METABELIAN p-GROUPS WITH FIVE GENERATORS AND ORDERS p^{12} AND p^{11}

In commemoration of G. A. Miller

BY

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1. Introduction

This paper continues the study of metabelian groups with elements of order p which are generated by five elements, and which are not direct products of abelian groups and metabelian groups with fewer generators. The problem is stated precisely and the method of investigation is explained in an earlier paper.[1] In that paper the existence and the distinctness of eighty-five such groups of orders from p^{15} to p^{11} were established. This paper will establish the completeness[2] of the list for these orders.

The considerations will all be geometric; nevertheless this is a paper about groups. The groups motivate the study of the complicated considerations required to determine invariants and to show in each case that a given set of invariants is sufficient to characterize a space. We shall be interested in planes and three-spaces in the finite nine-dimensional projective space S which is determined by the Plücker coordinates of the lines of a projective four-space X over GF(p). We classify planes and three-spaces of S under collineations of X.

2. Geometric formulation

We state the problem in geometric terms; the reader is referred to the earlier paper for consideration of the bearing of this study, and also for any proofs required for statements in this section.

Denote the five elements which generate G, any one of these groups, by $U_1, U_2, \cdots, U_5$. Designate commutators of pairs of U's as follows:

$$\begin{array}{lll}
U_2^{-1}U_1U_2 = U_1s_1, & U_3^{-1}U_2U_3 = U_2s_5, & U_4^{-1}U_3U_4 = U_3s_8, \\
U_3^{-1}U_1U_3 = U_1s_2, & U_4^{-1}U_2U_4 = U_2s_6, & U_5^{-1}U_3U_5 = U_3s_9, \\
U_4^{-1}U_1U_4 = U_1s_3, & U_5^{-1}U_2U_5 = U_2s_7, & \\
U_5^{-1}U_1U_5 = U_1s_4, & & U_5^{-1}U_4U_5 = U_4s_{10}.
\end{array}$$

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[1] *Finite metabelian groups and the lines of a projective four-space*, Amer. J. Math., vol. 73 (1951), pp. 539–555.

[2] Strictly, the paper establishes the completeness of a corrected list. Four groups, those connected with spaces of 9′, 20′, 20″, and 21′, were overlooked in the earlier paper. Spaces 20′ and 21′ were first noted by Dr. W. E. Koss and Mr. Peter Yff respectively.

If the s_i's are all independent, the group is of order p^{15}; all other groups satisfying the given conditions are quotient groups of this with respect to subgroups of the central $C = \{s_1, s_2, \cdots, s_{10}\}$.

Any element of G is $cU_1^{x_1}U_2^{x_2} \cdots U_5^{x_5}$, where c is an element of C and $x_1, x_2, \cdots, x_5$ are numbers in GF(p). To this element we let correspond the point $x_1, x_2, \cdots, x_5$ in a finite projective space X of four dimensions. A second element $c'U_1^{y_1}U_2^{y_2} \cdots U_5^{y_5}$ of G determines a second point $y_1, y_2, \cdots, y_5$ of X. The commutator of these two elements is $s_1^{a_1}s_2^{a_2} \cdots s_{10}^{a_{10}}$ where $a_1, a_2, \cdots, a_{10}$ are the Plücker line-coordinates of the line xy in X. These numbers can be used as the coordinates of a point in projective nine-space S over GF(p). Every point of S determines a cyclic subgroup of C, the central and the commutator subgroup of G of order p^{15}.

The points of S which correspond to commutators, or which correspond to lines of X, are points of the V_6^5 defined by $B_1 = B_2 = \cdots = B_5 = 0$, where

$$\begin{aligned}
B_1 &= a_1a_8 - a_2a_6 + a_3a_5 ,\\
B_2 &= a_1a_9 - a_2a_7 + a_4a_5 ,\\
B_3 &= a_1a_{10} - a_3a_7 + a_4a_6 ,\\
B_4 &= a_2a_{10} - a_3a_9 + a_4a_8 ,\\
B_5 &= a_5a_{10} - a_6a_9 + a_7a_8 .
\end{aligned}$$

We shall designate this locus by V.

Every group satisfying the given conditions will be obtained by setting certain elements of the commutator subgroup of the biggest group equal to identity. Elements dependent on those set equal to identity will constitute a subgroup of C and will correspond to a linear space in S. Different subgroups of the same order will correspond to subspaces of the same dimension; if these subspaces of S have different relations to V, then the corresponding quotient groups of G will be groups that are not simply isomorphic. We are to see that there are just 22 types of plane and 58 types of three-space in S; points and lines were discussed completely in the earlier paper.

We list some facts that will be needed in all that follows.

(1) The lines of a pencil in X determine the points of a ruling of V.

(2) A point P of S not on V is on a line joining two points of V; a choice of coordinate system in X will put P in the form 1, 0, 0, 0, 0, 0, 0, 1, 0, 0.

(3) Two points of V on a line with P not on V are images on V of two skew lines in X; these lines determine a three-space R in X; R depends on P only, and not on the points of V which were used to define it.

(4) The equation of R is $B_5x_1 - B_4x_2 + B_3x_3 - B_2x_4 + B_1x_5 = 0$, where the B's are those for the point P which determines R.

(5) We denote by Σ the five-space in S determined by the lines of a three-space in X; a point P in S is in one and only one Σ unless P is on V. Lines, planes, etc. in a Σ are called Σ-lines, Σ-planes, etc.

(6) A line in S not a Σ-line has one or no points on V; respective canonical forms are $k, 0, 0, l, 0, 0, 0, k, 0, 0$ and $k, l, 0, 0, 0, 0, l, k, 0, 0$.

(7) The line $k, l, 0, 0, 0, 0, l, k, 0, 0$ determines a unique point M on V such that the plane determined by M and the line is tangent to V at M. The six-dimensional space tangent to V at M contains planes, three-spaces, etc., which we shall call τ-planes, τ-three-spaces, etc.

(8) So much use will be made of the close connection between the canonical form $k, l, 0, 0, 0, 0, l, k, 0, 0$ in S and the frame of reference in X that we shall describe it briefly here. Let l be a line in S which is not a Σ-line and has no point on V. Let P_1 and P_2 be arbitrary points on l; let R_1 and R_2 be the corresponding three-spaces in X; let the plane of intersection of R_1 and R_2 be σ; let the images on V of the lines of σ be the points of the plane π; and let Σ_1 and Σ_2 be the five-spaces in S which contain P_1 and P_2 respectively. The polar of P_1 with respect to the intersection of V and Σ_1 intersects π in a line l_1; likewise P_2 determines a line l_2 in π. Lines l_1 and l_2 intersect in the point M. A line joining P_1 to a point Q_1 on l_2 and not M intersects V in a second point Q_1'; a line joining P_2 to a point Q_2 on l_1 and not M intersects V in Q_2'. The points M, Q_1, Q_2, Q_1', Q_2' are images on V of lines m, q_1, q_2, q_1', q_2' in X; these lines have the following relations: m, q_1, and q_2 are in the plane σ, and the intersection of q_1 and q_2 may be taken to be $A_1 = 1, 0, 0, 0, 0$; m and q_1 intersect at $A_2 = 0, 1, 0, 0, 0$; m and q_2 intersect at $A_3 = 0, 0, 1, 0, 0$; q_1' passes through A_3 and contains $A_4 = 0, 0, 0, 1, 0$; q_2' passes through A_2 and contains $A_5 = 0, 0, 0, 0, 1$. With this choice of a coordinate system in X and the corresponding determination of the coordinate system in S, the line takes the canonical form above. This rapid description shows the great arbitrariness in choosing a coordinate system to give a line the canonical form. By taking advantage of this arbitrariness we get a start in classifying planes.

