GEOMETRIC *ABA*-GROUPS

BY

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1. Introduction

A group of collineations of an incidence system will be called *acutely transitive* if it is transitive on the configurations consisting of an incident point and line. By an (*acutely transitive*) *representation* of a group G on an incidence system Σ will be meant a homomorphism of G onto an (acutely transitive) group of collineations of Σ.

A finite incidence system will be called a *2-design* if each point lies on the same number $h \geqq 2$ of lines, each line contains the same number $k \geqq 2$ of points, and each pair of points lies on exactly one line (of course, 2-designs are special balanced incomplete block designs). In this paper we characterize the finite groups admitting acutely transitive representations on 2-designs as the groups G containing subgroups A and B such that

(1) $G = ABA$,
(2) $AB \cap BA = A + B$, and
(3) $A \nsubseteq B$ and $B \nsubseteq A$.

Such a group we call a *geometric ABA-group*. Any doubly transitive group is a geometric ABA-group with $B{:}A \cap B = 2$. We observe that an acutely transitive group on a 2-design is necessarily primitive on the points, which means that in a geometric ABA-group, A is a maximal subgroup.

A finite group admits an acutely transitive representation θ on a finite projective plane π if and only if it is a geometric ABA-group satisfying

(3′) $A{:}A \cap B = B{:}A \cap B \geqq 3$,

in which case we call it a *projective ABA-group*. By using the Ostrom-Wagner theorem [7; Theorem 5] it is easy to see that then the additional condition

(4) $G = A + AxA$

is necessary and sufficient for π to be Desarguesian and $\theta(G)$ to contain the little projective group. As an application we show that a simple group satisfying Steinberg's axioms [10] with the symmetric group of degree 3 as Weyl group is necessarily a little projective group. We show that if a projective ABA-group has $A{:}A \cap B = n + 1$, where n is either an odd nonsquare, or $n = n_0^2$ with $n_0 \equiv -1 \pmod 4$, then the plane is Desarguesian, and $\theta(G)$ contains the little projective group.

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A necessary and sufficient condition that a group G be an *acutely regular* group of collineations of a 2-design, in the sense that it be acutely transitive and no element $\neq 1$ fix an incident point and line, is that G be a geometric ABA-group such that

(5) $A \cap B = 1.$

It is easy to see that this means precisely that G is an *independent ABA-group* as defined by Gorenstein [4]. Using Singer's theorem [8] we show that the only Desarguesian projective planes admitting acutely regular groups of collineations are the planes of orders 2 and 8. It follows that these are the only Desarguesian cyclic planes generated by *perfect residue difference sets* (cf. [6]). We prove moreover that an acutely transitive group on a Desarguesian projective plane, which is not acutely regular, contains the little projective group, thus determining all acutely transitive collineation groups of Desarguesian projective planes. Our proof depends on the above-mentioned fact that acutely transitive groups are primitive, and an application of Thompson's theorem [13].

2. Acutely transitive representations

An *incidence system* is a system consisting of two nonempty sets, the elements of one being called *points* and those of the other *lines*, together with a relation, called *incidence*, which may or may not hold between a given point and a given line. All incidence systems considered here will be assumed to satisfy the following two conditions:

(a) *each point is incident with at least one line,* and
(b) *each line is incident with at least one point.*

In the usual way we shall use such phrases as "P lies on L" and "L passes through P" to express the incidence of a point P and line L.

By a *collineation* of an incidence system Σ is meant a one-to-one mapping of the points onto the points and the lines onto the lines which preserves incidence. The collineations of Σ form a group, the *collineation group* of Σ. By a *representation* of a group G on an incidence system Σ we shall mean a homomorphism of G into the collineation group of Σ. Two such representations θ_1 and θ_2 of G on Σ_1 and Σ_2 will be called *equivalent* if there is an isomorphism σ of Σ_1 onto Σ_2 (this term being used in the obvious sense) which commutes with the action of G, i.e., which is such that the diagram

$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{\theta_1(g)} & \Sigma_1 \\ \sigma\downarrow & & \downarrow\sigma \\ \Sigma_2 & \xrightarrow{\theta_2(g)} & \Sigma_2 \end{array}$$

is commutative for each $g \in G$.

Given a group G and two subgroups A and B of G, an incidence system $\pi(G, A, B)$ can be constructed by taking as points the left cosets Ax of G modulo A and as lines the left cosets By of G modulo B. The point Ax and line By are to be taken as incident if the cosets Ax and By have an element in common. Replacing left cosets by right cosets produces an isomorphic incidence system.

There is a natural representation * of G on $\pi(G, A, B)$ which associates with each $g \in G$ the collineation g^* sending the point Ax onto the point Axg and the line By onto the line Byg. The kernel of this representation is the maximum normal subgroup of G contained in the intersection of A and B. If N is any normal subgroup of G contained in $A \cap B$, there is clearly a representation of G on $\pi(G/N, A/N, B/N)$ which is equivalent to *.

