TRIPLY TRANSITIVE GROUPS IN WHICH ONLY THE IDENTITY FIXES FOUR LETTERS

BY

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Doubly transitive groups in which only the identity fixes three letters have been the subject of recent investigations by Feit [1] and Suzuki [3], [4]. In the present paper we shall consider the corresponding class of triply transitive groups—that is, triply transitive groups in which only the identity fixes four letters—and shall prove the following:

THEOREM.[1] *If G_3 is a finite triply transitive group in which only the identity fixes* 4 *letters, then G_3 is one of the following*:

(a) *sharply 4-fold transitive, and hence is either the Mathieu group M_{11}, the symmetric group S_4 or S_5, or the alternating group A_6;*

(b) *sharply triply transitive, and hence is either the linear fractional group L_q over some $GF(q)$ or the group L_q' of transformations*

$$x \longrightarrow (ax^{\sigma(\Delta)} + b)/(cx^{\sigma(\Delta)} + d),$$

over some $GF(q^2)$, q odd, where $\Delta = ad - bc \neq 0$, and

$$\sigma(\Delta): x \longrightarrow \begin{cases} x & \text{if } \Delta \text{ is a square,} \\ x^q & \text{if } \Delta \text{ is a nonsquare;} \end{cases}$$

(c) *the full semilinear fractional group P_q of all transformations $x \longrightarrow (ax^{\alpha} + b)/(cx^{\alpha} + d)$ over some $GF(2^q)$ where q is a prime, $ad - bc \neq 0$, and α is an automorphism of $GF(2^q)$.*

Our proof relies heavily on the work of Feit and Suzuki as well as on some earlier results of Zassenhaus [6]. In Section 1 we list for the sake of clarity most of the known results which we shall need. Section 2 is devoted to an initial reduction of the theorem. In Sections 3 and 4, respectively, we then treat the cases that G_3 is of odd degree and even degree.

1. Summary of known results

If G_t is a t-fold transitive group of degree $n + t - 1$ on the letters P_1, $P_2, \cdots, P_{n-1}, Q_1, Q_2, \cdots, Q_t$, we shall denote by G_i the subgroup of G_t fixing $Q_{i+1}, Q_{i+2}, \cdots, Q_t$ for $i = 0, 1, 2, \cdots, t-1$. If no permutation

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[1] *Added in proof.* Recently Suzuki has completely classified doubly transitive groups of odd degree in which only the identity fixes three letters, while Ito has solved the corresponding problem for groups of even degree. Their classification shows that, apart from the known classical groups, the new simple groups of Suzuki are the only such doubly transitive groups. Thus our theorem will follow from known results, once it is verified that the simple groups of Suzuki do not have transitive extensions. It should be added, however, that the Ito-Suzuki proofs are not elementary.

except the identity fixes $t + 1$ letters, then G_1 is a transitive permutation group (on the remaining n letters) in which only the identity fixes 2 letters. By a theorem of Frobenius ([2], Theorem 16.8.8, p. 292) G_1 contains a transitive normal subgroup A of degree and order n such that $G_1 = G_0 A, G_0 \cap A = 1$, and G_0 induces a regular group of automorphisms of A. The above notation will be used throughout the paper (but, of course, only for $t = 2$ or 3).

We now list the main results which we shall need for our proof:

I (Witt [5]). *If G_t is t-fold transitive, there exist involutions σ_i in G_i for each $i \geqq 2$ such that*

(a) $(\sigma_i \sigma_{i+1})^3 \in G_0$,

(b) *σ_i normalizes G_j for $0 \leqq j \leqq i - 2$,*

(c) $G_i = G_{i-1}\{\sigma_i\}G_{i-1}, \quad i \geqq 2$.

In fact, σ_i can be chosen as any involution in G_i which interchanges Q_{i-1} and Q_i and fixes Q_j, for $j \neq i, i - 1$.

It will also be convenient to say that G_t is a *regular* t-fold transitive group if only the identity fixes $t + 1$ letters.

II (Zemmer [8]). *If $\sigma_i = u\sigma_i v$ where $u, v \in G_{i-1}$, then in fact $u, v \in G_{i-2}$.*

III (Zassenhaus [6]).

