - (\cup \mathbf{q}_i))^{-1}B$. There exist $t \in \mathbb{N}, a, b, c \in B$ with $c \notin \mathbf{q}_1 \cup \cdots \cup \mathbf{q}_r$ such that $\beta = a/y^t = b/c$. We may assume that either $t = 0$ (and we are done) or that $t > 0$ and $a \notin yB$. Since $yB = yA \cap B$, it follows that $\mathbf{q}_1, \dots, \mathbf{q}_r$ are the contractions to B of the minimal primes $\mathbf{p}_1, \dots, \mathbf{p}_r$ of yA in A. Since A is a Krull domain, $A = A[1/y] \cap A_{\mathbf{p}_1} \cap \cdots \cap A_{\mathbf{p}_n}$. Thus $\beta \in A$, and $a = y^t\beta \in yA \cap B = yB$, a contradiction. Thus $t = 0$ and $\beta = a \in B$. □

4.5 THEOREM. *With the setting and notation of* (4.1), *the intermediate rings* A *and* B *have the following properties*:

(1) *A and B are quasilocal Krull domains.*
(2) *$B \subseteq A$, with A dominating B.*
(3) $A^* = B^* = T^*$.
(4) *If B is Noetherian, then $B = A$.*

Moreover, if T is a unique factorization domain (*UFD*) *and y is a prime element of T, then B is a UFD.*

Proof. As noted in the proof of (4.4.5), A is a Krull domain. By (4.4.6), $B[1/y]$ is a localization of B_0. Since B_0 is a Krull domain, it follows that $B[1/y]$ is a Krull domain. By (4.4.7), B is the intersection of $B[1/y]$ and the DVR's $B_{\mathbf{q}_1}, \dots, B_{\mathbf{q}_r}$. Therefore B is a Krull domain. Items (2) and (3) are immediate from (4.4). If B is Noetherian, then B^* is faithfully flat over B, and hence $B = F(\tau_1, \dots, \tau_s) \cap B^* = A$. For the last statement, if T is a UFD, so is the localized polynomial ring B_0. By (4.4.6), $B[1/y] = S_0^{-1} B_0[1/y]$, which implies that $B[1/y]$ is also a UFD. By (4.4.2), y is a prime element of B; hence it follows from [Sa, (6.3), page 21] that B is a UFD. □

5. Limit-intersecting elements

As we state in the introduction, we are interested in the structure of $L \cap \widehat{R}$, for intermediate fields L between the fraction fields of R and $\widehat{R}$. This is difficult to determine in general. We show in Theorem 5.5 that each of the *limit-intersecting* properties of (5.1) implies $L \cap \widehat{R}$ is a directed union of localized polynomial ring extensions of R. These limit-intersecting properties are related to the idealwise independence concepts defined in [HRW1].

5.1 *Definition.* Let $(T, \mathbf{n})$ be a quasilocal Krull domain with fraction field F, let $0 \neq y \in \mathbf{n}$ be such that the y-adic completion $\widehat{(T, (y))} := (T^*, \mathbf{n}^*)$ of T is an analytically normal local Noetherian domain of dimension d. Assume that T^* and $\widehat{T}$ are weakly flat over T. Let $\tau_1, \dots, \tau_s \in \mathbf{n}^*$ be algebraically independent over F (as in (4.1)).

(1) The elements $\tau_1, \dots, \tau_s$ are said to be *limit-intersecting* in y over T provided the inclusion morphism $B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} \longrightarrow T_y^*$ is weakly flat (see (3.4)).
(2) The elements $\tau_1 \dots, \tau_s$ are said to be *residually limit-intersecting* in y over T provided the inclusion morphism $B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} \longrightarrow T_y^*$ is LF_1 (see (3.1)).
(3) The elements $\tau_1 \dots, \tau_s$ are said to be *primarily limit-intersecting* in y over T provided the inclusion morphism $B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} \longrightarrow T_y^*$ is flat, or LF_{d-1} (see (3.1)).

Since T_y^* and $\widehat{T}_y$ have dimension $d-1$, the condition LF_{d-1} is equivalent to primarily limit-intersecting, that is, to the flatness of the map $B_0 \longrightarrow T_y^*$.

5.2. *Remarks.* (1) The terms "residually" and "primarily" come from [HRW1]. We justify this terminology in (5.7) and (5.8). It is clear that primarily limit-intersecting implies residually limt-intersecting and residually limit-intersecting implies limit-intersecting.

(2) Since $\widehat{T}_y$ is faithfully flat over T_y^*, the statements obtained by replacing T_y^* by $\widehat{T}_y$ give equivalent definitions to those of (5.1) (see [HRW1, (6.1), (6.3)]).

(3) We remark that

$$B \longrightarrow T_y^* \text{ is weakly flat } \iff B \longrightarrow T^* \text{ is weakly flat.}$$

To see this, observe that by (4.4.2), every height-one prime of B containing y is the contraction of a height-one prime of T^*. If $\mathbf{p}$ is a height-one prime of B with $y \notin \mathbf{p}$, then $\mathbf{p}T^* \cap B = \mathbf{p}$ if and only if $\mathbf{p}T_y^* \cap B = \mathbf{p}$.

(4) Since by (4.4.6), B_y is a localization $S_0^{-1}B_0$ of B_0, and since the canonical maps $B_0 \longrightarrow T_y^*$ and $B \longrightarrow T_y^*$ factor through the localization at the powers of y, the elements $\tau_1, \dots, \tau_s$ are limit-intersecting in y over T if and only if the canonical map

$$S_0^{-1}B_0 = B_y \longrightarrow T_y^*$$

is weakly flat. In view of (5.3) below, we also have that $\tau_1, \dots, \tau_s$ are residually (resp. primarily) limit-intersecting in y over T if and only if the canonical map

$$S_0^{-1}B_0 = B_y \longrightarrow T_y^*$$

is LF_1 (resp. LF_{d-1} or equivalently flat).

(5) If $d = 2$, then obviously $LF_1 = LF_{d-1}$. Hence in this case primarily limit-intersecting is equivalent to residually limit-intersecting.

(6) Since $T \longrightarrow B_n$ is faithfully flat for every n, it follows [B, Chap. 1, Sec. 2.3, Prop. 2, p. 14] that $T \longrightarrow B$ is always faithfully flat. Thus if residually limit-intersecting elements exist over T, then $T \longrightarrow T_y^*$ must be LF_1. If primarily limit-intersecting elements exist over T, then $T \longrightarrow T_y^*$ must be flat.

(7) Items (3.6) and (4.2) show that in some situations there are no limit-intersecting elements in T^*. Indeed, if T is complete with respect to some nonzero ideal I, and y is outside every minimal prime over I, then every algebraically independent $\tau = \sum a_i y^i \in T^*$ fails to be limit-intersecting in y. To see this, choose an element $x \in I$, x outside every minimal prime ideal of yT; define $\sigma := \sum a_i x^i \in T$. Then $\tau - \sigma \in (x - y)T^* \cap T[\tau]$. Thus a minimal prime over $x - y$ in T^* intersects $T[\tau]$ in an ideal of height greater than one, because it contains $x - y$ and $\tau - \sigma$.

5.3. PROPOSITION. *Assume the notation and setting of* (4.1) *and let k be a positive integer with $1 \le k \le d - 1$. Then the following are equivalent*:

(1) *The canonical injection* ϕ: $B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{m}, \tau_1, \dots, \tau_s)} \longrightarrow T_y^*$ *is* LF_k.
(1′) *The canonical injection* ϕ_1: $B_0 := T[\tau_1, \dots, \tau_s]_{(\mathbf{m}, \tau_1, \dots, \tau_s)} \longrightarrow \widehat{T}_y$ *is* LF_k.
(2) *The canonical injection* ϕ': $U_0 := T[\tau_1, \dots, \tau_s] \longrightarrow T_y^*$ *is* LF_k.
(2′) *The canonical injection* ϕ_1': $U_0 := T[\tau_1, \dots, \tau_s] \longrightarrow \widehat{T}_y$ *is* LF_k.
(3) *The canonical injection* θ: $B_n := R[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \dots, \tau_{sn})} \longrightarrow T_y^*$ *is* LF_k.

