BANACH ALGEBRA AND SUMMABILITY

BY

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1. Several Banach algebras arise naturally in summability theory. These are algebras of infinite matrices $A = (a_{nk})$, $n, k = 1, 2, \cdots$. The study of these algebras may have two products: the generation of examples of Banach algebras with certain properties; and, on the other hand, applications to the theory of summability which has, since 1929, benefited greatly from the theory of Banach space and linear topological space in general. An example of such an application of Banach algebra is the observation that an element near the identity element has an inverse, which was applied to summability by Agnew and, for instance, was used extensively in [7].

2. Let $\| A \| = \sup_n \sum_k | a_{nk} |$. Then if $\| A \| < \infty$, A is an endomorphism (= continuous linear transformation) of m, the Banach space of all bounded complex sequences, and $\| A \|$ is its norm. However, much significance is attached to the way in which A transforms sequences not in m. Let Φ be the set of all matrices A with $\| A \| < \infty$. Then Φ is a Banach algebra; it is a closed proper subalgebra of the algebra of all endomorphisms of m. Let c, c_0 be the spaces of convergent and null sequences.

For each n, k, $| a_{nk} | \leqq \| A \|$ and so the entries are continuous linear functionals on Φ. Let T be the subset of Φ consisting of the triangular matrices, i.e. the matrices A such that $a_{nk} = 0$ if $k > n$. Since the entries are continuous, T is closed in Φ. It is clearly a subalgebra and *has its inverses*, i.e. $A \in \mathrm{T}$, $A^{-1} \in \Phi$ together imply $A^{-1} \in \mathrm{T}$.

Let Γ be the subset, clearly a subalgebra, of Φ consisting of the conservative matrices, i.e. matrices A such that $Ax \in c$ whenever $x \in c$. A matrix $A \in \Phi$ belongs to Γ if and only if $a_k = \lim_n a_{nk}$ exists for each k (in which case $\sum | a_k | < \infty$) and $\lim_n \sum_k a_{nk}$ exists. Since a_{nk} and $\sum_k a_{nk}$ are all linear functionals of unit norm, this shows that Γ is closed in Φ. We have also, [8], that Γ *contains its inverses*, i.e. $A \in \Gamma$, $A^{-1} \in \Phi$ *imply* $A^{-1} \in \Gamma$. It is in the proof of this result that we are forced (as far as we know) to consider the action of A on unbounded sequences.

For $A \in \Gamma$, let $\chi(A) = \lim_n \sum_k a_{nk} - \sum a_k$. Then χ is a multiplicative linear functional, [8]. If $\chi(A) = 0$, A is called co-null, otherwise co-regular. These terms were introduced in [6].

Let $\Delta = \Gamma \cap \mathrm{T}$, the set of conservative triangular matrices. Many of the classical summability methods are defined by matrices in Δ, for example the Cesàro, Nörlund and Hausdorff methods. We shall see, below, several fundamental structural differences between Δ and Γ, and individual differences

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between their members. For example, if $A \in \Delta$ and $A^{-1} \in \Delta$, then A must be *a triangle* (i.e. triangular and $a_{nn} \neq 0$ for each n). In this case A sums no divergent sequences, i.e. $Ax \notin c$ if $x \notin c$. For bounded x this is trivial since A and A^{-1} preserve c as operators on m; for unbounded x it is also true since $A^{-1}(Ax) = x$. The situation is quite different for Γ. If $A \in \Gamma$ and $A^{-1} \in \Gamma$, A sums no bounded divergent sequences, but may sum unbounded ones. Indeed A need not be 1–1 for sequences outside m; see [8], Theorem 7, for a regular matrix which is its own inverse but is not 1–1; also we may consider a "principal diagonal matrix"; see [7].

The crucial point is contained in this remark:

Remark. For $A, B \in \mathrm{T}$ *and any sequence* x, Bx *exists and* $A(Bx) = AB(x)$.