If a point Ax and line By are incident, there is an element g in $Ax \cap By$. Hence $Ax = Ag = Ag^*$ and $By = Bg = Bg^*$. This means that G^* is transitive on the configurations consisting of an incident point and line. Let us call a group of collineations of an incidence system Σ *acutely transitive* if it is transitive on the configurations consisting of an incident point and line. A representation θ of a group G on Σ will be called *acutely transitive* if $\theta(G)$ is acutely transitive. Using this terminology we have the first part of

LEMMA 1. *Given a group G with subgroups A and B, the natural representation of G on the incidence system $\pi(G, A, B)$ is acutely transitive. Conversely, any acutely transitive representation θ of a group G on an incidence system Σ is equivalent to the natural representation of G on $\pi(G, A, B)$, where A and B are the subgroups of G such that $\theta(A)$ and $\theta(B)$ fix respectively a point and a line incident with it.*

The second part is equally easy since acute transitivity implies transitivity on points and lines in view of the properties (a) and (b) assumed for incidence systems. Hence, by choosing a point P and a line L which are incident, and letting A and B be the subgroups of G such that $\theta(A)$ and $\theta(B)$ fix P and L respectively, an isomorphism of Σ onto $\pi(G, A, B)$ with the required property is obtained by mapping Px onto Ax and Ly onto By for each x, y in G.

In the rest of this section θ will denote an acutely transitive representation of a group G on an incidence system Σ, and A and B will denote the subgroups of G such that $\theta(A)$ and $\theta(B)$ fix respectively a point P and a line L through P. If X is a point or line of Σ and g is an element of G, we will write Xg for the image of X under the collineation $\theta(g)$. We observe now some simple correspondences between the structure of the group G relative to the subgroups A and B, and properties of the incidence system Σ.

Because of the choice of A and B, the lines through P are those of the form La with $a \in A$, and two such lines La and La' are equal if and only if $a \equiv a' \pmod{A \cap B}$. Similarly, the points on the line L are those of the form Pb with $b \in B$, and $Pb = Pb'$ if and only if $b \equiv b' \pmod{A \cap B}$. Since $\theta(G)$ is transitive on the points and lines of Σ, we therefore have

LEMMA 2. *Each point of Σ lies on $h = A:A \cap B$ lines, and each line of Σ carries $k = B:A \cap B$ points. The number v of points of Σ is given by $v = G:A$, and the number b of lines by $b = G:B$.*

Next we have

LEMMA 3. *Any two points of Σ lie on a line if and only if $G = ABA$.*

Proof. Assume that any two points lie on a line, and let x be an element of G. The points P and Px lie on a line M. Since P is on M, $M = La$, $a \in A$, and since Px is on M, $M = La'x$, $a' \in A$. Hence $a \equiv a'x \pmod{B}$, $x = a'^{-1}ba$, $b \in B$, and hence $G = ABA$.

Conversely, assume that $G = ABA$, and let Q be any point. Then $Q = Px$, $x \in G$, and $x = a'ba$, a', $a \in A$, $b \in B$. Hence $Q = Pa'ba = Pba$ lies on the line $Lba = La$, and so does P. That any two points lie on a line now follows from the fact that $\theta(G)$ is transitive on the points.

LEMMA 4. *Two distinct points of Σ lie on at most one line (i.e., Σ is a partial plane) if and only if $AB \cap BA = A + B$.*

Proof. First assume that Σ is a partial plane, and suppose that $a'b = b'a$ with a', a in A and b, b' in B. Write $x = a'b = b'a$. The line La contains the point P and also the point $Px = Pb'a$. Moreover, P and $Px = Pa'b = Pb$ are points of L. If $x \notin A$, $Px \neq P$, and hence, since Σ is a partial plane, $La = L$. Hence $a \in B$ and $x = b'a \in B$.

Conversely, assume that $AB \cap BA = A + B$. Let Q be a point $\neq P$ of L, and let M be a line through P. Then $Q = Pb$, $b \in B$, $b \notin A$, and $M = La$, $a \in A$. If M passes through Q, $M = La'b$, $a' \in A$, and hence there exists $b' \in B$ such that $b'a = a'b$. Since $b \notin A$, it follows that $a'b \notin A$, and hence by our assumption, that $b'a \in B$. Hence $a \in B$ and $M = La = L$. Thus L is the only line through P and $Q \neq P$ on L. It follows from the acute transitivity of $\theta(G)$ that no two points of Σ can have more than one line in common.

3. Representations on 2-designs

An incidence system in which the number of points and lines is finite will be called a 2-*design of type* (h, k) if

(1) each point lies on the same number $h \geqq 2$ of lines,
(2) each line contains the same number $k \geqq 2$ of points, and
(3) each pair of points lies on exactly one line.

If v is the number of points and b the number of lines, it is easy to see that

$$v = h(k - 1) + 1 \quad \text{and} \quad vh = bk.$$

Clearly $h \geqq k$. Moreover, $h = k$ if and only if the 2-design is a finite projective plane of order $n = h - 1$. (This is well known; see for example [5].)

A group possessing subgroups A and B such that

(i) $G = ABA$,
(ii) $AB \cap BA = A + B$, and
(iii) $A \not\subseteq B$ and $B \not\subseteq A$

will be called a *geometric ABA-group*. According to Lemmas 1 through 4 we have at once

PROPOSITION 1. *A finite group G admits an acutely transitive representation on a 2-design if and only if G is a geometric ABA-group.*

More precisely, if G is a geometric ABA-group, then the incidence system $\pi(G, A, B)$ is a 2-design, and the natural representation is acutely transitive.

Conversely, given an acutely transitive representation θ of G on a 2-design Σ, let A and B be the subgroups of G such that $\theta(A)$ and $\theta(B)$ fix an incident point and line respectively. Then the kernel of θ is the largest normal subgroup of G contained in $A \cap B$, G is a geometric ABA-group, and θ is equivalent to the natural representation of G on $\pi(G, A, B)$. The type of Σ is (h, k) where

$$h = A:A \cap B \quad \text{and} \quad k = B:A \cap B,$$

and the number v of points of Σ is $G:A = h(k-1)+1$. In order that $\theta(G)$ be doubly transitive on the points of Σ it is necessary and sufficient that $G = A + AxA$, $x \in G$.