(3′) *The canonical injection* θ_1: $B_n := R[\tau_{1n}, \dots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \dots, \tau_{sn})} \longrightarrow \widehat{T}_y$ *is* LF_k.
(4) *The canonical injection* ψ: $B \longrightarrow T^*_y$ *is* LF_k.
(4′) *The canonical injection* ψ: $B \longrightarrow \widehat{T}_y$ *is* LF_k.

Moreover, these statements are also all equivalent to LF_k *of the corresponding canonical injections obtained by replacing* B_0, U_0 *and* B *by* $B_0[1/y]$, $U_0[1/y]$ *and* $B[1/y]$.

Proof. We have:

$$U_0 \xrightarrow{\text{loc.}} B_0 \xrightarrow{\phi} T^*_y \xrightarrow{\text{f.f.}} \widehat{T}_y.$$

The injection ϕ'_1: $U_0 \longrightarrow \widehat{T}_y$ factors as ϕ': $U_0 \longrightarrow T^*_y$ followed by the faithfully flat injection $T^*_y \longrightarrow \widehat{T}_y$. Therefore ϕ' is LF_k if and only if ϕ'_1 is LF_k. The injection ϕ' factors through the localization $U_0 \longrightarrow B_0$ and so ϕ is LF_k if and only if ϕ' is LF_k.

Now set $U_n := T[\tau_{1n}, \dots, \tau_{sn}]$ for each $n > 1$ and $U := \bigcup_{n=0}^{\infty} U_n$. For each positive integer i, $\tau_i = y^n \tau_{in} + \sum_{i=0}^{n} a_i y^i$. Thus $U_n \subseteq U_0[1/y]$, and $U_0[1/y] = \bigcup U_n[1/y] = U[1/y]$. Moreover, for each n, B_n is a localization of U_n, and hence B is a localization of U.

We have

$$\begin{aligned} B[1/y] \longrightarrow T^*_y \text{ is } LF_k &\iff U[1/y] \longrightarrow T^*_y \text{ is } LF_k \\ &\iff U_0[1/y] \longrightarrow T^*_y \text{ is } LF_k \\ &\iff B_n[1/y] \longrightarrow T^*_y \text{ is } LF_k \\ &\iff B_0[1/y] \longrightarrow T^*_y \text{ is } LF_k. \end{aligned}$$

Thus

$$\begin{aligned} \psi\colon B \longrightarrow T^*_y \text{ is } LF_k &\iff U \longrightarrow T^*_y \text{ is } LF_k \\ &\iff \phi'\colon U_0 \longrightarrow T^*_y \text{ is } LF_k \\ &\iff \theta\colon B_n \longrightarrow T^*_y \text{ is } LF_k \\ &\iff \phi\colon B_0 \longrightarrow T^*_y \text{ is } LF_k. \end{aligned}$$

□

5.4 *Remarks.* (1) If $(T, \mathbf{n})$ is a one-dimensional quasilocal Krull domain, then T is a rank-one discrete valuation domain (DVR). Hence T^* is also a DVR and T^*_y is flat over $U_0 = T[\tau_1, \dots, \tau_s]$. Therefore, in this case, $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over T if and only if $\tau_1, \dots, \tau_s$ are algebraically independent over F.

(2) Let $\tau_1, \dots, \tau_s \in k[[y]]$ be transcendental over $k(y)$, where k is a field. Then $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over $k[y]_{(y)}$ by (1) above. In [HRW2, (3.3)], we show that if $x_1, \dots, x_m$ are additional indeterminates over $k(y)$, then $\tau_1, \dots, \tau_s$ are primarily limit-intersecting in y over $k[x_1, \dots, x_m, y]_{(x_1, \dots, x_m, y)}$.

(3) With the notation of (4.1), if B is Noetherian, then $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over T. For B Noetherian implies T^* as the (y)-adic completion of B is flat over B. Hence T^*_y is also flat over B, and it follows from (5.3) that $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over T.

(4) By the equivalence of (1) and (2) of (5.3), we see that $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over T if and only if the endpiece power series $\tau_{1n}, \ldots, \tau_{sn}$ are primarily limit-intersecting in y over T.

5.5 THEOREM. *With the setting and notation of* (4.1), *the following are equivalent*:

(1) *The elements $\tau_1, \ldots, \tau_s$ are limit-intersecting in y over T.*
(2) *The intermediate rings A and B are equal.*
(3) *$B \longrightarrow T^*_y$ is weakly flat.*
(4) *$B \longrightarrow T^*$ is weakly flat.*

Proof. (1)$\Rightarrow$(2). Since A and B are Krull domains with the same field of fractions and $B \subseteq A$ it is enough to show that every height-one prime ideal $\mathbf{p}$ of B is the contraction of a (height-one) prime ideal of A. By Theorem 4.4.3, each height-one prime of B containing yB is the contraction of a height-one prime of A.

Let $\mathbf{p}$ be a height-one prime of B which does not contain yB. Consider the prime ideal $\mathbf{q} = T[\tau_1, \ldots, \tau_s] \cap \mathbf{p}$. Since $B[1/y]$ is a localization of the ring $T[\tau_1, \ldots, \tau_s]$, we see that $B_{\mathbf{p}} = T[\tau_1, \ldots, \tau_s]_{\mathbf{q}}$ and thus $\mathbf{q}$ has height one in $T[\tau_1, \ldots, \tau_s]$. The limit-intersecting hypothesis implies $\mathbf{q}T^* \cap T[\tau_1, \ldots, \tau_s] = \mathbf{q}$ and there is a height-one prime ideal $\mathbf{w}$ of T^* with $\mathbf{w} \cap T[\tau_1, \ldots, \tau_s] = \mathbf{q}$. This implies that $\mathbf{w} \cap B = \mathbf{p}$ and thus also $(\mathbf{w} \cap A) \cap B = \mathbf{p}$. Hence every height-one prime ideal of B is the contraction of a prime ideal of A. Since A is birational over B, this prime ideal of A can be chosen to have height one.

(3) $\Longleftrightarrow$ (4). This is shown in (5.2.3).

(2)$\Rightarrow$(4). If $B = A = F \cap T^*$, then by (3.5) every height-one prime ideal of B is the contraction of a height-one prime ideal of T^*.

(4)$\Rightarrow$(1). If $B \hookrightarrow T^*$ is weakly flat so is the localization $B_y \hookrightarrow T^*_y$. Since $B_y = S_0^{-1} B_{0y}$ for a suitable multiplicative subset $S_0 \subseteq B_{0y}$ the embedding $B_{0y} \hookrightarrow T^*_y$ is weakly flat. Now (1) holds by (5.2.4).

5.6 *Remarks.* (1) If an injective morphism of Krull domains is weakly flat, then it is height-one preserving (3.5.2). Thus any of the equivalent conditions of (5.5) imply that $B \longrightarrow T^*$ is height-one preserving.

(2) In (5.5) if B is Noetherian, then by (4.5.4), $A = B$ and all the conclusions of (5.5) hold.

(3) In [HRW2, (4.4)], we give an example of a three-dimensional regular local domain R dominating $\mathbb{Q}[x, y, z]_{(x,y,z)}$ and having completion $\mathbb{Q}[[x, y, z]]$, such that there exists an element τ in the (y)-adic completion of R that is residually limit-

intersecting in y over R but fails to be primarily limit-intersecting in y over R. In particular, the rings A and B constructed using τ are equal, yet A and B are not Noetherian. We also show in [HRW2, (2.12)] that if R is a semilocal Noetherian domain, then $\tau_1, \dots, \tau_s \in yR^*$ are primarily limit-intersecting in y over R if and only if B is Noetherian. If this holds, we also have $B = A$.

We now give criteria for elements to be residually limit-intersecting or primarily limit-intersecting similar to those in [HRW1] for elements to be residually algebraically independent or primarily independent.