This remains true with Φ instead of T if x is assumed bounded, but not if x is not assumed bounded. Indeed *there exists* $A \in \Gamma$ *with* $A^{-1} \in \Gamma$, *and sequence* $x \neq 0$ *such that* $Ax = 0$, as was pointed out just preceding the remark.

3. Let us consider an algebra with identity I. Following [5] we define the radical to be the set of points A such that $I - BA$ has an inverse for every B. (In our case, $I - BA$ will occasionally have an inverse matrix which does not lie in the space considered, i.e. no inverse "exists".) We note the facts, given in [5]: $I - BA$ has an inverse if and only if $I - AB$ does. The radical is a two-sided ideal. If the algebra is a Banach algebra, the radical is closed. The algebra is called semisimple if the radical contains only 0. The spectrum of A is the set of scalars z such that $A - zI$ has no inverse. If A is in the radical, its spectrum contains only 0.

It is clear that *if f is a linear and multiplicative functional,* $f \neq 0$, *then for each A, $f(A)$ is in the spectrum of A* (since $f(A - IfA) = 0$ and $f^{\perp}$ is an ideal). Hence every element A in the radical has $f(A) = 0$. This, applied to Δ with $f = \chi$, proves that every matrix in the radical of Δ is co-null. Theorem 2, below, proves much more than this. We can also apply this remark to Δ or T with $f(A) = a_{nn}$, for a fixed n. This proves that every matrix in the radical of Δ or T has all its diagonal elements zero.

THEOREM 1. Φ *and* Γ *are semisimple.* T *and* Δ *are not semisimple.*

Given $A \in \Phi$, $A \neq 0$, let $x \in c$, $Ax \neq 0$. For example, if, for a certain n, k, we have $a_{nk} \neq 0$, we may take $x = \delta^k$, a sequence of zeros save for a one in the k^{th} place. Define $B \in \Gamma$ as follows: B consists entirely of zeros except for a single column, and $BAx = x$. Then $I - BA$ carries x into 0, hence is not 1–1 and has no inverse. This proves that A is not in the radical of Φ or Γ, and hence that these spaces are semisimple.

To prove the second part of the theorem is relatively easy. For example, a triangular matrix with a single nonzero element, and that one occurring below the main diagonal, is easily seen to be in the radical of T and Δ. However, we can give an exact description of these radicals.

THEOREM 2. *For both* T *and* Δ, *a matrix* A *is in the radical if and only if its diagonal elements are zero and* $\sum_k |a_{nk}|$ *is uniformly convergent. For* Δ *this last condition is equivalent to "A is coercive," i.e. Ax is convergent whenever x is bounded.*

The proof of the "if" part of this theorem was suggested by J. A. Schatz. The details of the proof were worked out by E. K. Dorff.

We proved just before the statement of Theorem 1, that a matrix in the radical of Δ or T has zero diagonal.

Next, we shall show that a radical matrix A must have the uniform convergence property mentioned.

Consider first the case in which A is conservative, i.e. $A \in \Delta$. Suppose that $\sum_k |a_{nk}|$ is not uniformly convergent. We first observe that A is not coercive since a coercive matrix must have, along with other properties, uniform convergence of the series mentioned. Thus there exists a bounded sequence x such that $y = Ax$ is divergent; y must be bounded since A has finite norm, hence y has two distinct limit points, say $y_{p(n)} \to \alpha$, $y_{q(n)} \to \beta$. We may assume $p(n) \uparrow \infty$, $q(n) \uparrow \infty$, $p(n) < q(n)$, and $|y_{p(n)} - y_{q(n)}| > \frac{1}{2}|\alpha - \beta|$ for each n. Let B be defined as follows: for $n < q(1)$ and all k, let $b_{nk} = 0$; for $q(r) \leqq n < q(r+1)$ let $b_{nk} = 0$ for $k \neq p(r)$, $k \neq q(r)$, while

$$b_{n,p(r)} = \frac{x_n - y_{q(r)}}{y_{p(r)} - y_{q(r)}}, \qquad b_{n,q(r)} = \frac{y_{p(r)} - x_n}{y_{p(r)} - y_{q(r)}}.$$