Concerning the final statement of Proposition 1, the condition that G be doubly transitive on its cosets modulo A is well known to be equivalent to the condition that the number of double cosets of G with respect to A be 2.

Proposition 1 enables us to pass freely from group-theoretic to synthetic considerations in the study of geometric ABA-groups, and to state our results in either group-theoretic or geometric form.

To obtain a first indication of the extent of the class of geometric ABA-groups let us observe that a 2-design of type $(h, 2)$ may be considered simply as a set of $v \geqq 3$ points, the lines being all pairs of points. An acutely transitive group on such a 2-design is simply a doubly transitive group on the points. Thus by Proposition 1 we have

PROPOSITION 2. *A group G has a homomorphic image which is a doubly transitive group of degree $\geqq 3$ if and only if G is a geometric ABA-group with $B:A \cap B = 2$.*

Next we prove two propositions which are very useful in analyzing the structure of geometric ABA-groups. First, an acutely transitive group need not be doubly transitive, as we shall see, but it is easy to prove

PROPOSITION 3. *An acutely transitive group G of collineations of a 2-design Σ is primitive on the points of Σ.*

Proof. Assume that G is imprimitive, and consider a partition of the points of Σ into imprimitive classes. From the acute transitivity of G it follows that each line of Σ meets each imprimitive class of points in 0 or t points, where t is a fixed number $\neq 1$. If P is a point of an imprimitive class C of points, each of the h lines through P meets C in t points, and every point $\neq P$ of C lies on exactly one of these lines. Hence the number v_C of points in C is given by

$$v_C = h(t-1)+1.$$

The number v of points of Σ is equal to gv_C, where g is the number of imprimitive classes of points. Hence, since $v = h(k - 1) + 1$, we have

$$h(k - 1) + 1 = g[h(t - 1) + 1]$$

which implies that

$$h[(k - 1) - g(t - 1)] = g - 1.$$

Since $g \geqq 2$, this implies that

$$h \leqq g - 1 < g < k,$$

contrary to the fact that $h \geqq k$.

COROLLARY. *In a geometric ABA-group, A is a maximal subgroup.*

Note that an acutely transitive group need not be primitive on the lines. For example, the full collineation group of a Desarguesian affine plane is certainly acutely transitive, and the pencils of parallel lines constitute imprimitive classes of lines.

Secondly, we have

PROPOSITION 4. *Let G be an acutely transitive group of collineations of a 2-design Σ, and let H be a normal subgroup of G such that*

(a) *H contains an element $\neq 1$ fixing a point, and*

(b) *no element $\neq 1$ of H fixes an incident point and line.*

Then H is a Frobenius group whose Frobenius kernel M is an elementary abelian normal subgroup of G, transitive on the points of Σ.

Proof. Choose a point P and a line L which are incident, and denote by G_P and G_L respectively the subgroups of G fixing P and L. If $H \subseteq G_P$, then, since H is normal in G, it is contained in every conjugate of G_P, and therefore fixes every point. But $H \neq 1$; hence $H \nsubseteq G_P$.

By Proposition 3, G_P is a maximal subgroup of G, and hence $G = HG_P$; it follows that H is transitive on the points of Σ. An element $\neq 1$ of H fixing two points would fix an incident point and line, contrary to hypothesis. Hence H is a Frobenius group whose Frobenius kernel M is regular and transitive on the points of Σ. Moreover $H = M(H \cap G_P)$ and $1 = M \cap G_P$.

Since M is a characteristic subgroup of H, it is normal in G. Moreover, since G_P is maximal in G, $G = MG_P$. If N is a characteristic subgroup $\neq 1$ of M, N is normal in G and $G = NG_P$. Since $M \cap G_P = 1$, we must have $N = M$. By Thompson's theorem [13] M is nilpotent, and therefore M is elementary abelian.

COROLLARY. *Under the hypotheses of Proposition 4, the number v of points of Σ is a prime power.*

We shall see later (§6) that in case Σ is a Desarguesian projective plane of order n, we must have $n = 2$ or 8 in the situation of Proposition 4. However, there exist proper normal subgroups of the full collineation groups of many affine planes, which satisfy conditions (a) and (b) of Proposition 4, and which are not acutely transitive.

We include the following proposition which may be proved by only a slight modification of the method used by Gorenstein in proving Theorem 4 of [4].

PROPOSITION 5. *Under the hypotheses of Proposition* 4, *if* $G_P \cap H$ *has even order, then the order of* $G_L \cap H$ *is* 1 *or* 2.

Proof. First note that since $G_P \cap G_L \cap H = 1$, we have

(1) *If* $ab = b'a'$, a, a' *in* $G_P \cap H$, b, b' *in* $G_L \cap H$, *then* $a = a' = 1$ *or* $b = b' = 1$.

Now assume that the order of $G_P \cap H$ is even, and let a_0 be an element of order 2 in this subgroup. Then $x\sigma = a_0^{-1}xa_0$ for $x \in M$ defines an automorphism σ of order 2 of M, having no fixed elements $\neq 1$. By Proposition 4 (or directly in this case by a result of Burnside) M is abelian, and hence $x(x\sigma)$ is fixed by σ. Hence $x(x\sigma) = 1$, i.e., $xa_0^{-1}xa_0 = 1$ or

(2) $xa_0 = a_0 x^{-1}$ *for any* $x \in M$.