3. The planes of S

(i) *Σ-planes in S.* There are Σ-planes in S; each such plane lies in the Σ determined by the lines of a three-space in X. In dealing with them we may neglect X and consider only the three-space. These planes were all determined in a previous paper.[3] The Σ-planes are

1. $k, l, 0, 0, m, 0, 0, 0, 0, 0$, the image of a plane of lines in X.
2. $k, l, m, 0, 0, 0, 0, 0, 0, 0$, the image of a bundle of lines in X.

[3] *Finite metabelian groups and Plücker line-coördinates*, Amer. J. Math., vol. 62 (1940), pp. 365–379.

3. $k, l, 0, 0, 0, 0, 0, m, 0, 0,$ which intersects V in two lines.
4. $k, l, m, 0, 0, 0, 0, k, 0, 0,$ which intersects V in one line.
5. $k, l, 0, 0, 0, m, 0, k, 0, 0,$ which intersects V in a conic.
6. $k, l, m, 0, 0, rl, 0, k, 0, 0$ (r not a square), which intersects V in a point.

The intersection of V and Σ is a four-dimensional hyperquadric. Any plane in Σ then intersects V in a conic or else lies wholly on V. The latter possibilities are 1 and 2. If the conic is not degenerate, the plane is 5; if the conic is degenerate with one vertex, it is 3 if the quadratic polynomial is factorable in GF(p), otherwise it is 6; if the conic has a line of vertices, the plane is 4. The proofs that planes having the properties listed can be put in the forms given are not attempted here; they are given, however, in the paper cited, and they are not hard to supply.

(ii) *A preliminary classification of planes not in any Σ.* A plane ρ which is not in any Σ contains points not on V, for a plane lying on V is determined by three points of V which are images of three lines in X that intersect in pairs, and three such lines either lie in a plane or pass through a point, in either of which events they lie in a three-space. Let ρ contain the point P which is not on V. P determines a five-space Σ, and ρ does not lie in Σ. If ρ contained as many as four points of V no three of which were collinear, then ρ would be a Σ-plane. One of the vertices of the diagonal triangle of the quadrangle determined by the four points would be not on V and so could be taken for P above. The three-space determined in X by P would contain the lines of which the four points of V are images, and so the corresponding Σ would contain ρ. Therefore any plane of S which is not a Σ-plane intersects V in 0, 1, 2, 3 points, in a line, or in a line and one additional point.

Unless ρ is a Σ-plane, it cannot contain two Σ-lines which intersect in a point not on V. Hence every ρ which is not a Σ-plane contains a line l which is not a Σ-line. If ρ intersects V in a line, then every l has a point on V; if ρ does not contain a line of V, then ρ contains an l which has no point on V. Hence, any plane ρ which is not a Σ-plane contains one or the other of the lines given in (6) of Section 2.

(iii) *Some transformations of S which leave a line fixed.* The planes ρ of S which contain 0, 1, 2, or 3 points of V all contain the line $k, l, 0, 0, 0, 0, l, k, 0, 0$ which we shall call P_1P_2 with P_1 given by $l = 0$ and P_2 by $k = 0$. Each such ρ is given by one additional point whose coordinates may be modified by using some of the freedom noted when we discussed the canonical form of P_1P_2. We give here three transformations of S into itself which leave the form of P_1P_2 unchanged. We employ the notation of Section 2.

For the first transformation we move A_4 along q_1' and A_5 along q_2', leaving $P_1, P_2, Q_1, Q_2, Q_1', Q_2'$ fixed. Denote this transformation by T_1. The effect of T_1 in X is described by the matrix of coefficients

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & b & 0 & 0 & 1 \end{bmatrix}$$

in expressions of the new coordinates in terms of the old. The matrix which follows is the description of T_1 in S by means of the matrix of coefficients in the expressions of the old coordinates in terms of the new; its elements are the properly ordered two-rowed minors of the inverse of the matrix above.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -ab & b & 0 & 0 & -a & 1 \end{bmatrix}.$$

The second transformation T_2 represents the changes in the coordinate systems brought about by moving Q_1 and Q_2 along the lines l_2 and l_1 respectively, still leaving P_1 and P_2 fixed. The points A_4 and A_5 are not determined by the Q's, but a combination of T_1 and T_2 will do all that can be done in that respect. The following transformation moves Q_1 to $\bar{Q}_1 = Q_1 + kM$ and Q_2 to $\bar{Q}_2 = Q_2 + lM$.

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & -k & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -l & 0 & 0 & 0 & 0 & 0 \\ -k & 0 & 1 & 0 & k^2 & -l & 0 & k & 0 & 0 \\ 0 & l & 0 & 1 & -l^2 & 0 & -l & 0 & k & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -kl & 0 & -k & -l & 0 & 1 \end{bmatrix}.$$

For the third transformation T_3 we let the point $aP_1 + P_2$ play the role of P_2 and determine a coordinate system so that P_1 and the new P_2 have coordinates in canonical form. There is arbitrariness in the choice of Q_1 and Q_2 as well as in the choice of A_4 and A_5. We shall carry out the selections which determine the matrices of T_3 in X and in S.

$P_1 = 1, 0, 0, 0, 0, 0, 0, 1, 0, 0.$ $\qquad$ $P_2 = a, 1, 0, 0, 0, 0, 1, a, 0, 0.$

$R_1: \quad x_5 = 0.$ $\qquad$ $R_2: \quad ax_1 + x_4 + a^2x_5 = 0.$

$$\sigma = \begin{cases} 0, 1, 0, 0, 0 \\ 0, 0, 1, 0, 0 \\ -1, 0, 0, a, 0. \end{cases} \qquad \pi = \begin{cases} 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 \\ 1, 0, 0, 0, 0, a, 0, 0, 0, 0. \\ 0, 1, 0, 0, 0, 0, 0, a, 0, 0 \end{cases}$$

$$l_1 = \begin{cases} 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 \\ a, -1, 0, 0, 0, a^2, 0, -a, 0, 0. \end{cases} \qquad l_2 = \begin{cases} 0, 0, 0, 0, 1, 0, 0, 0, 0, 0 \\ 1, 0, 0, 0, 0, a, 0, 0, 0, 0. \end{cases}$$

$M = 0, 0, 0, 0, 1, 0, 0, 0, 0, 0.$

$Q_1 = 1, 0, 0, 0, 0, a, 0, 0, 0, 0.$ $\qquad$ $Q_1' = 0, 0, 0, 0, 0, -a, 0, 1, 0, 0.$

$Q_2 = a, -1, 0, 0, 0, a^2, 0, -a, 0, 0.$ $\qquad$ $Q_2' = -2a, 0, 0, 0, 0, -a^2, -1, 0, 0, 0.$

$$q_1 = \begin{cases} 0, 1, 0, 0, 0, & A_2 \\ -1, 0, 0, a, 0, & A_1 \end{cases} \qquad q_2 = \begin{cases} -1, 0, 0, a, 0 \\ 0, -a, 1, 0, 0. \end{cases}$$

$$q_1' = \begin{cases} 0, -a, 1, 0, 0, & A_3 \\ 0, 0, 0, 1, 0, & A_4 \end{cases} \qquad q_2' = \begin{cases} 0, 1, 0, 0, 0 \\ 2a, 0, 0, -a^2, -1, & A_5 . \end{cases}$$

The A's at the right just above designate the points selected for the vertices of the new frame of reference in X. The matrix of T_3 in X is the set of A's in their proper order. The matrix in S is

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & -a^2 & 0 & -a & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a^2 & 1 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2a & 0 & 0 & 0 & 0 & a^2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 1 & 0 & 0 \\ -2a^2 & -2a & 0 & 0 & 0 & a^3 & a & a^2 & 1 & 0 \\ 0 & 0 & -2a & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(iv) *Planes with no point on V.* Among planes which contain the line P_1P_2 of (iii) are those with no point on V. These planes are

7. $k, l, 0, 0, 0, m, l, k + rm, m, 0.$
8. $k, l, m, 0, -rm, 0, l, k, 0, 0.$
9. $k, l, 0, 0, m, 0, l, k, 0, m.$

We proceed to show that if ρ is not a Σ-plane and has no point on V it can be put in one of these forms. The plane is determined by P_1, P_2, and a third point which may be taken to be

$$P_3 = 0, 0, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}.$$

We consider first the possibility that ρ is a τ-plane. The line P_1P_2 is in the space tangent to V at $M = 0, 0, 0, 0, 1, 0, 0, 0, 0, 0$. If P_3 is in that tangent space, $a_3 = a_4 = a_{10} = 0$. Any point in ρ is $P = kP_1 + lP_2 + mP_3$. Conditions that P be on V are

$$k^2 + a_8km - a_6lm = 0, \qquad a_9km - l^2 - a_7lm = 0,$$

$$(a_7a_8 - a_6a_9)m^2 + kl + a_7km + a_8lm = 0.$$

Eliminating m between the last two congruences, we get

$$a_9k^3 - a_7k^2l + a_8kl^2 - a_6l^3 = 0.$$

Since there exist irreducible cubic congruences, it follows that there exist τ-planes with no point on V. We note that the above conditions are independent of a_5. Moreover, neither a_6 nor a_9 is zero, and so a or b in T_1 can be selected so that P_3' has $a_5' = 0$.