(a) *If G_3 is sharply triply transitive, then G_3 is one of the groups L_q, L_q'.*

(b) *If G_2 is a regular doubly transitive group in which G_0 has even order, then G_2 is a subgroup of index $\leqq 2$ in a sharply triply transitive group.*

IV (Feit [1]). *If G_2 is a regular doubly transitive group of degree $n + 1$, then G_2 has index $\leqq 2$ in a sharply 3-fold transitive group, or either*

(a) *G_2 contains a normal subgroup of order $n + 1$, in which case G_2 is either sharply doubly transitive or the full semilinear affine group over some $GF(2^q)$, or*

(b) *$n = p^e = o(A)$ for some prime p, and A is non-Abelian.*

V (Feit [1]). *If G_2 is a regular doubly transitive group of degree $n + 1$ in which G_0 is of odd order, no subgroup of order $n + 1$ is normal in G_2, and the normal subgroup generated by A is G_2, then G_0 is cyclic and $x^{\sigma_2} = x^{-1}$ for all x in G_0.*

VI (Suzuki [3]). *If G_2 is a regular doubly transitive group of odd degree, then G_2 is either simple or sharply doubly transitive.*

VII (Suzuki [4]). *A nonsolvable group G whose order is divisible by 3 and in which the centralizer of every involution is a 2-group contains a composition factor which is isomorphic to some $LF(2, q)$.*

2. Reduction of the theorem

LEMMA 2.1. *It suffices to prove the theorem under the following assumptions:*

(a) *$o(G_0) = m > 1$, where m is odd.*

(b) *A is a non-Abelian p-group of order $n = p^e$.*

(c) *If M is a doubly transitive subgroup of G_2 containing A and of degree $n + 1$, then $M \cap G_0 \neq 1$, and M does not contain a normal subgroup of order $n + 1$.*

(d) *If G_3 is of even degree, then G_2 and G_3 are both simple.*

Proof. Suppose first that G_2 is a subgroup of index $\leqq 2$ in a sharply triply transitive group. If G_2 is itself sharply triply transitive, then G_3 is sharply 4-fold transitive, so that by a theorem of Jordan (Hall [2], Theorem 5.8.1, p. 73) G_3 is either M_{11}, S_4, S_5, or A_6. On the other hand, if G_2 is of index 2 in a sharply triply transitive group, G_3 will be of index 2 in a sharply 4-fold transitive group,[2] and hence G_3 is A_5, which is in fact sharply triply transitive.

Hence the theorem holds in either of these cases, and in particular if G_0 is of even order or $G_0 = 1$ by III. Thus we may suppose that (a) holds.

If G_2 contains a normal subgroup of order $n + 1$, then G_2 is the full semilinear affine group over some $GF(2^q)$ by IV. It is easy to see that G_3, being a transitive extension of G_2, must then be P_q.

Hence we may suppose that G_2 itself does not contain a normal subgroup of order $n + 1$ and is not of index $\leqq 2$ in a sharply triply transitive group, so that by IV, $n = p^e = o(A)$ and A is non-Abelian. Thus (b) holds.

Now let M be a doubly transitive subgroup of G_2 of degree $n + 1$ containing A. If $M \cap G_0 = 1$, M is sharply doubly transitive and hence is a Frobenius group (Hall [2], Theorem 20.7.1, p. 382). Thus $M = AB$, where $B \lhd M$ and A is a regular group of automorphisms of B. Since A is a non-Abelian p-group, A must be a generalized quaternion group (Hall [2], Lemma 20.7.4, p. 390). But then every element of G_0 fixes the unique element of order 2 in A, contrary to the fact that G_0 acts regularly on A. Thus $M \cap G_0 \neq 1$. On the other hand, if M contains a normal subgroup of order $n + 1$, M is the full semilinear affine group over some $GF(2^q)$ since $M \cap G_0 \neq 1$. But then A is cyclic contrary to (b), which proves (c).

Suppose finally that G_3 is of even degree. Then G_2, being of odd degree, is either simple or sharply doubly transitive by VI. Since the theorem has already been proved if G_2 is sharply doubly transitive, we may assume that G_2 is simple. If G_3 contains a normal subgroup N, it follows that $G_2 \cap N = 1$ and hence that $o(N) = n + 2$. Since the elements of A fix exactly 2 letters, it follows easily that N must be an elementary Abelian 2-group and hence $o(N) = n + 2 = 2^b$. But since G_3 is of even degree, $n = o(A) = 2^e$, forcing $e = 1$. Thus G_3 is S_4.

Henceforth we shall assume that G_3 satisfies all the conditions of the lemma.