A consequence of (1) and (2) is that

(3) $G_L \cap M = 1$.

For, if $x \in G_L \cap M$, (2) gives $xa_0 = a_0 x^{-1}$, and hence, since $a_0 \neq 1$, (1) gives $x = 1$.

Since $H = (G_P \cap H)M$, each $b \neq 1$ in $G_L \cap H$ can be written as $b = ax$ with $a \in G_P \cap H$ and $x \in M$, where $a \neq 1$ by (3). The automorphism τ of M defined by $x\tau = a^{-1}xa$ for $x \in M$ has no fixed points $\neq 1$. Let d be the order of a, then

$$b^{d-1} = (ax)^{d-1} = a^{-1}(x[\tau^{d-2} + \tau^{d-3} + \cdots + 1]) = a^{-1}x',$$

where $x' \in M$ is such that $(x\tau^{-1})x'$ is fixed by τ^{-1}. Hence $(x\tau^{-1})x' = 1$, $x' = (x\tau^{-1})^{-1}$, and hence $b^{d-1} = a^{-1}(x\tau^{-1})^{-1}$, whence by (2),

$$b^{d-1}a_0 = a^{-1}a_0(x\tau^{-1}).$$

But $x\tau^{-1} = axa^{-1} = ba^{-1}$, so $b^{d-1}a_0 = a^{-1}a_0\, ba^{-1}$, and hence $b^{d-1}a_0\, a = a^{-1}a_0\, b$. Since $b \neq 1$, (2) implies that $a = a_0$. Hence each $b \neq 1$ in $G_L \cap H$ can be written in the form $b = a_0\, x$, $x \in M$. If $b_1 \in G_L \cap H$, $b_1 \neq 1$, $b_1 = a_0\, y$, $y \in M$, then $b^{-1}b_1 = x^{-1}y \in G_L \cap M$. Hence by (3), $b_1 = b$, and therefore $G_L \cap H$ has order at most 2.

It may be pointed out that, as the subgroup fixing a letter in the Frobenius group H, the subgroup $G_L \cap H$ has a completely known structure [12].

4. Acutely regular representations

A group of collineations of an incidence system Σ will be called *acutely regular* if it is acutely transitive and no element $\neq 1$ fixes an incident point and line. A representation θ of a group G on Σ will be called *acutely regular* if $\theta(G)$ is acutely regular. Let A and B be the subgroups of G such that $\theta(A)$ and $\theta(B)$ fix respectively a point and a line through it. Then clearly θ is acutely regular if and only if $\theta(A) \cap \theta(B) = 1$. Hence by Proposition 1, we have

PROPOSITION 6. *A finite group G admits an acutely regular representation on a 2-design if and only if it is a geometric ABA-group such that $A \cap B$ is a normal subgroup of G.*

It is easily seen that a geometric ABA-group satisfies the condition $A \cap B = 1$ if and only if it is an *independent ABA-group* in the sense of Gorenstein [4]. Hence, according to Proposition 6, the independent ABA-groups are precisely the groups isomorphic to acutely regular groups of collineations of 2-designs. Applying Propositions 2 through 5 we obtain the following results for an independent ABA-group G:

(1) *G is a Frobenius group with A as maximal subgroup and with Frobenius kernel M an elementary abelian group such that $G = AM$, $1 = A \cap M$. The order of G is $h(k - 1) + 1$, where $h = A{:}1$ and $k = B{:}1$.*

(2) *If the order of A is even, then the order of B is 2, and G is a doubly transitive group in which no element $\neq 1$ fixes two letters.*

The groups in (2) have been completely classified (cf. [5, Theorem 20.7.1]). These results coincide with results of Gorenstein [4].

5. Representations on projective planes

The rest of this paper is mainly concerned with acutely transitive groups on finite projective planes. The correspondence between planes admitting such groups and a certain class of geometric ABA-groups is given by

PROPOSITION 7. *The following conditions concerning a finite group G are equivalent:*

(1) *G admits a sharply transitive representation on a finite projective plane.*

(2) *G is a geometric ABA-group such that $A{:}A \cap B = B{:}A \cap B \geqq 3$.*

(3) *G possesses subgroups A and B such that*

(a) $G = ABA = BAB$,

(b) $AB \cap BA = A + B$, *and*

(c) $G{:}A \geqq 2$ *and* $A{:}A \cap B \geqq 3$.

Proof. The equivalence of (1) and (2) is an immediate consequence of Proposition 1 together with the fact that the 2-designs of type (h, h) with $h \geqq 3$ are precisely the projective planes of order $n = h - 1$.

Assume (1). Then (a) and (b) of (3) follow from Lemmas 3 and 4 and their duals, while (c) is a consequence of the existence of a quadrangle in a projective plane. Reversing the argument we obtain that (3) implies that

$\pi(G, A, B)$ is a projective plane and hence that (3) implies (1), completing the proof of the proposition.

It is worth noting that finiteness did not enter into the proof of the equivalence of (1) and (3).