Now let ρ be the plane determined by P_1, P_2, and

$$P_3 = 0, 0, 0, 0, 0, a_6, a_7, a_8, a_9, 0$$

where the cubic $a_9\theta^3 - a_7\theta^2 + a_8\theta - a_6$ is irreducible. If we apply transformation T_3 with $a = 1$, the point P_3 goes into

$$P_3' = -2a_7 - 2a_9, -2a_9, 0, 0, 0, a_6 + a_7 + a_8 + a_9, a_7 + a_9, a_8 + a_9, a_9, 0.$$

The point in ρ whose first two (new) coordinates are zeros has for its nonzero coordinates a_6', a_7', a_8', a_9' which are the coefficients of the transform of the irreducible cubic by $\theta = \theta' - 1$. The interchange of P_1 and P_2 performs the same transformation on the cubic as does $\theta = 1/\theta'$; the transformation in X which leaves the vertices of the frame of reference fixed and changes the unit point to $d, 1, d, 1, d^2$ performs the transformation $\theta = d\theta'$ on the cubic. These transformations generate the linear fractional group on θ, and under this group all irreducible cubics are conjugate. Hence, in any τ-plane which has no point on V, points can be selected so that P_1 and P_2 are in canonical form and $P_3 = 0, 0, 0, 0, 0, 1, 0, r, 1, 0$, where $x^3 + rx - 1$ is an arbitrary irreducible polynomial. This is plane 7.

For any other plane on P_1P_2 the tangent space at M cannot contain P_3, and hence not all of a_3, a_4, and a_{10} are zero. We note that transformations T_1, T_2, and T_3 all leave a_4 unchanged, and that T_1 and T_2 leave a_3 and a_{10} unchanged also. We separate the planes into two classes: (1) those determined by P_3 with $a_4 = 0$, and (2) those determined by P_3 with $a_4 \neq 0$.

(1) Suppose $a_4 = 0$ and $a_{10} \neq 0$. We may apply T_3 with $a_3 - 2a_{10}a = 0$ and obtain $a_3' = 0$. Since ρ contains P_1 and P_2, it contains a point $P_3 = 0, 0, 0, 0, a_5, a_6, a_7, a_8, a_9, a_{10}$.[4] Application of T_2 will give $a_7 = a_8 = 0$, and T_1 will give $a_6 = a_9 = 0$. By proper choice of the unit point we obtain

(a) $P_3 = 0, 0, 0, 0, 1, 0, 0, 0, 0, 1.$

[4] We omit accents for the new coordinates; we wish only to differentiate here between coordinates which are zero and those that are not known to be zero.

The other planes of set (1) are those for which $a_{10} = 0$ and hence $a_3 \neq 0$. Applying T_2 with $2a_3k + a_8 = 0$ and $-a_3l + a_6 = 0$ gives $a_6 = 0$ and $a_1 = a_8$. Then ρ contains the point $P_3 = 0, 0, a_3, 0, a_5, 0, a_7, 0, a_9, 0$. When $a_9 \neq 0$, we may apply T_1 with $a_5 + ba_9 = 0$ and $-a_3a = a_7$ to remove a_5 and to make $a_2 = a_7$. When $a_9 = 0$, T_1 can be applied to make $a_2 = a_7$. In both cases, P_3 can be changed to a point which has $a_2 = a_7 = 0$. Thus we have the possibilities:

(b) $P_3 = 0, 0, a_3, 0, 0, 0, 0, 0, a_9, 0,$
(c) $P_3 = 0, 0, a_3, 0, a_5, 0, 0, 0, 0, 0.$

We note that in the case of (c) the line P_1P_3 is a Σ-line.

(2) Now suppose $a_4 \neq 0$. Then in consideration of T_3 we may suppose $a_{10} = 0$. We consider first those planes given by P_3 with $a_3 \neq 0$. With proper choice of k and l, T_2 gives $a_6 = a_9 = 0$. P_3 can be selected in ρ so that $a_7 = a_8 = 0$. Applying T_1 with proper choice of a and b will change a_1 and a_2 to zero. Hence, we have

(d) $P_3 = 0, 0, a_3, a_4, a_5, 0, 0, 0, 0, 0.$

Finally, suppose $a_3 = 0$. T_2 and a change of P_3 will remove a_2, a_7, a_8, and a_9, introducing $a_1 \neq 0$. We then have

$$P_3 = a_1, 0, 0, a_4, a_5, a_6, 0, 0, 0, 0.$$

T_1 can be used to remove a_1 and to remove a_5 if $a_6 \neq 0$. We have the possibilities:

(e) $P_3 = 0, 0, 0, a_4, 0, a_6, 0, 0, 0, 0,$
(f) $P_3 = 0, 0, 0, a_4, a_5, 0, 0, 0, 0, 0.$

In the case of (f), P_2P_3 is a Σ-line.

We shall now show that the plane determined by (a) contains no Σ-line, so that planes (c) and (f) are different from (a). Denote the plane given by (a) as $k, l, 0, 0, m, 0, l, k, 0, m$. A point $P = k, l, m$ in it determines the three-space

$$R:\quad (m^2 + kl)x_1 - lmx_2 + kmx_3 + l^2x_4 + k^2x_5 = 0.$$

If P is on the line $m = 0$ (i.e., the line P_1P_2), R is $klx_1 + l^2x_4 + k^2x_5 = 0$. If P is P_3, R is $x_1 = 0$. For no k and l can these be the same R, and hence a Σ-line in ρ does not pass through P_3. A Σ-line in ρ must therefore intersect P_1P_3 and P_2P_3 in distinct points. If P is on the line $l = 0$, R is $m^2x_1 + kmx_3 + k^2x_5 = 0$; if P is on $k = 0$, then R is $m^2x_1 - lmx_2 + l^2x_4 = 0$. These R's are the same only if the corresponding P's are the same. Hence, ρ contains no Σ-line.

We next show that the planes determined by (a), (b), (d), and (e) are the same, and those determined by (c) and (f) are the same; they are respectively planes 9 and 8 above.

The transformations used so far to simplify the coordinates of P_3 have all left the line P_1P_2 fixed; in order to go farther it will be convenient to change to a different P_1P_2. If in (b) we make the change $P_1' = P_1$, $P_2' = P_3$, $P_3' = P_2$, and then change the coordinate system so that P_1' and P_2' are in canonical form,[5] P_3' becomes 0, 0, 0, 0, 1, 0, 0, 0, 0, 1. If in (e) we interchange the rôles of P_1 and P_3, we obtain (a) again. In (d) we may take $P_1' = P_1 + P_2$, $P_2' = P_1 - P_2$, $P_3' = P_3$, and this will change (d) into (a). Hence, any plane in S which has no point on V, contains no Σ-line, and is not a τ-plane can be put in the form 8.