5.7 PROPOSITION. *With the setting and notation of* (4.1) *and* $s = 1$, *the following are equivalent*:

(1) *The element* $\tau = \tau_1$ *is residually limit-intersecting in* y *over* T.
(2) *If* $\widehat{P}$ *is a height-one prime ideal of* $\widehat{T}$ *such that* $y \notin \widehat{P}$ *and* $\widehat{P} \cap T \neq 0$, *then* $\mathrm{ht}(\widehat{P} \cap T[\tau]_{(\mathbf{n},\tau)}) = 1$.
(3) *For every height-one prime ideal* P *of* T *such that* $y \notin P$ *and for every minimal prime divisor* $\widehat{P}$ *of* $P\widehat{T}$ *in* $\widehat{T}$, *the image* $\bar{\tau}$ *of* τ *in* $\widehat{T}/\widehat{P}$ *is algebraically independent over the fraction field of* T/P.
(4) $B \longrightarrow T_y^*$ *is* LF_1 *and height-one preserving.*

Proof. For (1)$\Rightarrow$ (2), suppose (2) fails; that is, there exists a prime ideal $\widehat{P}$ of $\widehat{T}$ of height one such that $y \notin \widehat{P}$, $\widehat{P} \cap T \neq 0$, but $\mathrm{ht}(\widehat{P} \cap T[\tau]) \geq 2$. Let $\widehat{Q} := \widehat{P}\widehat{T}_y$. Then $Q := \widehat{Q} \cap T[\tau]_{(\mathbf{n},\tau)}$ has height greater than or equal to 2. But by the definition of residually limit-intersecting in (5.1), the injective morphism $T[\tau]_{(\mathbf{n},\tau)} \longrightarrow \widehat{T}_y$ is LF_1 and so by (3.1), $(T[\tau]_{(\mathbf{n},\tau)})_Q \longrightarrow \widehat{(\widehat{T}_y)_{\widehat{Q}}}$ is faithfully flat, a contradiction to $\mathrm{ht}(Q) > \mathrm{ht}(\widehat{P}) = \mathrm{ht}(\widehat{Q})$.

For (2)$\Rightarrow$ (1), the argument of (1) $\Rightarrow$ (2) can be reversed since $(T[\tau]_{(\mathbf{n},\tau)})_Q \longrightarrow (\widehat{T}_y)_{\widehat{Q}}$ is faithfully flat.

For (3)$\Rightarrow$ (2), again suppose (2) fails; that is, there exists a prime ideal $\widehat{P}$ of $\widehat{T}$ of height one such that $y \notin \widehat{P}$, $\widehat{P} \cap T \neq 0$, but $\mathrm{ht}(\widehat{P} \cap T[\tau]) \geq 2$. Now $\mathrm{ht}(\widehat{P} \cap T) = 1$, since LF_1 holds for $T \hookrightarrow \widehat{T}$. Thus, with $P = \widehat{P} \cap T$, we have $PT[\tau] < \widehat{P} \cap T[\tau]$; that is, there exists $f(\tau) \in (\widehat{P} \cap T[\tau]) - PT[\tau]$, or equivalently there is a nonzero polynomial $\bar{f}(x) \in (T/(\widehat{P} \cap T))[x]$ so that $\bar{f}(\bar{\tau}) = \bar{0}$ in $T[\tau]/(\widehat{P} \cap T[\tau])$, where $\bar{\tau}$ denotes the image of τ in $\widehat{T}/\widehat{P}$. This means that $\bar{\tau}$ is algebraic over the fraction field of $T/(\widehat{P} \cap T)$, a contradiction to (3).

For (2)$\Rightarrow$ (3), let $\widehat{P}$ be a height-one prime of $\widehat{R}$ such that $\widehat{P} \cap T = P \neq 0$. Since $\mathrm{ht}(\widehat{P} \cap T[\tau]) = 1$, $\widehat{P} \cap T[\tau] = PT[\tau]$ and $T[\tau]/(PT[\tau])$ canonically embeds in $\widehat{T}/\widehat{P}$. Thus the image of τ in $T[\tau]/PT[\tau]$ is algebraically independent over T/P.

For (1) $\Longleftrightarrow$ (4), we see by (5.3) that (1) is equivalent to the embedding ψ: $B \longrightarrow T_y^*$ being LF_1. Now (5.2.1) and (3.5.2) imply that when ψ is LF_1, it is also height-one preserving. $\square$

5.8 THEOREM. *Assume the setting and notation of* (4.1) *and in addition that* $(R, \mathbf{m}) := (T, \mathbf{n})$ *is excellent. The following are equivalent*:

(1) *The elements* $\tau_1, \dots, \tau_s$ *are primarily limit-intersecting in* y *over* R.
(2) *For every prime ideal* P *of* $B_0 := R[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)}$ *with* $y \notin P\widehat{R}$ *and* $\dim(B_0/P) \leq s$, *the extension* $P\widehat{R}$ *is primary for the maximal ideal of* $\widehat{R}$.

Proof. For (1)$\Rightarrow$(2), let $P \in \operatorname{Spec}(B_0)$ be such that $y \notin P\widehat{R}$ and $\dim(B_0/P) \leq s$. Suppose that $P\widehat{R}$ is not $\mathbf{m}\widehat{R}$-primary. Then there exists a minimal prime divisor $\widehat{Q}$ of $P\widehat{R}$ such that $y \notin \widehat{Q}$. It follows that $\operatorname{ht}(\widehat{Q}) \leq d - 1$, where $d = \dim(R)$. Put $Q := \widehat{Q} \cap B_0$; now $B_0 \longrightarrow \widehat{R}_y$ is LF_{d-1} and so the morphism

$$\phi_{\widehat{Q}} : (B_0)_Q \longrightarrow (\widehat{R}_y)_{\widehat{Q}\widehat{R}_y}$$

is faithfully flat. Hence by going-down [M2, Theorem 4, page 33], $\operatorname{ht}(Q) \leq d - 1$. But $P \subseteq Q$ and B_0 is catenary, so $d - 1 \geq \operatorname{ht}(Q) \geq \operatorname{ht}(P) \geq d$, a contradiction.

For (2)$\Rightarrow$(1), let $\widehat{P} \in \operatorname{Spec}(\widehat{R})$ with $\operatorname{ht}(\widehat{P}) \leq d - 1$. Put $P = \widehat{P} \cap B_0$ and $\mathbf{p} = \widehat{P} \cap R = P \cap R$. We show that the induced morphism

$$\phi_{\widehat{P}} : (B_0)_P \longrightarrow \widehat{R}_{\widehat{P}}$$

is faithfully flat. By [M1, (1) $\Longleftrightarrow$ (3) of Theorem 22.3] we have to verify two conditions:

(a) The morphism $\bar{\phi}_{\widehat{P}} : (B_0/\mathbf{p}B_0)_P \longrightarrow (\widehat{R}/\mathbf{p}\widehat{R})_{\widehat{P}}$ is faithfully flat.
(b) $\mathbf{p}(B_0)_P \otimes_{(B_0)_P} \widehat{R}_{\widehat{P}} \cong \mathbf{p}\widehat{R}_{\widehat{P}}$

Proof of (*a*). We observe that the ring $(B_0/\mathbf{p}B_0)_P$ is a localization of the polynomial ring $(R_\mathbf{p}/\mathbf{p}R_\mathbf{p})[\tau_1, \dots, \tau_n]$. Hence the ring $(B_0/\mathbf{p}B_0)_{\mathbf{p}B_0}$ is regular and so is the ring $(\widehat{R}/\mathbf{p}\widehat{R})_{\widehat{P}}$, since R is excellent. In particular, the ring $(\widehat{R}/\mathbf{p}\widehat{R})_{\widehat{P}}$ is Cohen-Macaulay, and [M1, Theorem 23.1] applies. Therefore we need only show the following dimension formula:

$$\dim(\widehat{R}/\mathbf{p}\widehat{R})_{\widehat{P}} = \dim(B_0/\mathbf{p}B_0)_P + \dim(\widehat{R}/P\widehat{R})_{\widehat{P}}.$$

Since $P\widehat{R}$ is contained in $\widehat{P}$ and $\operatorname{ht}(\widehat{P}) \leq d - 1$, our hypothesis implies that $\dim(B_0/P) > s$. (If $\dim(B_0/P) \leq s$, then $P\widehat{R}$ is $\mathbf{m}\widehat{R}$-primary.)