A finite group satisfying the equivalent conditions (1) through (3) of Proposition 7 will be called a *projective ABA-group*. By Propositions 3 and 7 we have

PROPOSITION 8. *An acutely transitive group of collineations of a finite projective plane is primitive on the points and on the lines. Equivalently, in a projective ABA-group the subgroups A and B are maximal.*

According to the Ostrom-Wagner theorem [7; Theorem 5], if a group of collineations of a finite projective plane is doubly transitive on the points, then the plane is Desarguesian and the group contains the little projective group. Hence by Propositions 1 and 7 we have

PROPOSITION 9. *A finite group G admits a representation θ on a finite Desarguesian projective plane such that $\theta(G)$ contains the little projective group if and only if G is a geometric ABA-group such that $G = A + AxA$.*

An application of this result is given in §8.

It is natural to ask whether the condition $G = A + AxA$ is needed in Proposition 8. Equivalently we have the following two questions:

(a) Is a finite projective plane admitting an acutely transitive group of collineations necessarily Desarguesian?

(b) Does an acutely transitive group of collineations of a Desarguesian projective plane necessarily contain the little projective group?

In the rest of this section we answer (a) affirmatively for certain special odd orders, and show that (b) holds for any Desarguesian plane of odd order. Sections 6 and 7 are devoted to showing that there are precisely two exceptions to (b) in the case of even order Desarguesian planes. We begin by proving

LEMMA 5. *An acutely transitive group G of collineations of a projective plane π of odd order n has nonabelian 2-Sylow subgroups.*

Proof. Since n is odd and $n + 1$ divides the order of G, a 2-Sylow subgroup T of G is $\neq 1$. Suppose T is abelian, and let S be a subgroup of T maximal with respect to the property that the fixed elements of S form a subplane π_0 of π. Then $S \neq T$; otherwise $T \subseteq G_P \cap G_L$ for any incident point P and line L of π_0, contrary to the fact that $2 \mid n + 1$ and $n + 1 = G_P{:}G_P \cap G_L$. Hence there exists φ in T, $\varphi \notin S$, such that $\varphi^2 \in S$, and then φ induces a collineation φ^* of order 2 on π_0. If φ^* fixes a subplane of π_0 pointwise, so does $\langle S, \varphi\rangle$, contrary to the maximality of S. Hence, by a theorem of Baer (cf. [5; Theorem 20.9.7]), φ^* is a perspectivity on π_0. Let T_1 be a 2-Sylow subgroup of G containing $\langle S, \varphi\rangle$ and contained in G_Q, where Q is a point on the axis of φ^*. Since T_1 is abelian, each ψ in T_1 permutes the fixed points of φ and hence fixes the axis M of φ^*. Hence $T_1 \subseteq G_M \cap G_Q$, which is impossible.

Using this lemma we prove first

PROPOSITION 10. *Let π be a finite projective plane of odd order n such that either*

(i) *n is not a square, or*

(ii) *$n = n_0^2$ with $n_0 \equiv -1 \pmod 4$.*

If π admits an acutely transitive group G, then π is Desarguesian, and G contains the little projective group.

Proof. If n is not a square, the result follows from a theorem of Wagner [14; Theorem 3]. Assume that $n = n_0^2$ with $n_0 \equiv -1 \pmod 4$, and assume that π admits no perspectivities. Then, if φ is an element of order 4 in G, the above-mentioned theorem of Baer implies that the fixed elements of φ^2 constitute a subplane π_0 of order n_0. φ permutes the fixed points of φ^2, and hence permutes the $N_0 = (n^2 + n + 1) - (n_0^2 + n_0 + 1)$ points of π not in π_0. Since φ^2 fixes no point not in π_0, each transitive class under $\langle\varphi\rangle$ contains 4 points, and we must have $N_0 \equiv 0 \pmod 4$. But $n_0 \equiv -1 \pmod 4$ implies $N_0 \equiv 2 \pmod 4$, giving a contradiction. Hence G contains no element of order 4, contrary to Lemma 5.

As a second application of Lemma 5 we have

PROPOSITION 11. *An acutely transitive group G of collineations of a finite Desarguesian projective plane π of odd order necessarily contains the little projective group.*

Proof. Let T be a 2-Sylow subgroup of G, and let Λ be the full group of projectivities of π. Since π has odd order, $T \neq 1$. If T contains no homologies, then $T \cap \Lambda = 1$; otherwise an element of order 2 in $T \cap \Lambda$ would be a homology. Hence T is cyclic, contrary to Lemma 5. Therefore T contains homologies, and G contains the little projective group by a theorem of Wagner [14].

6. Acutely regular projective planes

In this section we wish to establish the following

PROPOSITION 12. *Let G be an acutely transitive group of collineations of a Desarguesian projective plane π of finite order n. If G contains a normal subgroup H such that H contains an element $\neq 1$ fixing a point, but no element $\neq 1$ in H fixes an incident point and line, then G is acutely regular and either*

(i) *$n = 2$ and G has order $3 \cdot 7$, or*

(ii) *$n = 8$ and G has order $9 \cdot 73$.*

For the proof we need some information about projectivities cyclic on the points of a Desarguesian projective plane π. According to a theorem of Singer [8], π always admits such a projectivity; one may be obtained as follows

[9]. The order n of π is a prime power, $n = p^r$. Let K be a field of n elements, and let L be an extension of K of degree 3. We may identify the points of π with the one-dimensional subspaces of L considered as a vector space over K, and the lines with the two-dimensional subspaces. Multiplication by a primitive root w in L effects a linear transformation W of L over K, and hence induces a projectivity ω of π; it can be shown that ω permutes the points of π cyclically. We shall refer to a projectivity ω of π obtained in this way as a *Singer cycle* on π.