Interchange of P_1 and P_2 interchanges (c) and (f). Hence, any plane in S which has no point on V, is not a τ-plane, but which contains a Σ-line, can be put in the form 9. This concludes the determination of planes that do not intersect V.

(v) *Planes with* 1, 2, *or* 3 *points on* V. The planes with 1, 2, or 3 points on V all contain a line P_1P_2. The transformations in (iv) still pertain; the present planes were excluded by requiring that there be no point on V. By looking more closely at that requirement we determine the planes:

10. $k, l, 0, 0, m, 0, l, k, 0, 0$.
11. $k, l, 0, 0, 0, m, l, k, 0, 0$.
12. $k, l, m, 0, 0, 0, l, k, 0, 0$.
13. $k, l, 0, 0, 0, m, l - rm, k, 0, 0$ (r not a square).
14. $k, l, m, rm, 0, 0, l, k, 0, 0$.
15. $k, l, 0, 0, 0, 0, l, k, 0, m$.
16. $k + m, l, 0, 0, 0, 0, l, k, 0, 0$.
17. $k + m, l, m, 0, 0, 0, l, k, 0, 0$.
18. $k + m, l + m, 0, 0, 0, 0, l, k, 0, 0$.

When ρ is a τ-plane, it will be determined by P_1P_2 and the point $P_3 = 0, 0, 0, 0, a_5, a_6, a_7, a_8, a_9, 0$. The polynomial $f(\theta) = a_9\theta^3 - a_7\theta^2 + a_8\theta - a_6$ will now be reducible. The transformations on this polynomial in (iv) show that unless $f(\theta)$ is identically zero we may suppose a_6 or a_9 is not zero, and hence a_5 may be made zero. The one case it may not be made zero gives plane 10; this plane is obviously unique, since P_1P_2 determines the unique point $P_3 = M$. Plane 10 has one point on V and is tangent to V at that point.

The reducible $f(\theta)$ may be a cube as is given by 11.[6] This plane has one point on V and contains the tangent line $l = 0$.

If $f(\theta)$ is the product of a linear and an irreducible quadratic factor, the plane is 13 which has one point on V and no line tangent to V. If $f(\theta)$ is the product of a linear factor by the square of another, the plane is 16. For this

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[5] We omit the computation because of its length; it is exactly like that which determined the matrix T_3.

[6] In this case $f(\theta) = -1$. The transformation $\theta = 1/\theta'$ in (iv) applies, giving $f(\theta') = \theta'^3$.

the P_3 in the proper form gives $f(\theta)$ which is reduced to θ. $f(\theta) = 0$ has the root zero and the double root infinity. Plane 16 has two points on V. If $f(\theta)$ has three distinct linear factors, the plane is 18; it has three points on $V; f(\theta) = \theta^2 - \theta$.

When ρ is not a τ-plane, P_3 can be made to take one of the forms (a) to (f) of (iv) with the added possibility that some of the a's are zeros. Case (a) was obtained on the assumption that $a_{10} \neq 0$; if in this case $a_5 = 0$, we have plane 15. This plane has one point on V and no tangent line.

In cases (b) and (c) we have $a_3 \neq 0$. If in the respective cases $a_9 = 0$ and $a_5 = 0$, we have plane 12 which has one point on V and the line $l = 0$ tangent to V. If in (c) $r = -a_5/a_3$ is not a square, we have plane 8 with no point on V, but if r is a square we have plane 17, with two points on V.[7]

Both (e) and (f) reduce to 0, 0, 0, 1, 0, 0, 0, 0, 0, 0 which gives a ρ that is changed into 12 by interchanging P_1 and P_2.

In case (d) we could have $a_5 = 0$, in which case we have plane 14 if $r = a_4/a_3$ is not a square. This has one point on V and no tangent line. If r is a square, the plane will still have one point on V and no tangent line. The planes for r a square and r not a square are different. To see this, consider the plane

$$k,\ l,\ m,\ rm,\ 0,\ 0,\ l,\ k,\ 0,\ 0.$$

The three-space in X determined by a point k, l, m is

$$klx_1 - rkmx_2 - lmx_3 + l^2x_4 + k^2x_5 = 0.$$

By means of this relation every point of X determines a conic in ρ. Now, ρ has a special point, P_3, which is on V and is the image of a line p_3 in X. The points of p_3 determine the conics of a special pencil in ρ.

$$P_3 = 0, 0, 1, r, 0, 0, 0, 0, 0, 0 \quad \text{and} \quad p_3 = \begin{cases} 1, 0, 0, 0, 0 \\ 0, 0, 0, 1, r \end{cases},$$

and these points on p_3 give the conics $kl = 0$ and $l^2 + rk^2 = 0$. The special pencil of conics is $rk^2 + \lambda kl + l^2 = 0$. When r is not a square, every conic of the pencil consists of two distinct lines; when r is a square, there are two conics each of which is a line counted twice. This was the difference between planes 14 and 15 that was explained in the earlier paper. Since we have now found all planes which contain P_1P_2, it should follow that plane 15 and this last one with r a square are the same.[8] To see that they are the

[7] If ρ has two points on V, it contains a Σ-line, so we should expect it to come from (c) or (f).

[8] One reason for keeping the above canonical form for plane 15 is that it is in print; another reason is to exhibit one of the places where it would be easy to go astray in accounting for all the possibilities. It would not be hard to miss the fact that it makes a difference whether or not a_4/a_3 is a square. Plane 15 was found first, and many attempts were made to change 14 into 15 before they were looked at closely enough to see the difference explained above.

same, we notice that in the case of plane 15 the points P_1 and P_2 are one each on the two degenerate parabolas. Making this change in the case where a_4/a_3 is a square gives the form 15.

(vi) *Planes with a line on V that are not Σ-planes.* There are four planes, not Σ-planes, each of which contains a ruling of V:

19. $k, 0, 0, 0, 0, 0, 0, 0, l, m.$
20. $k, 0, 0, 0, l, 0, 0, k, m, 0.$
21. $k, 0, 0, 0, 0, 0, l, k, m, 0.$
22. $k, 0, 0, 0, 0, 0, 0, k, l, m.$

Plane 19 has the line $k = 0$ and the point $l = m = 0$ on V; any plane with a line and a point on V can be put in the form 19. For the line P_2P_3 determines a pencil of lines in X which may be taken to be in the plane $\sigma = A_3A_4A_5$ with vertex of the pencil at A_5. The other point P_1 in ρ and on V determines a line p_1 in X. The line p_1 cannot intersect the plane σ for then P_1, P_2, and P_3 would all be in a five-space Σ determined by the lines of a three-space in X and ρ would be a Σ-plane. Hence, A_1 and A_2 may be selected on p_1, and ρ takes the form 19.

Let ρ be a plane, not a Σ-plane, intersecting V in one line P_2P_3 only, and let P_1 be a point of ρ not on V. There is no more than one Σ-line in ρ on P_1; hence there is no more than one line through P_1 tangent to V. Therefore, ρ is in no more than one of the spaces tangent to V at points of P_2P_3.

Suppose ρ is tangent to V at the point P_2. Then since any tangent is conjugate to any other, we may take P_1P_2 to be

$$k, 0, 0, 0, l, 0, 0, k, 0, 0.$$

The points of the line P_2P_3 image the lines of a pencil in a plane σ in X. P_1 determines the three-space R_1 in X; R_1 does not contain σ and hence intersects it in the line p_2. p_3 is a line in σ and intersects R_1 only at its intersection with p_2. We wish to show that this intersection can be taken to be A_3. The line p_2 is $\begin{cases} 0, 1, 0, 0, 0 \\ 0, 0, 1, 0, 0 \end{cases}$. The point P_1 is on Q_1Q_1' where

$$Q_1 = 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 \quad \text{and} \quad Q_1' = 0, 0, 0, 0, 0, 0, 0, 1, 0, 0.$$

If we take a new $q_1' = \begin{cases} 0, a, b, 0, 0 \\ 0, 0, 0, 1, 0 \end{cases}$, we have

$$Q_1' = 0, 0, 0, 0, 0, a, 0, b, 0, 0 \quad \text{and} \quad Q_1 = b, 0, 0, 0, 0, -a, 0, 0, 0, 0.$$

Hence, Q_1' and Q_1 can be selected so that P_1P_2 is in the above form and so that p_3 passes through $A_3 = 0, 0, 1, 0, 0$, the intersection of p_2 and q_1'. Then A_5 may be taken on p_3, not in R_1. ρ is then in the form 20; ρ is tangent to V at P_2.