The matrix B belongs to Δ since $b_{nk} = 0$ for $k > n$; $\| B \| \leqq 8M/|\alpha - \beta|$, where M is an upper bound for all $|x_n|$, $|y_n|$; $\sum_k b_{nk} = 1$ for each $n \geqq q(1)$, and each column of B terminates in a string of zeros. (Thus, in fact, B is regular.) Also By is a sequence whose n^{th} term, for $n \geqq q(1)$ is x_n. Hence $(I - BA)x = x - By$ is a sequence terminating in zeros, a convergent sequence; call it z. Thus the matrix $I - BA$ is a conservative triangular matrix which transforms a divergent sequence, x, into a convergent one, z. Thus $I - BA$ cannot have an inverse, D, in Δ, since Dz would have to be convergent, while $Dz = x$. Hence A is not in the radical of Δ. (The above construction is inspired by [1].)

In the second case where A is not necessarily conservative, let us assume that A is in the radical and that $\sum_k |a_{nk}|$ is not uniformly convergent. We can find $\delta > 0$ and sequences $m(i) \uparrow \infty$, $n(i) \uparrow \infty$ such that $\sum_{k=m(i)}^{m(i+1)} |a_{n(i)k}| > \delta$ for $i = 1, 2, \cdots$. Let B be a diagonal matrix with $b_{nn} = 1$ for $n = n(1)$, $n(2), \cdots$, $b_{nn} = 0$ otherwise. Then BA, a row submatrix of A, has no row submatrix with the uniform convergence property. Similarly, form C such that CBA is a row submatrix of BA with the sequence of row sums convergent. Then form $DCBA$ with convergent first column, $EDCBA$ with convergent second column, and so on. Each of $B, C, D, E, \cdots$ is diagonal with zeros and ones on the diagonal. Let F be the matrix whose first row is the first row of A, whose second row is the second row of BA, whose third row is the third row of CBA, and so on; F is a submatrix of A, thus $F = GA$ with G

diagonal with zeros and ones on the diagonal; F is conservative but not coercive. As above we can find $H \in \Delta$ such that $I - HF$ has no inverse in Δ. Then it has no inverse in T for Γ contains its inverses in Φ and so Δ contains its inverses in T. Thus F is not in the radical of T. But $F = GA$, and this contradicts the assumption that A is in the radical and the fact that the radical is an ideal.

We now turn to the proof that if $\sum_k | a_{nk} |$ is uniformly convergent and if $a_{nn} = 0$ for each n, then A belongs to the radical of T or Δ, whichever is appropriate.

First let us notice that A is a limit point of right-finite matrices, i.e. matrices B with $b_{nk} = 0$ for $k > m$, m being some number depending only on B. (This is not to be confused with row-finite, in which m depends on n.) This is true since given $\varepsilon > 0$ we choose K such that $\sum_{k>K} | a_{nk} | < \varepsilon$ for all n. Let $b_{nk} = a_{nk}$ for $k \leqq K$, $b_{nk} = 0$ otherwise. Then B is right-finite and $\| B - A \| < \varepsilon$.

Since the radical is closed, we have only to prove that any right-finite matrix with zero diagonal is in the radical. Since the radical is closed under addition, it is sufficient to prove that a matrix with only one nonzero column and zero diagonal is in the radical. This we now do. Let m be a positive integer, and let $a_{nk} = 0$ for all n, k except when $k = m$; let $a_{nm} = 0$ for $n \leqq m$. Let B be an arbitrary matrix in T. (If $A \in \Delta$, take $B \in \Delta$.) Let $D = I - BA$. Clearly $d_{nn} = 1$ and D is triangular; hence D^{-1} exists. Our next step is to see that $\| D^{-1} \| < \infty$. Let x be an arbitrary bounded sequence, and $y = D^{-1}x$. Then $x = Dy = y - BAy$; hence

$$y_n = x_n + y_m \sum_{r=m+1}^{n} b_{nr} a_{rm} \quad \text{and} \quad | y_n | \leqq | x_n | + | y_m | \cdot \| B \| \cdot \| A \|.$$