3. Odd degree

We define a maximal sequence of subgroups $M^{(i)}$ of G_2, $i = 0, 1, 2, \cdots, t$

[2] For then G_2 is a projective unimodular group of dimension 2, and the only such groups with transitive extension are those for which the corresponding full projective group has a transitive extension.

as follows: $M^{(0)} = G_2$, while $M^{(i)}$ is the smallest proper normal subgroup of $M^{(i-1)}$ which contains A.

LEMMA 3.1. *Each $M^{(i)}$ is doubly transitive of degree $n + 1$ and A is not normal in $M^{(i)}$. Furthermore, $G_0 \cap M^{(i)}$ is characteristic in G_0 for all i.*

Proof. We shall prove by induction that $M^{(i)}$ is doubly transitive of degree $n + 1$. Assume the statement for $M^{(i-1)}$. Then $M^{(i-1)}$ is a primitive group, and hence $M^{(i)}$, being normal in $M^{(i-1)}$, is at least transitive on $n + 1$ letters. But the subgroup of $M^{(i)}$ fixing a point contains A, which is transitive on the remaining n letters. Thus $M^{(i)}$ is doubly transitive of degree $n + 1$. Clearly A cannot be normal in $M^{(i)}$ since A fixes a letter.

Since G_0 is a regular group of automorphisms of odd order, its Sylow subgroups are all cyclic and $G_0 = RS$, where R and S are cyclic of relatively prime order and $R \lhd G_0$ (Zassenhaus [7], Satz 5). It is easy to see that G_0 contains at most one normal subgroup of any given order, which is necessarily of the same form as G_0. It follows that a subnormal series of G_0 is in fact a characteristic series. By definition of $M^{(i)}$ the subgroups $G_0 \cap M^{(i)}$ form a subnormal series of G_0, and hence $G_0 \cap M^{(i)}$ is characteristic in G_0, as asserted.

Now set $M = M^{(t)}$, $M_1 = M \cap G_1$ and $M_0 = M \cap G_0$. Since $M_1 \supset A$ and $G_1 = G_0 A$, we have $M_1 = M_0 A$.

LEMMA 3.2. *If $x \in M_0$, then $x^{\sigma_3} = x^{-1}$.*

Proof. Since M is doubly transitive on the same letters as G_2, M contains an involution σ_2' which interchanges Q_1 and Q_2. By I, we may assume that $\sigma_2 = \sigma_2'$ and hence that σ_2 is in M. Now by Lemma 2.1, $o(M_0) = m_0 > 1$, m_0 is odd, and M contains no normal subgroup of order $n + 1$. Furthermore by construction the normal closure of A in M is M itself. It follows therefore from V that M_0 is cyclic and $x^{\sigma_2} = x^{-1}$ for all x in M_0.

Now σ_3 normalizes G_0 and $(\sigma_2 \sigma_3)^3 \in G_0$ by I. Since M_0 is characteristic in G_0 by the preceding lemma, σ_3 normalizes M_0. Since G_0 has odd order, $\sigma_2 \sigma_3$ induces an automorphism of M_0 of odd order. But M_0 is cyclic, and consequently its automorphism group is Abelian, whence $\sigma_2 \sigma_3$ induces an automorphism of M_0 of order dividing 2. We conclude that σ_2 and σ_3 must induce the same automorphism of M_0, and hence that $x^{\sigma_3} = x^{-1}$ for all x in M_0, proving the lemma.

To prove the theorem we shall now show that e is even. Since σ_3 induces automorphisms of G_1 and G_0, and A is characteristic in G_1, σ_3 induces an automorphism of A.

If we consider any characteristic series of A whose factor groups are each elementary Abelian p-groups of type $(p, p, \cdots, p)$, M_0 and σ_3 will induce automorphisms on each factor group, and it will suffice to show that each of these factor groups has order p to an even power. Since our argument is the same for each factor, we shall prove the assertion only for the last term A_1 in the series, A_1 being a characteristic elementary Abelian subgroup of A of type $(p, p, \cdots, p)$.

The holomorph of M_0 and σ_3 is completely reducible on A_1 regarded as a vector space over $GF(p)$. Hence if H_i, $i = 1, 2, \cdots, r$, denote the minimal $\{M_0, \sigma_3\}$-invariant subgroups of A_1, A_1 is the direct product of the H_i, and each H_i is the direct product of either one or two minimal M_0-invariant subgroups, since σ_3 has order 2 and normalizes M_0. Each of these M_0-invariant subgroups has the same order, say p^f. If each H_i is the product of two minimal M_0-invariant subgroups, then clearly $o(A_1) = p^{2fr}$, and our assertion follows. Thus we may assume H_1, say, is also a minimal M_0-invariant subgroup.