Now suppose ρ intersects V in a line and is not tangent to V at any point of the line. Let P_1 be a point of ρ not on V; let P_2P_3 be the ruling of V; let σ and R_1 be as above. The intersection q of R_1 and σ does not pass through the vertex of the pencil p_2p_3 for then ρ would be in the space tangent to V at a point of P_2P_3. Hence, q intersects p_2 and p_3 at distinct points. The point Q, on V, may or may not be such that QP_1 is a tangent to V. If it is not, then QP_1 meets V in a point Q'. By selecting A_1 and A_2 on q', A_3 on p_2 and q, A_4 on p_3 and q, and A_5 on p_2 and p_3, we have the canonical form 22. This plane is not a τ-plane.

If Q above is on the polar of P_1, a coordinate system in R_1 can be selected so that

$$P_1 = 1, 0, 0, 0, 0, 0, 0, 1, 0, 0 \quad \text{and} \quad Q = 0, 0, 0, 0, 1, 0, 0, 0, 0, 0.$$

The line common to σ and R_1 is A_2A_3. A_5 can be taken at the vertex of the pencil p_2p_3, which is not in R_1. The plane ρ is then plane 21 which is in the space tangent to V at Q. This completes the determination of the types of plane in S.

4. Some collineations of S leaving certain planes unchanged in form

In the determination of the types of plane in S it was necessary to obtain more information about lines than was required to determine the types of line. Likewise, in the determination of types of three-space it will be necessary to have more information about certain of the types of plane. A three-space with certain relations to V can often readily be seen to contain a plane of a certain type. Knowing that a plane of a given type is present, we know that a coordinate system can be selected to exhibit it in a particular form. Usually that can be done in many ways. That it could be done at all was enough to fix a canonical form for the type, but to determine a canonical form for the three-space that will give the plane the canonical form for its type generally will require a special selection of the frame of reference in the plane. It may thus become necessary to know all possible selections of the coordinate system to present a given plane in canonical form. The collineations that were found necessary in classifying the three-spaces are collected in this section.

(i) *The plane* $k, l, 0, 0, 0, 0, l, k, 0, m$. This plane intersects V in the point $P_3 \colon k = l = 0$ only; it is not a τ-plane and contains no Σ-line. We ask how much is the freedom of choice of P_1 and P_2 if the form is to remain unchanged.

We note first that the lines P_1P_3 and P_2P_3 are completely determined by the plane's relation to V. Let $P = (k, l, m)$ be any point of the plane. For P we have the following:

$$B_1 = k^2, \qquad B_2 = -l^2, \qquad B_3 = km, \qquad B_4 = lm, \qquad B_5 = kl.$$

The three-space R in X determined by P is $klx_1 - lmx_2 + kmx_3 + l^2x_4 + k^2x_5 = 0$. If k, l, m are given, this defines R. If $x_1, x_2, \cdots, x_5$ are given, this defines a conic in the plane. The point P_3, being on V, is the image of a line in X, namely, the line $\begin{cases} 0, 0, 0, 1, 0 \\ 0, 0, 0, 0, 1 \end{cases}$.

This pencil of points in X, which has a special relation to the plane, determines a pencil of conics in the plane, namely, $k^2 + \lambda l^2 = 0$. The pencil of conics contains the two degenerate parabolas $k^2 = 0$ and $l^2 = 0$, given by $\lambda = 0$ and $\lambda = \infty$, respectively. Hence, the lines $k = 0$ and $l = 0$ are special lines in the plane. If the plane is to have the given form, P_1 and P_2 must be selected on these lines.

If P_1 and P_2 are left fixed, the coordinate system can be changed, still leaving the coordinates of P_1 and P_2 unchanged. Transformations T_1 and T_2 do this. Neither T_1 nor T_2 leaves P_3 unchanged. Hence, if we wish the plane to retain the above form, choice of P_1 and P_2, necessarily on the special lines, determines the coordinate system excepting that there is left some freedom in the choice of the unit point.

We give the transformation resulting from the choices $P_1' = P_1 + aP_3$ and $P_2' = P_2 + bP_3$.

$$T_4 = \begin{bmatrix} 1 & 0 & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b & -a & b^2 & -a^2 & 1 & 0 & a & b & 0 & -ab \\ 0 & 0 & -a & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 & 1 & 0 & 0 & -b \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 1 & 0 & -a \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(ii) *The plane* $k, k, 0, 0, 0, 0, l, m, 0, 0$. This is one form of the plane with three points on V. A transformation which leaves every point of the plane fixed is

$$T_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 1 & 0 & ab & -a & 0 & -a & 0 & 0 \\ -c & 0 & 0 & 1 & -ac & 0 & -a & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -bc & c & 0 & 0 & -b & 1 \end{bmatrix}.$$

(iii) *The plane* $k, l, 0, 0, 0, 0, l, m, 0, 0$. This is a form of plane 16; it is useful in dealing with three-spaces with two points on V. A transformation which leaves every point of the plane fixed is

$$T_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 1 & 0 & ab & -a & 0 & 0 & 0 & 0 \\ -c & a & 0 & 1 & -a^2 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -bc & c & 0 & -a & -b & 1 \end{bmatrix}.$$

The points P_1 and P_3 are obviously special points in the plane, being on V; the line P_1P_2 is special, being tangent to V at P_1. A transformation of the plane into itself which keeps the form could only move P_2 along the line P_1P_2. Such is

$$T_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 1 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & bc & c & 0 & 0 & 0 & 0 & 0 & b & 1 \end{bmatrix}.$$

(iv) *The plane* $k, l, m, 0, 0, 0, l, k, 0, 0$. The line P_1P_3 is tangent to V at P_3. If the form of the plane is to remain unchanged, P_1 must remain on that line. We note that T_3 leaves P_3 unchanged, and hence P_2 may be moved along the line P_1P_2. Then if we combine T_3 with a transformation which moves P_1 along P_1P_3 and leaves P_2 fixed, we will have a transformation which leaves the plane in canonical form with P_1P_2 any line not on P_3, and P_2 any point, except P_1, on that line. Even then we can change the coordinate system by applying T_1 with $a = 0$. The following transformation moves P_1 to $P_1' = P_1 + aP_3$ and P_2 to $P_2' = P_2 + bP_3$.

$$T_8 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 1 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(v) *The plane* $k, l, 0, 0, 0, 0, 0, 0, 0, m$. This is a plane with a ruling and an additional point on V; it is not tangent to V at any point of the intersection. The line of V represents a pencil of lines in X with vertex at A_1; the pencil lies in the plane $A_1A_2A_3$; the other point on V is the image of the line A_4A_5 in X. If the form of the plane is left unchanged, the point A_1, the plane $A_1A_2A_3$, and the line A_4A_5 must be left unchanged. The most general transformation in X is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & b & 0 & 0 \\ c & d & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & f & 1 \end{bmatrix}.$$

The corresponding transformation in S is

$$T_9 = \begin{bmatrix} 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & e & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ ad-c & a-bc & 0 & 0 & 1-bd & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & ae & 0 & 1 & e & b & be & 0 \\ 0 & 0 & af & a & 0 & f & 1 & bf & b & 0 \\ 0 & 0 & c & ce & 0 & d & de & 1 & e & 0 \\ 0 & 0 & cf & c & 0 & df & d & f & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-ef \end{bmatrix}.$$