We need to determine the normalizer of a Singer cycle ω on π. Let $\mathfrak{A}$ denote the automorphism group of the field L. Each $A \in \mathfrak{A}$ may be considered as a nonsingular semilinear transformation of L considered as a vector space over K, and hence induces a collineation α of π. If $wA = w^t$, then for $x \in L$,

$$x(A^{-1}WA) = [(xA^{-1})w]A = x(wA) = xw^t = xW^t.$$

Hence $A^{-1}WA = W^t$, A normalizes $\langle W\rangle$, and α normalizes $\langle\omega\rangle$. Thus if $\bar{\mathfrak{A}}$ denotes the image of $\mathfrak{A}$ in the group of collineations of π, the normalizer of $\langle\omega\rangle$ contains $\bar{\mathfrak{A}}$.

On the other hand, let τ be a collineation of π normalizing $\langle\omega\rangle$, and let T be a nonsingular semilinear transformation of L over K inducing τ. Since $\tau^{-1}\omega\tau = \omega^a$, $T^{-1}WT = \lambda W^a$ with $\lambda \neq 0$ in K. Write $\lambda = w^b$; then $T^{-1}WT = W^{a+b}$. Now $1T = w^c$ for some c; let $T^* = TW^{-c}$. Then $1T^* = 1$ and $T^{*-1}WT^* = W^{a+b}$. Hence $T^* = W^{-1}T^*W^{a+b}$, and

$$wT^* = 1T^*W^{a+b} = w^{a+b}.$$

It follows by induction that for any s,

$$w^sT^* = (w^{a+b})^s = (wT^*)^s,$$

and hence that T^* is a field automorphism of L. Denoting by τ^* the collineation of π induced by T^* we have $\tau^* = \tau\omega^{-c}$, so $\tau = \tau^*\omega^c$. We have proved that the normalizer of $\langle\omega\rangle$ is equal to $\bar{\mathfrak{A}}\langle\omega\rangle$. Certainly $\bar{\mathfrak{A}} \cap \langle\omega\rangle = 1$, for if $A \in \mathfrak{A}$ induces ω^t for some t, then there is a $\lambda \neq 0$ in K such that $xA = \lambda\cdot xw^t$ for all $x \in L$. Hence $1 = 1A = \lambda w^t$, and therefore $xA = x$ and $A = 1$. Moreover $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are isomorphic, for, if $A \in \mathfrak{A}$ induces the identity collineation of π, there is a $\mu \neq 0$ in K such that $xA = \mu x$ for all $x \in L$. Then $1 = 1A = \mu$, $xA = x$, and $A = 1$.

The essential conclusion from the above discussion for our purposes is

LEMMA 6. *Given a Singer cycle ω on a Desarguesian projective plane π of order $n = p^r$, p prime, the index of $\langle\omega\rangle$ in its normalizer in the full collineation group of π is $3r$.*

Now we can give a

Proof of Proposition 12. If n is odd, Proposition 10 implies that G contains the little projective group Π. We cannot have $\Pi \subseteq H$, since no element $\neq 1$ of H is to fix an incident point and line. Hence, since Π is simple, $H \cap \Pi = 1$, which implies that every element of H commutes with every element of Π. By assumption there is an $h \neq 1$ in H fixing a point P, and P must be the only fixed point of h. Hence every element of Π fixes P, which is impossible. Therefore n is even, $n = 2^r$.

By Proposition 4, G contains an elementary abelian normal subgroup M transitive on the points of π. The order of M is, on the one hand, equal to the number $n^2 + n + 1$ of points of π, and on the other hand, a prime power, q^α. Thus $n^2 + n + 1 = q^\alpha$, which implies that $\alpha = 1$ and $n^2 + n + 1 = q$ since n is a 2-power. It follows that q exactly divides the order of the full collineation group Γ of π, and hence that M is a q-Sylow subgroup of Γ. Hence M is conjugate to the subgroup $\langle\omega\rangle$ generated by a Singer cycle ω, and we may assume that $M = \langle\omega\rangle$ and that G normalizes $\langle\omega\rangle$. But $G:M = s(n + 1)$, where s is the order of the subgroup of G fixing an incident point and line. Hence by Lemma 6, $s(n + 1)$ divides $3r$, and we must have $s = 1, n = 2$ or 8, proving the proposition.

COROLLARY. *The only finite Desarguesian projective planes admitting acutely regular groups are those of orders* 2 *and* 8.

That the planes of orders 2 and 8 actually admit acutely regular groups is readily seen; their abstract structure is the following:

$$G = \langle x, \alpha\rangle, \qquad x^N = 1, \qquad \alpha^{-1}x\alpha = x^2,$$

where $N = 7$ if $n = 2$, and $N = 73$ if $n = 8$. These groups are independent projective ABA-groups, and are the only ones giving rise to Desarguesian projective planes.

According to §4, the order of A in an independent ABA-group must be odd, and hence a finite projective plane admitting an acutely regular group must have even order n. Moreover, the Frobenius kernel is elementary abelian, and transitive on the points, so the number $n^2 + n + 1$ of points must be a prime power, and a prime if n is a power of 2. The three smallest possible values of n not ruled out by the Bruck-Ryser theorem [5] are $n = 2, 8$, and 18. The planes of orders 2 and 8 admit acutely regular groups. Bruck showed in [2] that no plane of order 18 can exist which admits a regular transitive abelian group. Hence there is no plane of order 18 admitting an acutely regular group.