Claim. $\operatorname{ht}(P)$ in B_0 is equal to $\operatorname{ht}(P\widehat{R})$ in $\widehat{R}$; if $\widehat{W} \in \operatorname{Spec}(\widehat{R})$ is a minimal prime divisor of $P\widehat{R}$, then $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(P)$.

Proof of claim. Let $t_1, \dots, t_s$ be indeterminates over $\widehat{R}$, let $S := R[t_1, \dots, t_s]_{(\mathbf{m}, t_1, \dots, t_s)}$ and consider the commutative diagram

$$\begin{array}{ccccc} & & S := R[t_1, \dots, t_s]_{(\mathbf{m}, t_1, \dots, t_s)} & \longrightarrow & \widehat{S} \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\lambda}\downarrow \\ R & \longrightarrow & B_0 := R[\tau_1, \dots, \tau_s]_{(\mathbf{m}, \tau_1, \dots, \tau_s)} & \longrightarrow & \widehat{R}, \end{array}$$

where λ is the surjection with kernel $(t_1 - \tau_1, \dots, t_s - \tau_s)$, and α is the restriction, which is an isomorphism. Let $Q \in \operatorname{Spec}(S)$ correspond to $P \in \operatorname{Spec}(B_0)$ (that is, $Q := \alpha^{-1}(P)$ and let $\widehat{V} = \lambda^{-1}(W)$. Then $\widehat{V}$ is minimal over $(Q, \{t_i - \tau_i\})$ in $\widehat{S_s}$. We have that $\operatorname{ht}(Q) = \operatorname{ht}(P) < d$, $y \notin Q$ and $\dim(S/Q) > s$. Let $h = d - \operatorname{ht}(P)$; that is, $\operatorname{ht}(P) = \operatorname{ht}(Q) = d-h$ and $\dim(S/Q) = s+h$. Now choose $s_1, \dots, s_h \in S$ such that $I = (Q, s_1, \dots, s_h)S$ has height d in S. Now $\dim(B_0/(P, \alpha(s_1), \dots, \alpha(s_h))B_0)) = s$. Thus $(P, \alpha(s_1), \dots, \alpha(s_h))\widehat{R}$ is primary for the maximal ideal of $\widehat{R}$ by the hypothesis. Thus $J = \lambda^{-1}(P, \alpha(s_1), \dots, \alpha(s_h)) = (Q, s_1, \dots, s_h, \{t_i - \tau_i\})$ is primary for the maximal ideal of $\widehat{S}$. Therefore $\operatorname{ht}(J) = s + d$. But $(\widehat{V}, s_1, \dots, s_h)\widehat{S} \supseteq J\widehat{S}$, and so $\operatorname{ht}(\widehat{V}) \geq s+d-h$. Also $\operatorname{ht}(\widehat{V}) \leq \operatorname{ht}(Q)+s = d-h+s$. That is, $\operatorname{ht}(\widehat{V}) = s+d-h$. Now $\operatorname{ht}(\widehat{W}) = d - h = \operatorname{ht}(P)$, so $\operatorname{ht}(P) = \operatorname{ht}(P\widehat{R})$.

We proceed with the proof of (5.8) as follows. Let $\widehat{W} \in \operatorname{Spec}(\widehat{R})$ be a minimal prime divisor of $P\widehat{R}$ contained in $\widehat{P}$. Then

$$\begin{aligned}
\dim(\widehat{R}/P\widehat{R})_{\widehat{P}} &= \dim(\widehat{R}_{\widehat{P}}) - \operatorname{ht}(P\widehat{R}_{\widehat{P}}) \\
&= \dim(\widehat{R}_{\widehat{P}}) - \operatorname{ht}(\widehat{W}) \\
&= \dim(\widehat{R}_{\widehat{P}}) - \operatorname{ht}(P(B_0)_P) \\
&= \dim(\widehat{R}_{\widehat{P}}) - \operatorname{ht}(\mathbf{p}\widehat{R}_{\widehat{P}}) - (\operatorname{ht}(P(B_0)_P) - \operatorname{ht}(\mathbf{p}(B_0)_P)) \\
&= \dim(\,(\widehat{R}/\mathbf{p}\widehat{R})_{\widehat{P}}) - \dim(\,(B_0/\mathbf{p}B_0)_P).
\end{aligned}$$

Proof of (b). Since $R_{\mathbf{p}} \longrightarrow (R - \mathbf{p})^{-1}(B_0)$ is a flat extension we have that

$$\mathbf{p}(B_0)_P \cong \mathbf{p}R_{\mathbf{p}} \otimes_{R_{\mathbf{p}}} (B_0)_P.$$

Therefore

$$\mathbf{p}(B_0)_P \otimes_{(B_0)_P} \widehat{R}_{\widehat{P}} \cong (\mathbf{p}R_{\mathbf{p}} \otimes_{R_{\mathbf{p}}} (B_0)_P) \otimes_{(B_0)_P} \widehat{R}_{\widehat{P}} \cong \mathbf{p}R_{\mathbf{p}} \otimes_{R_{\mathbf{p}}} \widehat{R}_{\widehat{P}} \cong \mathbf{p}\widehat{R}_{\widehat{P}}$$

where the last isomorphism is implied by the flatness of the canonical morphism $R_{\mathbf{p}} \longrightarrow \widehat{R}_{\widehat{P}}$. $\square$

5.9 *Remark.* It would be interesting to know if a similar statement to that given in (5.8) also holds without the hypothesis that $T = R$ is an excellent normal Noetherian domain, i.e., if T is a quasilocal Krull domain as in (4.1) does condition (1) in (5.8) imply condition (2)?

We have the following transitive property of limit-intersecting elements.

5.10 PROPOSITION. *Assume the setting and notation of* (4.1). *Also assume that* $s > 1$ *and for all* $j \in \{1, \dots s\}$, *set* $A(j) := F(\tau_1, \dots, \tau_j) \cap \widehat{T}$. *Then the following statements are equivalent*:

(1) $\tau_1, \dots, \tau_s$ *are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in* y *over* T.

(2) *For all* $j \in \{1, \dots, s\}$, *the elements* $\tau_1, \dots, \tau_j$ *are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in* y *over* T *and the elements* $\tau_{j+1}, \dots, \tau_s$ *are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in* y *over* $A(j)$.

(3) *There exists a* $j \in \{1, \dots, s\}$, *such that the elements* $\tau_1, \dots, \tau_j$ *are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in* y *over* T *and the elements* $\tau_{j+1}, \dots, \tau_s$ *are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in* y *over* $A(j)$.

Proof. Set $B(j) := \bigcup_{n=1}^{\infty} T[\tau_{1n}, \dots, \tau_{jn}]_{(\mathbf{n}, \tau_{1n}, \dots, \tau_{jn})}$. It is clear that (2) $\Longrightarrow$ (3).

For (3) $\Longrightarrow$ (1), items (5.5) and (5.2.1) imply that $A(j) = B(j)$ under each of the conditions on $\tau_1, \dots, \tau_j$. The definitions of $\tau_{j+1}, \dots, \tau_s$ limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$ together with (5.2.4) imply the equivalence of the stated flatness properties for each of the morphisms

$$
\begin{aligned}
\varphi_1\colon A(j)[\tau_{j+1}, \dots, \tau_s]_{(-)} &\longrightarrow A(j)^*_y = T^*_y \\
\varphi_2\colon (A(j)[\tau_{j+1}, \dots, \tau_s]_{(-)})_y &\longrightarrow T^*_y \\
\varphi_3\colon (B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)})_y &\longrightarrow T^*_y \\
\varphi_4\colon (T[\tau_1, \dots, \tau_s]_{(-)})_y &\longrightarrow T^*_y \\
\varphi_5\colon T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} &\longrightarrow T^*_y.
\end{aligned}
$$

The respective flatness properties for φ_5 are equivalent to the conditions that $\tau_1, \dots, \tau_s$ be limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T. Thus (3) $\Longrightarrow$ (1).