(vi) *The plane* $k, l, 0, 0, 0, m, l, k, 0, 0$. The line P_1P_3 is tangent to V at P_3; P_2 is an arbitrary point not on P_1P_3. Transformations T_2 and T_3 leave the form of the plane unchanged, and T_1 with $a = 0$ does also. The following transformation moves P_1 and P_2 along the lines P_1P_3 and P_2P_3.

$$T_{10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & -b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -a & 0 & 1 & 0 & 0 \\ -a^2 & a & 0 & 0 & 0 & ab & -a & -b & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(vii) *The plane* $k, l, 0, 0, 0, m, l + m, k, 0, 0$. This is a τ-plane with P_3 on V; $l + m = 0$ is a Σ-line; there is no line tangent to V. The following transformation moves P_1 along the Σ-line.

$$T_{11} = \begin{bmatrix} d & 0 & 0 & 0 & -ad & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 & -ad & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & ad \\ 0 & 0 & 0 & 0 & d^2 & 0 & 0 & 0 & 0 & 0 \\ ad & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\ -ad & 0 & 0 & 0 & 0 & a^2d & d^2 & 0 & 0 & 0 \\ 0 & ad & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & -ad & 0 & 0 & 0 & 0 & 0 & a^2d & d & 0 \\ 0 & 0 & -ad & -ad & 0 & 0 & 0 & 0 & 0 & d \end{bmatrix}, (d = 1 + a^2).$$

We shall have use for another transformation, which leaves P_1 fixed but moves P_2 along P_2P_3.

$$T_{12} = \begin{bmatrix} d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -b & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (d = b + 1).$$

(viii) *The plane* $k, l, m, 0, -rm, 0, l, k, 0, 0$. This plane has no point on V; it contains the Σ-line P_1P_3. If the form of the plane is to remain unchanged, P_1 and P_3 must remain on the Σ-line. P_3 is determined by P_1, since they are conjugates with respect to the hyperquadric in which Σ intersects V. The transformation T_3, which moves P_2 along the line P_1P_2, leaves P_1 and P_3 unchanged; T_1 with $a = 0$ also leaves P_3 unchanged. We give a transformation which moves P_1 along P_1P_3 and leaves P_2 fixed. This with T_3 will allow us to select any line in the plane, except P_1P_3, for P_1P_2. If $P_1' = P_1 + aP_3$, then $P_3' = -aP_1 + P_3$.

$$T_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(ix) *The plane* $k, l, 0, 0, 0, m, l, k + \alpha m, m, 0$. This is a τ-plane, with no point on V if $x^3 + \alpha x - 1$ is irreducible. In determining the canonical form it was necessary to move P_2 along P_1P_2, to interchange P_1 and P_2, and to change the unit point; it was not necessary to change the line P_1P_2. This is a τ-plane, and it contains no Σ-line; any line in it can be taken for P_1P_2. The point P_3 is determined by P_1P_2. There is only one point on a given line that can serve for P_1 and give the canonical cubic, because the group of transformations of the line into itself is exactly the group of linear fractional transformations of x. In order to show that P_1 may be taken to be any point in the plane, it is necessary only to show that a change of the line P_1P_2 in the pencil on P_1, leaving P_1 fixed, changes the polynomial in x. For, since every line has a P_1 and no point is the P_1 of more than one line, every point must be the P_1 of some line. The following transformation has $P_1' = P_1$ and $P_2' = aP_2 + P_3$.

$$T_{14} = \begin{bmatrix} a^3 & -a & 0 & 0 & 0 & -a^2(\alpha a-1) & 0 & \alpha a-1 & 0 & 0 \\ a(\alpha a-1) & a^2 & 0 & 0 & 0 & -(\alpha a-1) & 0 & a(\alpha a-1) & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(a^2+\alpha^2 a-\alpha) & a & 0 & 0 & 0 & 0 & 0 & \alpha a-1 \\ 0 & 0 & 0 & 0 & \delta & 0 & 0 & 0 & 0 & 0 \\ a^2 & -1 & 0 & 0 & 0 & a^4 & 0 & -a^2 & 0 & 0 \\ -\alpha a^2 & \alpha & 0 & 0 & 0 & -a^3 & a^2 & a & -1 & 0 \\ \alpha a-1 & a & 0 & 0 & 0 & a^2(\alpha a-1) & 0 & a^3 & 0 & 0 \\ -\alpha(\alpha a-1) & -\alpha a & 0 & 0 & 0 & -a(\alpha a-1) & \alpha a-1 & -a^2 & a & 0 \\ 0 & 0 & -a(\alpha a-1) & -1 & 0 & 0 & 0 & 0 & 0 & a^2 \end{bmatrix},$$

$$\delta = a^3 + \alpha a - 1.$$

This transforms the point P_2 into

$$-a, \alpha + a^2, 0, 0, 0, -a^3 - (\alpha a - 1)^2, a^2, a - \alpha(\alpha a - 1), -1, 0.$$

The point P_3' is

$$0, 0, 0, 0, 0, a^3 + (\alpha a - 1)^2, \alpha, -2a + a(\alpha a - 1), 1, 0.$$

The corresponding cubic is

$$x^3 - \alpha x^2 + a(\alpha a - 3)x - [a^3 + (\alpha a - 1)^2] = 0.$$

Since this cannot be transformed into $x^3 + \alpha x - 1 = 0$ by a change that leaves P_1P_2 and also the point P_1 fixed, it follows that P_1 may be taken to be any point in the plane, and then P_2 and P_3 may be determined so that the plane has the above canonical form.

(x) *The plane* $k, l, 0, 0, 0, m, 0, 0, 0, 0$. This is a Σ-plane in the five-space determined by the lines in R: $x_5 = 0$; it intersects V in the two lines $l = 0$ and $m = 0$. The two lines on V determine two pencils of lines in R; the planes of the pencils in R intersect in a line which belongs to both pencils. To obtain the above form, A_1 and A_2 are selected at the vertices of the two pencils,

A_3 in the plane of the pencil with vertex at A_1, and A_4 in the plane of the other. A_3 may be moved about its plane, and likewise A_4, and also A_5 may be selected anywhere outside of R without affecting the form of the plane. These changes are made by

$$T_{15} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ bg+dh-f & -g & -h & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -ag-ch+e & 0 & 0 & 0 & -g & -h & 1 & 0 & 0 & 0 \\ ad-bc & c & -a & 0 & d & -b & 0 & 1 & 0 & 0 \\ \delta_1 & e-ch & ah & -a & f-dh & bh & -b & -h & 1 & 0 \\ \delta_2 & cg & e-ag & -c & dg & f-bg & -a & g & 0 & 1 \end{bmatrix},$$

$$\delta_1 = af - adh + bch - be, \qquad \delta_2 = cf - bcg + adg - de.$$

(xi) *The plane* $k, l, m, -m, 0, 0, l, k, 0, 0$. This plane contains P_3 on V; it contains no special line; any line not on P_3 can be taken for P_1P_2. We give a transformation which moves P_1 and P_2 along the lines P_1P_3 and P_2P_3 respectively.

$$T_{16} = \begin{bmatrix} 1 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & b & -b^2 & a^2 & 1 & -b & 0 & 0 & -a & ab \\ 0 & 0 & b & 0 & 0 & 1 & 0 & 0 & 0 & -a \\ 0 & 0 & 0 & b & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 1 & -b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(xii) *The plane* $k, l, 0, 0, m, 0, l, k, 0, m$. This plane involves the most complex considerations of all because it has no points or lines that are obviously special, and there is no point on V specially related to it as, for example, in the case of a τ-plane. Yet its relation to V does determine a special locus in the plane.