Thus y is bounded, and D^{-1} is a matrix which preserves boundedness, i.e. $\| D^{-1} \| < \infty$. Hence $D^{-1} \in \mathrm{T}$ and so A is in the radical of T. In case $A \in \Delta$, then $D \in \Delta$ and $D^{-1} \in \mathrm{T}$; as mentioned in §2 then $D^{-1} \in \Delta$. Hence A is in the radical of Δ.

It is perhaps of interest to observe that *the matrices in the radical are totally continuous operators on* m since they are limits of operators of finite rank.

4. Contained in the proof of Theorem 2 is the following result:

LEMMA 4.1. *In* T, *if* A *has an inverse* (*in* T), *then so does* $A + B$ *for any right-finite* $B \in \mathrm{T}$ *such that* $A + B$ *is a triangle.*

Related to this is

LEMMA 4.2. *Let* $A \in \Delta$ *and* $| a_{nn} | = 1$ *for all* n, *and let* $\sum | a_k | > \| A \| - 2$. *Then* $A^{-1} \in \Delta$, *and* A *sums no divergent sequences. The result is best possible in that the condition* $| a_{nn} | = 1$ *cannot be dropped and the* 2 *cannot be increased.*

Proof. Let $\sum_{k=1}^{m-1} | a_k | > t > \| A \| - 2$; then for large n, $\sum_{k=1}^{m-1} | a_{nk} | > t$, and so $\sum_{k=m}^{n-1} | a_{nk} | = \sum_{k=1}^{n} | a_{nk} | - 1 - \sum_{k=1}^{m-1} | a_{nk} | < \| A \| - 1 - t < 1$, and A differs by a right-finite matrix from a matrix in the unit neighborhood of I. By Lemma 4.1, the result follows. To see that it cannot be improved in the way mentioned, consider first the Cesàro matrix. It satisfies $\sum | a_k | > \| A \| - 2$ but not $a_{nn} = 1$. Consider also the matrix A with $a_{nn} = a_{n,n-1} = 1$, $a_{nk} = 0$ otherwise. This matrix has $a_{nn} = 1$ and $\sum | a_k | = \| A \| - 2$.

The result can be improved by replacing $a_{nn} = 1$ by a condition of the type $\lim a_{nn} = 1$ with appropriate modifications.

An application of Lemma 4.2 is the known result that x is convergent if $\{x_n + \sum_{k=1}^{n-1} a_k x_k\}$ is convergent, $\sum | a_k | < \infty$; for the corresponding matrix A has $\sum | a_k | = \| A \| - 1$.

5. The set of co-null matrices is a two-sided ideal in, and a maximal linear subspace of Γ (since χ is linear and multiplicative) but is not, clearly, an ideal in Φ. Indeed there exists co-null A with right inverse in Φ (it could not have a left inverse in Φ); see [8]. This shows that χ cannot be extended to Φ so as to be linear and multiplicative.

We examine other subsets of Γ for the property of being an ideal. For any sequence x, let (x) be the set of conservative matrices which sum x, i.e. transform x into a convergent sequence. Clearly $(x) = \Gamma$ if and only if x is convergent, or equivalently $I \in (x)$ if and only if $(x) = \Gamma$.

LEMMA 5.1. *$(x) \subset (y)$ if and only if $(x) = (y)$ or $(y) = \Gamma$.*

This result is essentially due to Brudno, and t Erdös and Rosenbloom, who prove it for bounded x; see [4]. If x is unbounded, choose $A \in \Gamma$ such that A transforms into convergent sequences only sequences of the form $\alpha x + u$, where α is a scalar, and u is convergent. See [7] for existence of A. Since $A \in (y)$, it follows that $y = \alpha x + u$.