For (1) $\Longrightarrow$ (2), we go backwards. The statement of (1) for $\tau_1, \dots, \tau_s$ is equivalent to the respective flatness property for φ_5. This is equivalent to φ_4 and thus φ_3 having the respective flatness property. By (5.2.4), $B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)} \longrightarrow T^*_y$ has the appropriate flatness property. Also $B(j) \longrightarrow B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)}$ is flat, and so $B(j) \longrightarrow T^*_y$ has the appropriate flatness property. Thus $\tau_1, \dots, \tau_j$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T. Therefore $A(j) = B(j)$, and so $A(j) \longrightarrow T^*_y$ has the appropriate flatness property. It follows that $\tau_{j+1}, \dots, \tau_s$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$. □

6. Some examples

Let $R = \mathbb{Q}[x, y]_{(x,y)}$, the localized polynomial ring in two variables x and y over the field $\mathbb{Q}$ of rational numbers. Then $\widehat{R} = \mathbb{Q}[[x, y]]$, the formal power series ring in x and y, is the $\mathbf{m} = (x, y)R$-adic completion of R. In [HRW1], an element $\tau \in \widehat{\mathbf{m}} = (x, y)\widehat{R}$ is defined to be *residually algebraically independent over R* if τ is algebraically independent over the fraction field of R and for each height-one prime $\widehat{P}$ of $\widehat{R}$ such that $\widehat{P} \cap R \neq (0)$, the image of τ in $\widehat{R}/\widehat{P}$ is algebraically independent over the fraction field of $R/(\widehat{P} \cap R)$. It is shown in [HRW1, Theorem 4.4], that if τ is residually algebraically independent over R and L is the fraction field of $R[\tau]$, then $L \cap \widehat{R}$ is the localized polynomial ring $R[\tau]_{(\mathbf{m},\tau)}$.

In this section we present several examples of residually algebraically independent elements.

6.1 THEOREM. *Let $\sigma \in x\mathbb{Q}[[x]]$ and $\rho \in y\mathbb{Q}[[y]]$ be such that the following two conditions are satisfied:*

(*i*) *σ is algebraically independent over $\mathbb{Q}(x)$ and ρ is algebraically independent over $\mathbb{Q}(y)$.*

(*ii*) $\operatorname{trdeg}_{\mathbb{Q}}\mathbb{Q}(y, \{\frac{\partial^n \rho}{\partial y^n}\}_{n\in\mathbb{N}}) > \operatorname{trdeg}_{\mathbb{Q}}\mathbb{Q}(x, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$.

Then $\tau := \sigma + \rho$ is residually algebraically independent over $\mathbb{Q}[x, y]_{(x,y)}$.

Before proving Theorem 6.1, we establish the existence of elements σ and ρ satisfying properties (i) and (ii) of Theorem 6.1. Let $\sigma = e^x - 1 \in \mathbb{Q}[[x]]$ and choose for ρ a hypertranscendental element in $\mathbb{Q}[[y]]$. Recall that a power series $\rho = \sum_{i=0}^{\infty} b_i y^i \in \mathbb{Q}[[y]]$ is called *hypertranscendental* over $\mathbb{Q}[y]$ if the set of all partial derivatives $\{\frac{\partial^n \rho}{\partial y^n}\}_{n\in\mathbb{N}}$ is infinite and algebraically independent over $\mathbb{Q}(y)$. (Two examples of hypertranscendental elements are the Gamma function and the Riemann Zeta function.[5]) Thus σ, ρ satisfy the conditions of Theorem 6.1

Alternatively, let $\sigma = e^x - 1$ and $\rho = e^{(e^y - 1)} - 1$. The conditions of Theorem 6.1 follow from [Ax].

In either case, Theorem 6.1 implies that $\tau := \sigma + \tau$ is residually algebraically independent, and we have the following corollary.

6.2 COROLLARY. *There exists an explicitly defined element $\tau \in (x, y)\mathbb{Q}[[x, y]]$ such that τ is residually algebraically independent over $\mathbb{Q}[x, y]_{(x,y)}$. Therefore the localized polynomial ring $\mathbb{Q}[x, y, \tau]_{(x,y,\tau)}$ is the intersection $\mathbb{Q}(x, y, \tau) \cap \widehat{R}$.*

Proof of 6.1. To show that the element $\tau = \sigma + \rho$ is residually algebraically independent over $R = \mathbb{Q}[x, y]_{(x,y)}$, we introduce the intermediate ring

$$D := \mathbb{Q}(x, \sigma) \cap \mathbb{Q}[[x]].$$

Then D is an excellent discrete valuation domain with completion $\widehat{D} = \mathbb{Q}[[x]]$, and D has transcendence degree 2 over $\mathbb{Q}$. There is a convenient way to describe D as a directed union of polynomial rings in two variables over $\mathbb{Q}$: Set

[5]The exponential function is, of course, far from being hypertranscendental.

$\sigma := \sum a_i x^i$, where $a_i \in \mathbb{Q}$. Then the n^{th}-endpiece for σ, defined as in (2.3), satisfies $\sigma_n = x(\sigma_{n+1} + a_{n+1})$ and D can be obtained as

$$D = \varinjlim_{n\to\infty} \mathbb{Q}[x, \sigma_n]_{(x,\sigma_n)} = \bigcup_{n=1}^{\infty} \mathbb{Q}[x, \sigma_n]_{(x,\sigma_n)}.$$

The displayed statement follows by (5.4.1): Every element of $\mathbb{Q}[[x]]$ which is algebraically independent over $\mathbb{Q}(x)$ is also primarily limit-intersecting over the discrete valuation domain $\mathbb{Q}[x]_{(x)}$.

Since $\sigma \in Dx$, the maximal ideal of D is (x). The structure morphism

$$\mathbb{Q}[x, \sigma_n]_{(x,\sigma_n)} \longrightarrow \mathbb{Q}[x, \sigma_{n+1}]_{(x,\sigma_{n+1})}$$

is defined by the relation $\sigma_n \mapsto x(\sigma_{n+1} + a_{n+1})$.

The ring $T := D[y]_{(x,y)}$ is between R and its completion $\widehat{R}$ and has completion $\widehat{T} = \widehat{R}$:

$$R = \mathbb{Q}[x, y]_{(x,y)} \longrightarrow T = D[y]_{(x,y)} \longrightarrow \widehat{R} = \widehat{T} = \mathbb{Q}[[x, y]]$$

$$\begin{tikzcd}
 & \widehat{R} = \mathbb{Q}[[x,y]] \arrow[d, dash] & \\
 & T := D[y]_{(x,y)} \arrow[dl, dash] \arrow[dr, dash] & \\
D := \mathbb{Q}(x,\sigma) \cap \mathbb{Q}[[x]] = \cup \mathbb{Q}[x,\sigma_n]_{(x,\sigma_n)} & & R := \mathbb{Q}[x,y]_{(x,y)}
\end{tikzcd}$$

The rings of the example

To show that $\tau := \sigma + \rho$ is residually algebraically independent over R, let $\widehat{Q}$ be a height-one prime ideal of $\widehat{R}$ and assume that $P := \widehat{Q} \cap R \neq 0$. Let $W := \widehat{Q} \cap T$. It is easy to see for $P = (x)$ or $P = (y)$ that the image $\bar{\tau}$ of τ in $\bar{\widehat{R}} = \mathbb{Q}[[x, y]]/\widehat{Q}$ remains algebraically independent over $\bar{R} = \mathbb{Q}[x, y]_{(x,y)}/P$. We show:

6.3. Proposition. *Let $P \in \operatorname{Spec}(R)$ and $\widehat{Q} \in \operatorname{Spec}(\widehat{R})$ be height-one primes as in the paragraph above with $P \neq (x)$ and $P \neq (y)$. Then $\bar{\tau}$ is transcendental over $\bar{T} := T/W$, and the set $\{\bar{\sigma}, \bar{\rho}\}$ is algebraically independent over $\bar{R}$. In particular $\tau = \sigma + \rho$ is residually algebraically independent over R.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{Q}[[y]] \arrow[r, "\psi_y"] & \bar{\widehat{R}} = \widehat{R}/\widehat{Q} & \mathbb{Q}[[x]] \arrow[l, "\psi_x"'] \\
 & \bar{T} = T/W \arrow[u] & \\
\mathbb{Q}[y]_{(y)} \arrow[uu] \arrow[r, "\phi_y"] & \bar{R} = R/P \arrow[u] & \mathbb{Q}[x]_{(x)} \arrow[l, "\phi_x"'] \arrow[uu]
\end{tikzcd}$$

All morphisms in the diagram are injective and we obtain:

(a) The ring $\bar{R}$ is algebraic over the rings $\mathbb{Q}[x]_{(x)}$ and $\mathbb{Q}[y]_{(y)}$ since $\operatorname{trdeg}_{\mathbb{Q}}(\bar{R}) = 1$.
(b) The ring $\bar{\widehat{R}}$ is finite over both rings $\mathbb{Q}[[x]]$ and $\mathbb{Q}[[y]]$.

To complete the proof of (6.3) we prove the following claim:

6.4 *Claim.* $\operatorname{trdeg}_{\mathbb{Q}}(\bar{T}) = 2$, and thus $\operatorname{trdeg}_{\bar{R}}(\bar{T}) = 1$.

Proof of (6.4). Let $W_0 = W \cap \mathbb{Q}[x, y, \sigma]$. Since $D[1/x]$ is a localization of $\mathbb{Q}[x, \sigma]_{(x,\sigma)}$, we see that $\bar{T}[1/\bar{x}]$ is a localization of $\mathbb{Q}[x, y, \sigma]/W_0$. Now W_0 has height one because $x \notin W_0$. This shows that $\operatorname{trdeg}_{\mathbb{Q}}(\bar{T}) = 2$.

Proof of (6.3) *continued.* We have seen that the element $\bar{\sigma}$ is algebraically independent over $\bar{R}$. Now $\bar{\sigma} \in \bar{T}$, and $\operatorname{trdeg}[T : \mathbb{Q}] = 1 + \operatorname{trdeg}[D : \mathbb{Q}] = 3$, whereas $\operatorname{trdeg}[R : \mathbb{Q}] = 2$. Also $\operatorname{trdeg}[\bar{T} : \mathbb{Q}] \leq 2$, and $\operatorname{trdeg}[\bar{R} : \mathbb{Q}] \leq 1$. Thus to show that $\bar{\tau}$ is transcendental over $\bar{T}$ is equivalent to showing that the set $\{\bar{\sigma}, \bar{\rho}\}$ is algebraically independent over $\bar{R}$. In order to show this we make use of the differential properties of the functions σ and ρ. We first pass to the embeddings of the fraction fields:

$$\begin{tikzcd}
\mathbb{Q}((\bar{y})) \arrow[r] & \mathcal{Q}(\bar{\widehat{R}}) := \mathcal{Q}(\mathbb{Q}[[\bar{x}, \bar{y}]]) \\
 & \mathcal{Q}(\bar{T}) := \mathbb{Q}(\bar{y}, \bar{\sigma}, \bar{x}) \arrow[u] \\
\mathbb{Q}(\bar{y}) \arrow[uu] \arrow[r] & \mathcal{Q}(\bar{R}) := \mathbb{Q}(\bar{y}, \bar{x}) \arrow[u]
\end{tikzcd}$$

We have $\mathcal{Q}(\bar{R}) = \mathbb{Q}(\bar{y}, \bar{x})$ and $\mathcal{Q}(\bar{T}) = \mathbb{Q}(\bar{y}, \bar{\sigma}, \bar{x})$ where $\bar{x}$ is algebraic over $\mathbb{Q}(\bar{y})$, and $\bar{y}$ and $\bar{\sigma}$ are algebraically independent over $\mathbb{Q}$. Let $\widehat{d}$, respectively d, denote the partial derivative map $\frac{\partial}{\partial \bar{y}}$ on $\mathbb{Q}((\bar{y}))$, respectively on $\mathbb{Q}(\bar{y})$. Note that d is the restriction of $\widehat{d}$ to $\mathbb{Q}(\bar{y})$. Since all the horizontal field extensions are separable algebraic, $\widehat{d}$ and d extend uniquely to derivations $\widehat{d_1}$ and d_1 of $\mathcal{Q}(\bar{\widehat{R}}) := \mathcal{Q}(\mathbb{Q}[[\bar{x}, \bar{y}]])$, respectively $\mathcal{Q}(\bar{R}) := \mathbb{Q}(\bar{x}, \bar{y})$. Again d_1 is the restriction of $\widehat{d_1}$ to $\mathbb{Q}(\bar{x}, \bar{y})$. Suppose that the height-one prime ideal P in $R = \mathbb{Q}[x, y]_{(x,y)}$ is generated by the prime element $p(x)$ given by

$$p(x) := \sum_{i=0}^{m} a_i(y)x^i \in \mathbb{Q}[x, y], \text{ where } a_i(y) \in \mathbb{Q}[y].$$

Then $p(\bar{x}) = 0$. We assign the notation $p'(\bar{x})$:

$$p'(\bar{x}) := \frac{\partial p}{\partial \bar{x}} = \sum_{i=1}^{m} i a_i(\bar{y})\bar{x}^{i-1} \neq 0,$$

because $\bar{x}$ is separable over $\mathbb{Q}[\bar{y}]$. Also (since $0 = d_1(p(\bar{x}))$)

$$\begin{aligned} d_1(p(\bar{x})) &= \frac{\partial p(\bar{x})}{\partial \bar{y}} = \sum_{i=0}^{m}[\frac{\partial a_i(\bar{y})}{\partial \bar{y}}\bar{x}^i + a_i(\bar{y})i(\bar{x})^{i-1}d_1(\bar{x})] \\ &\Longrightarrow -\sum_{i=0}^{m} \frac{\partial a_i(\bar{y})}{\partial \bar{y}}\bar{x}^i = d_1(\bar{x})\sum_{i=0}^{m} i a_i(\bar{y})(\bar{x})^{i-1} \\ &= d_1(\bar{x})p'(\bar{x}) = \widehat{d_1}(\bar{x})p'(\bar{x}). \end{aligned}$$

Thus, we have shown that $p'(\bar{x})\widehat{d_1}(\bar{x}) \in \bar{\widehat{R}}$.

Next we show:

6.5 *Claim.* For every element $\lambda \in \bar{\widehat{R}}$ we have that $p'(\bar{x})\widehat{d_1}(\lambda) \in \bar{\widehat{R}}$.