If H_1 is identified with the additive group and M_0 with a subgroup of the multiplicative group of $GF(p^f)$, σ_3 will then induce an automorphism of $GF(p^f)$, which by the preceding lemma will be of order 2. But $GF(p^f)$ possesses such an automorphism only if f is even. We conclude that $o(A_1)$ is even, and hence that e is even.

Now $o(G_2) = mp^e(p^e + 1)$, where m and p are odd and e is even, and hence $o(G_2) = 2s$, where s is odd. But then by a theorem of Burnside (Hall [2], Theorem 14.3.1, p. 203), G_3 contains a normal subgroup N of order s. Since G_1 is of odd order, $G_1 \subset N$, and it follows that N is doubly transitive. But this is impossible since N is of odd order.

4. Even degree

In this case the theorem depends upon the following lemma:

LEMMA 4.1. *The centralizer of every involution in G_3 is a 2-group.*

Proof. Since $n + 2$ is even, an involution a of G_3 fixes either no letters or two letters. But if a fixes no letters, it is the product of $\frac{1}{2}(2^e + 2) = 2^{e-1} + 1$ transpositions. Since G_3 is simple, $e > 1$, and hence a is an odd permutation. But then the even permutations of G_3 form a normal subgroup of index 2, contrary to the simplicity of G_3. Thus a fixes two letters and consequently is conjugate to an element of G_1. Since all the involutions in G_1 lie in A, which in the present case is a 2-Sylow subgroup of G_1, we may assume without loss of generality that a is in A.

Suppose now that x is in the centralizer of a, so that $x^{-1}ax = a$. If $x \in G_1$, clearly x is in A. If $x \in G_2$, but $x \notin G_1$, then $x = u\sigma_2 v$, where $u, v \in G_1$, and consequently

$$v^{-1}\sigma_2 u^{-1}au\sigma_2 v = a,$$

or

$$\sigma_2 = vav^{-1}\sigma_2 u^{-1}au.$$

But then $vav^{-1} \in G_0$ by II. This is impossible since then $vav^{-1} \in G_0 \cap A = 1$.

Suppose finally that $x \notin G_2$ so that $x = u\sigma_3 v$ with u, v in G_2. As above we find that $vav^{-1} \in G_1$, and hence that $vav^{-1} = a_1 \in A$. If $v \notin G_1$, then $v = r\sigma_2 s$, where $r, s \in G_1$, whence $\sigma_2 = r^{-1}a_1 r\sigma_2 sas^{-1}$. Thus $r^{-1}a_1 r \in G_0$ which is again impossible. It follows that $v \in G_1$ and similarly that $u \in G_1$. Since σ_3 normalizes G_1, $x = w\sigma_3$, $w \in G_1$. But then $x^2 = w^{1+\sigma_3}$. Now $w = ga_2$, where $g \in G_0$ and $a_2 \in A$, and hence $x^2 = ga_2\, g^{\sigma_3}a^{\sigma_3} = gg^{\sigma_3}a'$, where $a' \in A$.

But now since G_2 is simple, the subgroup M constructed in Section 3 is G_2, and hence by Lemma 3.2, $gg^{\sigma_3} = 1$. Thus $x^2 = a'$ is in A, and consequently x is a 2-element, proving the lemma.

Since G_3 is triply transitive, $3 \mid o(G_3)$; also, G_3 is simple, so that by VII, G_3 is isomorphic as a group (*not* necessarily as a permutation group) to $LF(2, q)$ for some q.

If q is odd, the 2-Sylow subgroups of $LF(2, q)$ are dihedral. Hence if $G_3 = LF(2, q)$, the subgroup A, being of index 2 in a 2-Sylow subgroup of G_3, is either cyclic or dihedral. But if $o(A) \geqq 8$, the center of A contains a unique element of order 2 which is necessarily left fixed by G_0, contrary to the fact that G_0 acts regularly on A. Thus A must be dihedral of order 4, and $o(G) = 3\cdot 4\cdot 5\cdot 6$. Indeed, in this case $G_3 = A_6 = LF(2, 9)$.

On the other hand, a simple comparison of $o(G_3) = m\cdot 2^e(2^e + 1)(2^e + 2)$ with $e > 1$ and $o(LF(2, 2^b)) = (2^b - 1)2^b(2^b + 1)$ shows that G_3 and $LF(2, 2^b)$ never have the same order. This completes the proof of the theorem.

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