THEOREM 3. *In Γ, (x) is a right ideal if and only if x is convergent, a left ideal if and only if x is bounded. $(x) \cap \Delta$ is, in Δ, a right ideal if and only if x is convergent, but is always a left ideal.*

If x is bounded, we simply consider it as a point of m, and since we are dealing with endomorphisms of m, the result is trivial since $A(Bx) = (AB)x$ for any $A, B \in \Phi$. The same is true for $A, B \in \Delta$, whether x is bounded or not since A is row-finite. See the earlier remark.

Next we assume that x is unbounded and show that (x) is not a left ideal in Γ. There exists, [7], $A \in (x)$ such that A has a two-sided inverse, in fact $\| A - I \| < 1$.

Next we assume that x is divergent and show that (x) is not a right ideal. Buck [2, 3] has shown that if A is any regular matrix and x is divergent, there exists a subsequence of x not summable by A. Let B be the matrix such that Bx is the subsequence of x; then $AB \notin (x)$ and the result follows.

LEMMA 5.2. *Let x be such that (x) is a closed subset of Γ ($(x) \cap \Delta$ is a closed subset of Δ), then $f(A) = \lim Ax$ for $A \in (x)$ defines a continuous function f on (x) (on $(x) \cap \Delta$).*

Here $\lim Ax$ means the limit of the convergent sequence Ax. The result follows from the Banach-Steinhaus closure theorem and the facts

$$f(A) = \lim_n \lim_m \sum_{k=1}^{m} a_{nk} x_k, \qquad \left| \sum_{k=1}^{m} a_{nk} x_k \right| \leqq \max_{1 \leqq k \leqq m} | x_k | \cdot \| A \|.$$

THEOREM 4. *(x) is closed in Γ if and only if x is bounded. $(x) \cap \Delta$ is closed in Δ if and only if x is bounded.*

Let x be bounded. Let $u_n(A) = \sum_{k=1}^{\infty} a_{nk} x_k$. Then

$$| u_n(A) | \leqq (\sup | x_n |) \| A \|.$$

Thus $\{u_n\}$ is a uniformly bounded sequence of functionals, and so (x) is closed, being the set of all A such that $\lim_n u_n(A)$ exists.

For x unbounded, let r be any number, and n a subscript such that $| x_n | > r$. Let $A \in \Delta$ be a matrix with only one nonzero column, the n^{th}, which has n zeros and then ones. Then $\| A \| = 1$, $| \lim Ax | = | x_n | > r$. By Lemma 5.2, (x) is not closed. This completes the proof. We could have shown that (x) is not closed in Γ for unbounded x by noting that $I \in \overline{(x)}$; but this is false for Δ since $\| I - A \| < 1$, $A \in \Delta$, implies that A sums no divergent sequences. Since we can, in fact, find a regular matrix $A \in (x)$ with $\| I - A \|$ as small as we wish, this yields a similar result for the subset of Γ consisting of multiplicative matrices, i.e. such that $a_k = 0$ for all k. (They are called multiplicative since for $x \in c$, Ax converges to a constant multiple, viz. $\lim_n \sum_k a_{nk}$, of $\lim x$.) But this is not so for the subset of Δ consisting of multiplicative matrices; call this subset M.

THEOREM 4′. *$(x) \cap M$ is closed in M if and only if x is bounded.*

Let x be unbounded. As for the previous result we need merely show that *given any positive integer s, there exists $A \in (x) \cap M$ with $\| A \| = 1$, $\lim Ax = s$.* (This cannot be found in [1], although it contains many examples of this type.)

Let $\{x_{p_n}\}$ be a subsequence of x with $| x_{p_n} | > \max (s, n)$. For $n < p_1$ let $a_{nk} = \delta_{nk}$ (the Kronecker delta), for $p_r \leqq n < p_{r+1}$, let $a_{nk} = 0$ for $k \neq p_r$, $a_{nk} = s/x_{p_r}$ if $k = p_r$.