For any point $P = (k, l, m)$ in the plane we have

$$B_1 = k^2, \qquad B_2 = -l^2, \qquad B_3 = km, \qquad B_4 = lm, \qquad B_5 = m^2 + kl.$$

Setting the B's equal to zero gives five conics in the plane. These conics are linearly independent and determine a unique conic apolar to them. This absolute conic is $C\colon m^2 - 2kl = 0$. C depends only on the plane; it does not depend on the coordinate system, for a change of coordinates would change the conics among conics of the linear set, and C is apolar to all of them. The points P_1 and P_2 are on C, and P_3 is the pole of the line P_1P_2 with respect to

C. We shall show that P_1 and P_2 can be taken to be any two points of C, and then if P_3 is taken to be the pole of P_1P_2, a coordinate system can be selected so that the plane is in the canonical form.

We look for the relations of P_1, P_2, and P_3 to V which characterize the canonical form.

$$P_1 = 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, \qquad P_2 = 0, 1, 0, 0, 0, 0, 1, 0, 0, 0,$$
$$P_3 = 0, 0, 0, 0, 1, 0, 0, 0, 0, 1.$$

P_1P_2 is in the space tangent to V at

$$M = 0, 0, 0, 0, 1, 0, 0, 0, 0, 0.$$

The point P_3 is on the line joining M to a second point on V,

$$Q_3' = 0, 0, 0, 0, 0, 0, 0, 0, 0, 1.$$

P_1 is on the line joining two points of V:

$$Q_1 = 1, 0, 0, 0, 0, 0, 0, 0, 0, 0 \quad \text{and} \quad Q_1' = 0, 0, 0, 0, 0, 0, 0, 1, 0, 0.$$

P_2 is on the line joining

$$Q_2 = 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 \quad \text{and} \quad Q_2' = 0, 0, 0, 0, 0, 0, 1, 0, 0, 0.$$

Corresponding to points Q_1, Q_1', Q_2, Q_2', M, Q_3' on V are lines q_1, q_1', q_2 q_2', m, q_3' in X. These lines have incidences which have been described earlier (Section 2) for the first five. The sixth line q_3' intersects q_1' and q_2'. These relations make it possible to select the frame of reference in X to give the canonical form.

We now prove that P_3 is the only point in the plane, not on P_1P_2, such that the line joining it to M has a second point on V. The points of the line joining M to an arbitrary point of the plane are

$$kr, lr, 0, 0, mr + 1, 0, lr, kr, 0, mr \qquad (r = 0, 1, \cdots, p - 1, \infty).$$

Conditions that this point be on V are $B_1 = k^2r^2 = 0$, $B_2 = l^2r^2 = 0$, $B_3 = kmr^2 = 0$, $B_4 = lmr^2 = 0$, $B_5 = (m^2 + kl)r^2 + mr = 0$. If $m = 0$, these equations are all quadratic with a double root zero (where they are not identically zero) corresponding to the fact that a line joining M to a point of P_1P_2 is a tangent to V. If $m \neq 0$, the last equation has a term of the first degree in r; hence the others must be identically zero, and hence $k = l = 0$. Therefore, there will be a second point of V on the line only if $(k, l, m) = (0, 0, 1)$.

Any line in the plane is $ak + bl - cm = 0$. This line is in the space tangent to V at the point[9]

$$M' = bc^2, ac^2, b^2c, -a^2c, (2ab + c^2)c, b^3, -a(ab + c^2), -b(ab + c^2), a^3, -abc.$$

[9] This point is obtained as the intersection of the polar spaces of $(c, 0, a)$ and $(0, c, b)$.

Conditions on $P = (k, l, m)$ derived from requiring $M' + rP$ to be on V for some r give $P = (b, a, c)$. This is a necessary condition on P_3 and the line P_1P_2 if the plane is to have the canonical form. A further condition is that P_1P_2 must cut C in two points, i.e., $c^2 - 2ab$ must be a square, not zero.

Conversely, if $c^2 - 2ab$ is a square, not zero, and P_1 and P_2 are intersections of $ak + bl - cm = 0$ with $m^2 - 2kl = 0$, then Q_1, Q_1', Q_2, Q_2' can be determined so that P_1, P_2, P_3 have the required coordinates. If P_3 is moved along the line P_1P_3, which is tangent to C at P_1, and P_2 is moved along C to the polar of the new P_3, and if then a coordinate system exists such that P_1, P_2, P_3 have the above form, it will follow that P_3 may be taken to be any point of the plane outside C; the result comes from the fact that P_1 and P_2 enter symmetrically in relation to P_3, to C, and also in relation to the frame of reference in X.

We give the transformation which leaves P_1 fixed, moves P_3 to $P_1 + cP_3$, $c \neq 0$, and moves P_2 along C.

$$T_{17} = \begin{bmatrix} 5c^5 & 0 & 6c^5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4c^3 & 6c^3 & 6c^3 & 0 & 2c^3 & 4c^3 & 0 & 3c^3 & 0 & 0 \\ 0 & 0 & c^6 & 0 & 0 & 5c^6 & 0 & 0 & 0 & 0 \\ 3c^2 & 5c^2 & 4c^2 & 2c^2 & 4c^2 & 4c^2 & 3c^2 & 6c^2 & 0 & 6c^2 \\ 2c^4 & 0 & c^4 & 0 & 4c^4 & 4c^4 & 0 & 5c^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3c^7 & 0 & 0 & 0 & 0 \\ 2c^3 & 0 & c^3 & 0 & c^3 & 3c^3 & 6c^3 & 3c^3 & 0 & 3c^3 \\ 0 & 0 & c^5 & 0 & 0 & 6c^5 & 0 & 5c^5 & 0 & 0 \\ 6c & 6c & 5c & 2c & 6c & 3c & 5c & 5c & 3c & 2c \\ 0 & 0 & 6c^4 & 0 & 0 & 3c^4 & 0 & 4c^4 & 0 & 4c^4 \end{bmatrix} \qquad (\text{for } p = 7).$$

It may be verified that T_{17} puts

$$\begin{aligned} 1, 0, 0, 0, 0, 0, 0, 1, 0, 0 \quad &\text{into} \quad 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, \\ 1, 2c^2, 0, 0, 2c, 0, 2c^2, 1, 0, 2c \quad &\text{into} \quad 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, \\ 1, 0, 0, 0, c, 0, 0, 1, 0, c \quad &\text{into} \quad 0, 0, 0, 0, 1, 0, 0, 0, 0, 1. \end{aligned}$$

5. Three-spaces which intersect V in at least one point

(i) *Introduction.* The three-spaces most easily dealt with are those having large intersections with V; one of the two three-spaces with no point on V requires more work than all the others, and for this reason the two are separated from them.

There is one three-space S_3 which will not be included in our list because it leads to a group that has been excluded. This S_3 lies wholly on V. Since every pair of points in S_3 is the image of a pair of intersecting lines in X, all of these lines must pass through a point. If this point is taken to be A_1, then A_2, A_3, A_4, A_5 may be selected arbitrarily, except that all five A's must be linearly independent, and then S_3 will be $k, l, m, n, 0, 0, 0, 0, 0, 0$. The group of order p^{11} defined by this three-space is given by the additional relations:

$s_1 = s_2 = s_3 = s_4 = 1$; it is the direct product of the metabelian group $\{U_2, U_3, U_4, U_5\}$ of order p^{10} and the cyclic group $\{U_1\}$.

(ii) *Three-spaces containing a Σ-plane cutting V in a nondegenerate conic.*

1. $k, 0, m, 0, n, 0, 0, l, 0, 0,$ the ruled quadric $kl + mn = 0$ on V.
2. $k, l, m, 0, n, -l, 0, k, 0, 0,$ the quadric $k^2 + l^2 + mn = 0$ on V.
3. $k, l, n, 0, 0, l, 0, m, 0, 0,$ the cone $km - l^2 = 0$ on V.
4. $n, n, 0, 0, k, m, 0, 0, m, l,$ the conic $kl - m^2 = 0, n = 0$ and the line $l = m = 0$ on V.
5. $k, l + n, m, 0, n, n, l, k + n, 0, 0,$ the conic $k^2 + kn - n^2 + mn = 0,$ $l = 0$, and the point $1, 1, 0, -1$ on V.
6. $n, 0, 0, 0, k, m, 0, n, m, l,$ the conic $kl - m^2 = 0, n = 0$ on V.