Proof of (6.5). Let $\widehat{q}(x, y) \in \mathbb{Q}[[x, y]]$ be a prime element generating $\widehat{Q}$. Since x and y are not contained in P, the element $\widehat{q}(x, y)$ is regular in x (in the sense of Zariski-Samuel [ZS, p.145]). Thus by [ZS, Corollary 1, p. 145] the element $\widehat{q}(x, y)$ can be written as

$$\widehat{q}(x, y) = \epsilon(x, y)(x^n + \widehat{b}_{n-1}(y)x^{n-1} + \cdots + \widehat{b}_0(y)),$$

for some unit $\epsilon(x, y) \in \mathbb{Q}[[x, y]]$, where each $\widehat{b}_i(y) \in \mathbb{Q}[[y]]$. Now $\widehat{Q}$ is also generated by $\epsilon^{-1}\widehat{q}$, and thus $\bar{\widehat{R}} = \mathbb{Q}[[x, y]]/\widehat{Q}$ is a finite free $\mathbb{Q}[[\bar{y}]]$-module with basis $1, \bar{x}, \dots, \bar{x}^{n-1}$. Thus every element $\lambda \in \bar{\widehat{R}}$ can be written as

$$\lambda = \widehat{c}_{n-1}(\bar{y})\bar{x}^{n-1} + \cdots + \widehat{c}_1(\bar{y})\bar{x} + \widehat{c}_0(\bar{y}), \text{ where } \widehat{c}_i \in \mathbb{Q}[[\bar{y}]].$$

This implies

$$\widehat{d_1}(\lambda) = \widehat{d_1}(\bar{x})(\widehat{(n}-1)\widehat{c}_{n-1}(\bar{y})\bar{x}^{n-2} + \cdots + \widehat{c_1}(\bar{y})) + \sum_{i=0}^{n-1} \widehat{d_1}(\widehat{c_i}(\bar{y}))\bar{x}^i.$$

Now the sum expression on the right is in $\hat{\bar{R}}$. But also, by the earlier argument, $p'(\bar{x})\widehat{d_1}(\bar{x}) \in \hat{\bar{R}}$ and so $p'(\bar{x})\widehat{d_1}(\lambda) \in \hat{\bar{R}}$. □

Note. For convenience we drop the bars on x, y, σ, ρ. For the remainder of this proof, x, y are considered in $\hat{\bar{R}}$. Thus we rewrite the last result as: $p'(x)\widehat{d_1}(\lambda) \in \hat{\bar{R}}$.

6.6 *Claim.* $\widehat{d_1}(\sigma) = d_1(x)\frac{\partial\sigma}{\partial x}$ and for all $n > 1$, $\widehat{d_1}^{\,n}(\sigma)$ is a linear combination of $\frac{\partial^i\sigma}{\partial x^i}$ over $\mathbb{Q}(x, y) = \mathcal{Q}(\bar{R})$, where $1 \leq i \leq n$. (Note that $\widehat{d_1} : \mathcal{Q}(\hat{\bar{R}}) \longrightarrow \mathcal{Q}(\hat{\bar{R}})$ and that its restriction $\widehat{d_1}|_{\mathcal{Q}(\bar{R})} : \mathcal{Q}(\bar{R}) \mapsto \mathcal{Q}(\bar{R})$ is a derivation of $\mathcal{Q}(\bar{R})$.)

Proof of (6.6). For all $m \in \mathbb{N}$ we have

$$\sigma = \sum_{i=1}^{m} a_i x^i + x^{m+1}\lambda \quad \text{where } \lambda = \sum_{i=m+1}^{\infty} a_i x^{i-(m+1)} \in \hat{\bar{R}} \text{ and } a_i \in \mathbb{Q}.$$

Therefore

$$p'(x)\widehat{d_1}(\sigma) = p'(x)d_1(x)\sum_{i=1}^{m} i a_i x^{i-1} + p'(x)d_1(x)x^m\lambda + x^{m+1}p'(x)\widehat{d_1}(\lambda).$$

By Claim 6.5,

$$p'(x)\widehat{d_1}(\sigma) - p'(x)d_1(x)\frac{\partial\sigma}{\partial x} \in (x^m)\hat{\bar{R}} \quad \text{for all } m \in \mathbb{N}.$$

Since we are in a domain, it follows that $\widehat{d_1}(\sigma) = d_1(x)\frac{\partial\sigma}{\partial x}$, as desired. The second statment of (6.6) follows by induction.

Completion of proof of (6.3). The field $\mathcal{Q}(\bar{T}, \{\frac{\partial^n\sigma}{\partial x^n}\}_{n\in\mathbb{N}}) = \mathbb{Q}(y, \sigma, x, \{\frac{\partial^n\sigma}{\partial x^n}\}_{n\in\mathbb{N}})$ is closed under $\widehat{d_1}$ and has the same transcendence degree over $\mathbb{Q}$ as the field $\mathbb{Q}(x, \{\frac{\partial^n\sigma}{\partial x^n}\}_{n\in\mathbb{N}})$. Now $\widehat{d_1}$ extends to the algebraic closure of $\mathbb{Q}(y, \sigma, x, \{\frac{\partial^n\sigma}{\partial x^n}\}_{n\in\mathbb{N}})$ uniquely. If τ is algebraic over $\mathcal{Q}(\bar{T})$, then the set $\{\frac{\partial^n\tau}{\partial y^n}\}_{n\in\mathbb{N}}$ is contained in the algebraic closure of the field $\mathcal{Q}(\bar{T}, \{\frac{\partial^n\sigma}{\partial x^n}\}_{n\in\mathbb{N}})$. But this is impossible, since the transcendence degree of $\mathbb{Q}(y, \{\frac{\partial^n\sigma}{\partial x^n}\}_{n\in\mathbb{N}})$ is too large.

An alternative way of saying this is as follows: The fraction field of $\bar{T}$, $\mathcal{Q}(\bar{T}) = \mathbb{Q}(\bar{y}, \bar{\sigma}, \bar{x})$ is generated by $\bar{y}, \bar{\sigma}$, and $\bar{x}$, which are all mapped into $\mathcal{Q}(\bar{T})$ under $\widehat{d_1}$. This shows that the field $\mathcal{Q}(\bar{T})$ is closed under the derivation $\widehat{d_1}$. Moreover, since the rationals are contained in $\mathcal{Q}(\bar{T})$, this derivation extends uniquely to the algebraic closure. If $\bar{\rho}$ is algebraic over $\mathcal{Q}(\bar{T})$ all its partial derivatives are algebraic over the same field, a contradiction to our assumption that $\{\frac{\partial^n \bar{\rho}}{\partial y^n}\}$ is an algebraically independent set of elements over $\mathbb{Q}$. Now $\mathbb{Q}((\bar{y})) \hookrightarrow \mathcal{Q}(\widehat{\bar{R}})$ and $\widehat{d_1}$ is the partial derivative: $\mathbb{Q}((\bar{y})) \to \mathbb{Q}((\bar{y}))$. This shows that $\bar{\rho}$ is transcendental over $\mathcal{Q}(\bar{T})$ and hence $\bar{\tau}$ is transcendental over $\bar{R}$.

This completes the proof of Proposition 6.3 and Theorem 6.1. $\square$

6.7 *Example.* The element $\tau = \sigma + \rho$ is residually algebraically independent over $R = \mathbb{Q}[x, y]_{(x,y)}$. Thus by [HRW1, Theorem 4.4], we have

$$\mathbb{Q}(x, y, \tau) \cap \widehat{R} = \mathbb{Q}[x, y, \tau]_{(x,y,\tau)}.$$

Since $\dim(R) = 2$, by Theorem 5.6 [HRW1], the element τ is also primarily independent over $R = \mathbb{Q}[x, y]_{(x,y)}$ in the sense of [HRW1, Definition 3.1], that is, for every prime ideal P of $S = R[\tau_1, \dots, \tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)}$ such that $\dim(S/P) \le n$, the ideal $P\widehat{R}$ is $\mathbf{m}\widehat{R}$-primary.

6.8 *Example.* For $S := \mathbb{Q}[x, y, z]_{(x,y,z)}$, the construction of (6.3) yields an example of a height-one prime ideal $\widehat{P}$ of $\widehat{S} = \mathbb{Q}[[x, y, z]]$ in the generic formal fiber of S such that

$$\mathcal{Q}(S) \cap (\widehat{S}/\widehat{P}) = S.$$

Proof. Let $\widehat{P} := (z - \tau) \subseteq \mathbb{Q}[[x, y, z]]$, where τ is the element of Theorem 6.1. Then $\mathbb{Q}(x, y, z) \cap \widehat{S}/\widehat{P}$ can be identified with the intersection $\mathbb{Q}(x, y, \tau) \cap \mathbb{Q}[[x, y]]$ of (6.1). Therefore

$$\mathbb{Q}(x, y, z) \cap (\widehat{S}/\widehat{P}) = S = \mathbb{Q}[x, y, z]_{(x,y,z)}.$$

The prime ideal $\widehat{P}$ is not maximal in the generic formal fiber of $S = \mathbb{Q}[x, y, z]_{(x,y,z)}$, since every prime ideal maximal in the generic formal fiber of a polynomial ring in one variable over a two-dimensional ring has height 2. $\square$

Example 6.8 demonstrates that the strong connection between the maximal ideals of the generic formal fiber of a localized polynomial ring and certain birational extensions of this localized polynomial ring does not extend to prime ideals nonmaximal in the generic formal fiber of this ring. (See [HRS] for more details.)