If a finite projective plane admits an acutely regular group with cyclic Frobenius kernel M, M must have prime order. It is easily seen that the planes of this type are precisely the cyclic planes generated by the *perfect residue difference sets* studied by E. Lehmer [6]. Hence the above corollary can be restated: *The only Desarguesian projective planes generated by perfect residue difference sets are those of orders* 2 *and* 8. The next 2-power value of n after 8 such that $n^2 + n + 1$ is a prime is $n = 512$, giving $n^2 + n + 1 = 262657$. A computation [6] showed that no perfect residue difference set exists for this value.

7. Acutely transitive groups on Desarguesian projective planes

The little projective group of a Desarguesian projective plane is doubly transitive and hence certainly acutely transitive. Therefore the following theorem completes the determination of the acutely transitive groups of collineations of finite Desarguesian projective planes.

THEOREM 1. *An acutely transitive group G of collineations of a finite Desarguesian projective plane π of order n contains the little projective group with precisely two exceptions, namely the acutely regular groups on the planes of orders* 2 *and* 8.

For the proof we need

LEMMA 7. *The order of a nontrivial finite permutation group in which no element $\neq 1$ fixes two letters has a factor $\neq 1$ in common with the degree of G.*

Proof. Let G be a permutation group of the m letters $1, 2, \cdots, m$, and let G_i be the subgroup of G fixing the letter i. Since no element $\neq 1$ of G fixes two letters, $G_i \cap G_j = 1$ for $i \neq j$. Assume that the order of G is prime to its degree m. If $g \in G$ does not belong to any G_i, then the cycles of g all have length > 1. If t is the length of the shortest cycle in g, then g^t fixes at least t letters; hence $g^t = 1$, and every cycle has length t. Hence t divides the order of G and the degree m of G, whence $t = 1$, a contradiction. Hence each $g \in G$ belongs to some G_i, and the subgroups $G_1, \cdots, G_m$ constitute a *partition* of G. Since every conjugate of a G_i is a G_j, and $G_i \cap G_j = 1$ for $i \neq j$, it follows that each G_i is its own normalizer, and hence that the partition is trivial by a result of Baer [1]. This means that $m = 1$, contrary to the assumption that G be nontrivial.

Proof of Theorem 1. According to Proposition 11, if the order n of π is odd, G contains the little projective group. We therefore assume that n is a 2-power, $n = 2^r$. Let $K = \Lambda \cap G_P \cap G_L$, where Λ is the group of projectivities of π, and G_P and G_L are the subgroups of G fixing respectively a point P and a line L through P. If $K = 1$, consider $H = G \cap \Lambda$; H is a normal subgroup of G and

$$H \cap G_P \cap G_L = \Lambda \cap G_P \cap G_L = 1.$$

If $H \subseteq G_P$, then H fixes every point, so $H = 1$ and $G{:}1$ divides r, which is clearly impossible. Hence $H \nsubseteq G_P$, and $G = G_P H$ since G_P is maximal. If $G_P \cap H = 1$, then $G{:}H = G_P{:}1$, a multiple of $n + 1$, while on the other hand, $G{:}H = G{:}(G \cap \Lambda) = G\Lambda{:}\Lambda$, a divisor of r, implying that $n + 1$ divides r, which is impossible. Therefore $H \cap G_P \neq 1$, and we infer from Proposition 12 that $n = 2$ or 8 and G is acutely regular.

Finally, assume that $K \neq 1$, and regard K as a permutation group on the n points $\neq P$ of L. If there is an element $\neq 1$ of K fixing every point of L, this element must be a perspectivity, and G contains the little projective group by [14; Theorem 3]. We assume therefore that K is faithful on the points $\neq P$ of L. If an element $\neq 1$ of K fixes two of these points, it must be the identity since $K \subseteq \Lambda$. Hence Lemma 7 implies the existence of an element of order 2 in K. By Baer's theorem this must be an elation, and hence G contains the little projective group.

8. Bruhat decompositions

In [10, 11], Steinberg showed that each of the simple groups obtained by Chevalley [3] and Steinberg [11] admits what we shall call a *Bruhat decomposition*. Precisely, we shall say that a group G admits a *Bruhat decomposition* with Weyl group $\mathfrak{W}$ if it possesses subgroups U, H, and W satisfying the following conditions (a) through (j).

(a) *H normalizes U.*

(b) *H is a normal subgroup of W.*

(c) *$W/H \cong \mathfrak{W}$ is a finite group.*

(d) *For each $w \in \mathfrak{W}$, U has subgroups U'_w* and *U''_w such that $U = U'_w U''_w$.*

For each $w \in \mathfrak{W}$, choose a representative $\omega(w)$ in W.