6. Agnew, [1], has given several results of this type: *if x and y are sequences, x divergent, then there exists a regular matrix A with $y = Ax$.* He remarks that A cannot be expected to be triangular. In fact, as we shall now see, the situation is worse with respect to triangular matrices, in that we cannot even expect to come close to y with Ax. This is clearly related to the semisimplicity of Φ and lack thereof of T; for if, given arbitrary x, y, we could find $A \in \mathrm{T}$ with $Ax = y$, then, given $B \in \mathrm{T}$, let $x = By$, choose A with $Ax = y$, and so $(I - AB)y = 0$, and B is not in the radical of T; hence T would be semisimple.

THEOREM 5. *Let x be a bounded divergent sequence, y a bounded sequence; then there exists $A \in \Delta$ such that $y - Ax$ has a finite number of nonzero terms, and hence is convergent. There exist unbounded sequences x, y such that $y - Ax$ is divergent for every $A \in \Delta$.*

The matrix mentioned in the first part of the theorem is the one called B, constructed in the proof of Theorem 2 so that By has x_n for its n^{th} term for n sufficiently large. For our present purpose we have interchanged x, y.

For the second part of the theorem, let B be the matrix given by $b_{n,n-1} = \varepsilon_n$, $n = 1, 2, \cdots$; $b_{nk} = 0$ for $k \neq n - 1$; where $\sum |\varepsilon_n| < \infty$. Let y be a sequence such that $x = By$ is unbounded. By Theorem 2, B is in the radical of Δ; hence, for any $A \in \Delta$, $I - AB$ has an inverse and so does not transform any divergent sequence into c. In particular $(I - AB)y \notin c$, and the theorem is proved.

Let s be the set of all sequences. Agnew's results, mentioned above, state that the map from Γ to s given by $A \to Ax$ is onto, x being a fixed unbounded sequence. Indeed it is onto from the subset of Γ consisting of the regular matrices. Theorem 5 says that it is onto s/c as a map of Δ if x is bounded and divergent, but need not be onto s/c if x is unbounded.

7. The maximal group G of a Banach algebra is the set of elements with inverses. The elements which are in the closure of the maximal group are of some general interest. Mercer's theorem states that the Cesàro matrix is such an element. The following example is instructive. Let A_s be the multiplicative matrix given by $a_{nn} = 1$, $a_{n,n-1} = s$, $a_{nk} = 0$ otherwise. Then for $0 \leqq s < 1$, $\| I - A_s \| = s < 1$; hence $A_s^{-1} \in \Delta$, and $A_s \in G$. For $s > 1$, let B be the matrix gotten from A by omitting the first row. Then $B \in \Gamma$, $\| I - s^{-1}B \| = 1/s < 1$; hence $B^{-1} \in \Gamma$, and, [7], Bx convergent implies that $x = u + \alpha v$, where $u \in c$, α is a scalar, and v is a fixed unbounded sequence. (Here we also have used the fact that the set of sequences v such that $Bv = 0$ is one-dimensional.) But Bx is convergent if and only if Ax is. Hence for $0 \leqq s < 1$ and $s > 1$, A_s sums no bounded divergent sequences; and in the first case, $0 \leqq s < 1$, A_s belongs to the maximal group.

The dividing line $s = 1$ yields the matrix A_1 which sums the bounded divergent sequence $\{(-1)^n\}$. We conjecture that this behaviour is general: i.e., let A be a matrix in Δ with $a_{nn} \neq 0$ and with A not in the maximal group (i.e. A sums divergent sequences); then (we conjecture) A *sums bounded divergent sequences if and only if it is in the closure of the maximal group.* Some writers have classed as pathological those matrices not summing any bounded divergent sequences. The conjecture would relegate the nonpathological matrices to a thin veneer on the maximal group. As an indication of the possible difficulty of the problem we note the proof of Mercer's theorem.

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