6.9 *Example.* Again let $S = \mathbb{Q}[x, y, z]_{(x,y,z)}$. With a slight modification of Example 6.8, we exhibit a prime ideal $\widehat{P}$ in the generic formal fiber of S which *does* correspond to a nontrivial birational extension; that is, the intersection ring

$$A := \mathcal{Q}(S) \cap \widehat{S}/\widehat{P}$$

is a spot over S.

Proof. Let τ be the element from Theorem 6.1. Let $\widehat{P} = (z - x\tau) \subseteq \mathbb{Q}[[x, y, z]]$. Since τ is transcendental over $\mathbb{Q}(x, y, z)$, the prime ideal $\widehat{P}$ is in the generic formal fiber of S. The ring S can be identified with a subring of $\widehat{S}/\widehat{P} \cong \mathbb{Q}[[x, y]]$ by considering $S = \mathbb{Q}[x, y, x\tau]_{(x,y,x\tau)}$. By reasoning similar to that of Example 6.8,

$$\mathcal{Q}(S) \cap \mathbb{Q}[[x, y]] = \mathbb{Q}(x, y, \tau) \cap \mathbb{Q}[[x, y]] = \mathbb{Q}[x, y, \tau]_{(x,y,\tau)}.$$

The ring $\mathbb{Q}[x, y, \tau]_{(x,y,\tau)}$ is then the essentially finitely generated birational extension of S defined as $S[z/x]_{(x,y,z/x)}$. □

Example 6.9 is of interest in connection with [HRS], where it is shown that if the prime ideal $\widehat{P}$ of $\mathbb{Q}[[x, y, z]]$ is maximal in the generic formal fiber of $S = \mathbb{Q}[x, y, z]_{(x,y,z)}$, then the intersection ring $\mathbb{Q}(x, y, z) \cap \mathbb{Q}[[x, y, z]]/\widehat{P}$ is well understood; whereas the last two examples show if $\widehat{P}$ is not maximal in the generic formal fiber, then the intersection ring can be almost anything.

6.10 *Example.* Let $\sigma \in x\mathbb{Q}[[x]]$ and $\rho \in y\mathbb{Q}[[y]]$ be as in Theorem 6.1. If $D := \mathbb{Q}(x, \sigma) \cap \mathbb{Q}[[x]] = \bigcup_{n=1}^{\infty} \mathbb{Q}[x, \sigma_n]_{(x,\sigma_n)}$ and $T := D[y]_{(x,y)}$, so T is regular local with completion $\widehat{T} = \mathbb{Q}[[x, y]]$, then the element ρ is primarily limit-intersecting in y over T.

Proof. We show that the morphism $\phi_y\colon T[\rho] \longrightarrow \mathbb{Q}[[x, y]]_y$ is LF_1; that is, the induced map $\phi_{\widehat{P}} : T[\rho]_{\widehat{P}\cap T[\rho]} \longrightarrow \mathbb{Q}[[x, y]]_{\widehat{P}}$ is flat for every height-one prime ideal $\widehat{P}$ of $\mathbb{Q}[[x, y]]$ with $y \notin \widehat{P}$. It is equivalent to show for every height-one prime $\widehat{P}$ of $\mathbb{Q}[[x, y]]$ that $\widehat{P} \cap T[\rho]$ has height ≤ 1. If $\widehat{P} = (x)$, the statement is immediate, since ρ is algebraically independent over $\mathbb{Q}(y)$. Next we consider the case $\widehat{P} \cap Q[x, y, \sigma] = (0)$. Since $\mathbb{Q}(x, y, \sigma) = \mathbb{Q}(x, y, \sigma_n)$ for every positive integer n, $\widehat{P} \cap Q[x, y, \sigma] = (0)$ if and only if $\widehat{P} \cap Q[x, y, \sigma_n] = (0)$. Moreover, if this is true, then since the fraction field of $T[\rho]$ has transcendence degree one over $\mathbb{Q}(x, y, \sigma)$, then $\widehat{P} \cap T[\rho]$ has height ≤ 1. The remaining case is where $P := \widehat{P} \cap Q[x, y, \sigma] \neq (0)$ and $xy \notin \widehat{P}$. By Proposition 6.3, $\bar{\rho}$ is transcendental over $\bar{T} = T/(\widehat{P} \cap T)$, and this is equivalent to $\mathrm{ht}(\widehat{P} \cap T[\tau]) = 1$. (For an alternative proof see [HRW2], (3.5).) □

Still referring to ρ, σ, σ_n as in (6.1) and (6.10) and using the fact that σ is primarily limit-intersecting in y over T, we have

$$A := \mathcal{Q}(T)(\rho) \cap \mathbb{Q}[[x, y]] = \varinjlim T[\rho_n]_{(x,y,\rho_n)} = \varinjlim \mathbb{Q}[x, y, \sigma_n, \rho_n]_{(x,y,\sigma_n,\rho_n)}$$

where the endpieces ρ_n are defined as in (2.3); viz., $\rho := \sum_{n=1}^{\infty} b_i y^i$ and $\rho_n = \sum_{i=n+1}^{\infty} b_i y^{i-n}$. The philosophy here is that sufficient "independence" of the algebraically independent elements σ and ρ allows us to explicitly describe the intersection ring A.

The previous examples have been over localized polynomial rings, where we are free to exchange variables. The next example shows, over a different regular local domain, that an element in the completion with respect to one regular parameter x may be residually limit-intersecting with respect to x whereas the corresponding element in the completion with respect to another regular parameter y may be transcendental but fail to be residually limit-intersecting.

6.11 *Example.* There exists a regular local ring R with $\widehat{R} = \mathbb{Q}[[x, y]]$ such that $\sigma = e^x - 1$ is residually limit-intersecting in x over R, whereas $\gamma = e^y - 1$ fails to be limit-intersecting in y over R.

Proof. Let $\{\omega_i\}_{i\in I}$ be a transcendence basis of $\mathbb{Q}[[x]]$ over $\mathbb{Q}(x)$ such that

$$\{e^{x^n}\}_{n\in\mathbb{N}} \subseteq \{\omega_i\}_{i\in I}.$$

Let D be the discrete valuation ring

$$D = \mathbb{Q}(x, \{\omega_i\}_{i\in I, \omega_i \neq e^x}) \cap \mathbb{Q}[[x]].$$

Obviously, $\mathbb{Q}[[x]]$ has transcendence degree 1 over D. The set $\{e^x\}$ is a transcendence basis of $\mathbb{Q}[[x]]$ over D. Let $R = D[y]_{(x,y)}$. By (5.4.1), the element $\sigma = e^x - 1$ is residually limit-intersecting in x over D. Moreover, by [HRW2, (3.3)], σ is also residually limit-intersecting over $R := D[y]_{(x,y)}$. However, the element $\gamma = e^y - 1$ is not residually limit-intersecting in y over R. To see this, consider the height-one prime ideal $P := (y - x^2)\mathbb{Q}[[x, y]]$. The prime ideal $W := P \cap R[\tau]_{(x,y,\tau)}$ contains the element $\gamma - e^{x^2} - 1 = e^y - e^{x^2}$. Therefore W has height greater than one and γ is not residually limit-intersecting in y over R. $\square$

Note that the intersection ring $\mathcal{Q}(R)(\tau) \cap \mathbb{Q}[[x, y]]$ is a regular local ring with completion $\mathbb{Q}[[x, y]]$ by Valabrega [V].

Added in Proof. Since completing this article, the authors have obtained additional results related to [HRW2, (2.12)] cited in (5.6.3); these new results will appear in *Noetherian domains inside a homomorphic image of a completion*, J. Algebra.

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