(e) *$\omega(w) U'_w \omega(w)^{-1} \subseteq U$.*

(f) *$G = \sum_{w \in \mathfrak{W}} HU\omega(w)U''_w$, and in the representation $g = hu\omega(w)u''$, $h \in H$, $u \in U$, $w \in \mathfrak{W}$ and $u'' \in U''_w$, each factor is unique.*

(g) *There is a distinguished set $\mathfrak{F}$ of elements of $\mathfrak{W}$ of period 2 which generates $\mathfrak{W}$.*

(h) *For $w \in \mathfrak{F}$, $HU + HU\omega(w)U''_w$ is a subgroup of G.*

(i) *For $w \in \mathfrak{F}$ and $s \in \mathfrak{W}$, $U''_w \not\subseteq U'_s$ implies $U''_w \subseteq U'_{sw}$.*

(j) *There is an $x \in U$ such that $x \in U'_w$ implies $w = 1$.*

The group $PSL_3(q)$, i.e., the little projective group of the Desarguesian projective plane of order q, admits a Bruhat decomposition with the symmetric group S_3 of degree 3 as Weyl group. This is the only finite simple group with this property, for we can prove

THEOREM 2. *A finite group G admitting a Bruhat decomposition with the symmetric group of degree 3 as Weyl group has a representation θ on a finite Desarguesian projective plane such that $\theta(G)$ contains the little projective group. Hence if G is simple, it is isomorphic with $PSL_3(q)$ for some q.*

Proof. First, writing $\omega = \omega(w)$, we have that

(1) *If $w \in \mathfrak{F}$ and x is an element $\neq 1$ of U''_w, then $\omega x \omega^{-1} \in HU\omega U''_w$.*

Otherwise, by (h) we have $\omega x \omega^{-1} \in HU$. Then $\omega x \omega^{-1} = hu$, $h \in H$, $u \in U$, so that $\omega x = hu\omega$, and using (f) we see that $x = 1$.

By (1), (d), and (e) we have

(2) *$u \in H$, $\omega u \omega^{-1} \in U$ implies $u \in U'_w$.*

Next

(3) *If w and s are distinct members of $\mathfrak{F}$, then $U''_w \subseteq U'_s$.*

If not, then by (i), $U''_w \subseteq U'_{sw}$. Choose $\omega = \omega(w)$ and $\sigma = \omega(s)$. Then $\omega(sw) = h\sigma\omega$ for some $h \in H$. Then $U''_w \subseteq U'_{sw}$ implies $h\sigma\omega U''_w \omega^{-1}\sigma^{-1}h^{-1} \subseteq U$, and hence $\omega U''_w \omega^{-1} \subseteq \sigma U \sigma^{-1}$ (since $\sigma^2 \in H$). Using (j) we can find $x \neq 1$ in U''_w. Applying (1) and (2) we get

$$h_1 u_1 \omega v''_w = u_2 \quad \text{or} \quad h_2 u_3 \sigma y''_s$$

with $h_i \in H$, $u_i \in U$, $v''_w \in U''_w$, and $y''_s \in U''_s$. In either case we contradict (f).

Now let w and s be distinct members of $\mathfrak{F}$. (Since $W = S_3$, there must be at least two members.) Choose $\omega = \omega(w)$ and $\sigma = \omega(s)$, and set

$$A = HU + HU\omega U''_w, \quad \text{and} \quad B = HU + HU\sigma U''_s.$$

These are subgroups of G by (h). We prove that G is a geometric ABA-group by establishing the conditions (3) of Proposition 7. First we must show that

(4) $G = ABA = BAB$.

Certainly ABA contains the sets HU, $HU\omega U''_w$, and $HU\sigma U''_s$. Hence

$$\begin{aligned} HU\omega\sigma U''_{ws} &= (HU\omega)(\sigma)(U''_{ws}) \subseteq ABA, \\ HU\sigma\omega U''_{sw} &= (HU)(\sigma)(\omega U''_{sw}) \subseteq ABA, \quad \text{and} \\ HU\omega\sigma\omega U''_{wsw} &= (HU\omega)(\sigma)(\omega U''_{wsw}) \subseteq ABA. \end{aligned}$$

Consequently $G = ABA$, and by symmetry, $G = BAB$.

Next we must show that

(5) $AB \cap BA = A + B$.

Suppose that $g \in AB \cap BA$. Since $g \in AB$, g is an element of HU, $HU\sigma U''_s$, $(HU\omega U''_w)(HU)$, or $(HU\omega U''_w)(HU\sigma U''_s)$. The first three subsets are all in $A + B$. The last is equal to $HU\omega U''_w \sigma U''_s$. We know from (3) that $\sigma U''_w \sigma^{-1} \subseteq U$, and hence $\sigma^{-1} U''_w \sigma \subseteq U$. Hence

$$HU\omega U''_w \sigma U''_s \subseteq HU\omega\sigma U = HU\omega\sigma U''_{ws}.$$

Similarly $g \in BA$ implies $g \in A + B$ unless $g \in HU\sigma\omega U''_{sw}$. Then using (f) we have $AB \cap BA = A + B$.

To complete the proof that G is a geometric ABA-group we observe that, since $G \neq A$, and since $U''_w \neq 1$ and $A \cap B = UH$, we have

(6) $G{:}A \geqq 2$ *and* $A{:}A \cap B \geqq 3$.

We now infer from Proposition 7 that G admits an acutely transitive representation θ on a finite projective plane. To show that the plane is Desarguesian and that $\theta(G)$ contains the little projective group, we must show that the condition of Proposition 9 is satisfied, namely, that

(7) $G = A + AxA$.

Since $A\sigma A \supseteq UH\sigma U \supseteq UH\sigma U''_s$, and since $G = ABA$, we have that $G = A + A\sigma A$. This completes the proof of Theorem 2.

It is worth noting that the finiteness comes in only in the application of the Ostrom-Wagner theorem. That is, we have proved that an arbitrary group admitting a Bruhat decomposition with S_3 as Weyl group has a homomorphic image which is a doubly transitive group of collineations of a projective plane. We have been unable to determine whether the plane is Desarguesian in the infinite case.

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