If the intersection of S_3 and V contains a nondegenerate conic, the plane of the conic will be a Σ-plane. Hence, the spaces in this set all contain Σ-planes at least; the first three are actually Σ-three-spaces, in the Σ determined by the lines of the three-space $x_5 = 0$ in X.

If S_3 lies in a Σ, the intersection of V and Σ cuts it in a quadric; if the quadric is degenerate, it can be at worst a cone with a single vertex, since we insist that some plane of S_3 intersect V in a nondegenerate conic. Suppose the quadric is not degenerate and that it has rulings. Let P_1 be an arbitrary point of the quadric; let P_2 and P_3 be arbitrary points, one on each of the rulings through P_1; and let P_4 be the intersection of two other rulings, one through P_2 and the other through P_3. Corresponding to these four points of V are four lines p_1, p_2, p_3, p_4 in X. p_1 intersects p_2 and p_3 and does not intersect p_4. p_4 intersects p_2 and p_3, and p_2 does not intersect p_3. p_1, p_2, and p_3 determine a three-space, and p_4 lies in it; this three-space determines in S the Σ in which S_3 lies. We select a frame of reference in X as follows: A_1 is on p_1 and p_2; A_2 is on p_1 and p_3; A_3 is on p_2 and p_4; A_4 is on p_3 and p_4; A_5 is anywhere outside the three-space already determined. Then S_3 will have the form 1.

Let S_3 intersect V in a nondegenerate quadric which has no rulings. S_3 contains a plane which cuts the quadric in a nondegenerate conic; this plane is a Σ-plane. Let P be a point of this plane not on V; a line joining P to a point of the quadric not in the plane cuts V twice or else is a tangent, and hence the line is a Σ-line. The quadric and S_3 are thus seen to be in a Σ. A coordinate system can be selected so that the plane of the conic is

$$k, 0, m, 0, n, 0, 0, k, 0, 0.$$

The three-space in X determined by a point of the plane is $x_5 = 0$. Hence, S_3 is in the five-space $a_4 = a_7 = a_9 = a_{10} = 0$. Any point of S_3 is

$$a_1, a_2, a_3, 0, a_5, a_6, 0, a_8, 0, 0.$$

S_3 contains a point $P_2' = 0, a_2, 0, 0, 0, a_6, 0, a_8, 0, 0$. The polar spaces of $P_3 = 0, 0, 1, 0$ and $P_4 = 0, 0, 0, 1$ with respect to V are respectively $a_5 = a_7 = a_9 = 0$ and $a_3 = a_4 = a_{10} = 0$. Both contain P_1 and P_2'. P_1 is not on

V, and hence the line P_1P_2' contains a point P_2 conjugate to P_1 with respect to V. For this point we have $a_1 + a_8 = 0$, and hence

$$P_2 = a_1, a_2, 0, 0, 0, a_6, 0, -a_1, 0, 0.$$

A change of coordinates:

$$A_1' = a_2A_1 + a_1A_4, \quad A_4' = a_2A_1 - a_6A_4, \qquad A_i' = A_i, \quad i \neq 1, 4,$$

and a proper choice of the unit point gives $P_2 = 0, 1, 0, 0, 0, r, 0, 0, 0, 0,$ r not a square. This is space 2.

Let S_3 intersect V in a cone, and let the vertex of the cone be P_4. Every point of S_3 is in the space tangent to V at P_4. Let ρ be a plane which cuts the cone in a conic C. Let P_1 and P_2 be points of C, and let P_3 be the pole of P_1P_2 with respect to C. Then p_1 and p_2 are two skew lines in X, and p_4 intersects both of them. If A_1 is the intersection of p_1 and p_4, A_4 the intersection of p_2 and p_4, A_2 an arbitrary point not A_1 on p_1, and A_3 an arbitrary point not A_4 on p_2, we have

$$P_1 = 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \qquad P_2 = 0, 0, 0, 0, 0, 0, 0, 1, 0, 0,$$
$$P_4 = 0, 0, 1, 0, 0, 0, 0, 0, 0, 0.$$

The three-space containing p_1 and p_2 is $x_5 = 0$. Consequently,

$$P_3 = a_1, a_2, a_3, 0, a_5, a_6, 0, a_8, 0, 0.$$

Since P_3 is in the space tangent to V at P_4, $a_5 = 0$. Since P_1P_3 is tangent to V at P_1, $a_8 = 0$; and since P_2P_3 is tangent to V at P_2, $a_1 = 0$. By rotating the plane of C on P_1P_2 we may move P_3 to the point $0, a_2, 0, 0, 0, a_6, 0, 0, 0, 0,$ and then by a choice of the unit point we may make $a_2 = a_6 = 1$. This gives 3.

Whenever S_3 lies in a five-space Σ, S_3 will intersect V in a quadric. We have taken care of all such S_3's except such as contain a plane of V. It has seemed desirable to consider S_3's with planes on V separately. The remaining spaces under the present heading all intersect at least one Σ in a plane. S_3 cannot contain a second Σ-plane, for the intersection of the two planes would contain points not on V and S_3 would lie in the Σ determined by such a point.

Suppose S_3 contains two points on V besides the points of the conic. Neither of the two points can be in the plane of the conic, since no Σ-plane intersects V in a conic and an additional point. Denote the line on the two points by L. L intersects the plane of the conic in a point which must be on the conic, for otherwise S_3 would be in the Σ determined by that point. L then has three points on V and hence lies wholly on V. If P_4 is taken to be a point on L not on the conic, P_2 as the point on L and the conic, P_3 on the conic, and P_1 the pole of P_2P_3 with respect to the conic, coordinates can be chosen so that we have 4.

Suppose next that S_3 contains a conic C and an additional point P_4 on V, but contains no ruling of V. Coordinates can be selected so that the plane of C is $k, 0, l, 0, m, 0, 0, k, 0, 0$. The equation of C is $k^2 + lm = 0$. The points of C are $kl, 0, l^2, 0, -k^2, 0, 0, kl, 0, 0$. The lines of X imaged on these points are $\begin{cases} l, 0, k, 0, 0 \\ 0, k, 0, l, 0 \end{cases}$. These lines are rulings of one set of the quadric $x_1x_2 - x_3x_4 = 0$, $x_5 = 0$. Any point of the plane of C, not on C, determines the three-space R: $x_5 = 0$, which contains the above quadric. P_4 is not in R, but it intersects R in a point. The point of intersection cannot be on the quadric, since S_3 contains no ruling of V. We may take A_5 to be on p_4, and the intersection of p_4 and R to be $a_4, a_7, a_9, a_{10}, 0$; since this point is not on the quadric, $a_4a_7 - a_9a_{10} \neq 0$. $P_4 = 0, 0, 0, a_4, 0, 0, a_7, 0, a_9, a_{10}$. We show that S_3 cannot be in the space tangent to V at any of its points. If $B = b_1, b_2, \cdots, b_{10}$ is a point of V such that the plane $n = 0$ of S_3 is in the tangent space at B, it is easily seen that $B = b_1, b_2, 0, 0, 0, b_6, 0, -b_1, 0, 0$. The requirement that P_4 be in the tangent space at B gives $a_4a_7 - a_9a_{10} = 0$, which is not so. We determine a canonical form for S_3. Let K be the point in which p_4 intersects R. Through K take a line t in R which intersects the quadric in two points; these points will lie on two distinct rulings of the quadric which are imaged on V on two points of C; let these two points be P_2 and P_3. Denote the pole of P_2P_3 with respect to C by P_1. The polar space of P_1 with respect to V does not contain the point T, which is the image on V of the line t in X, for otherwise S_3 would lie in the space tangent to V at T. Hence, the line joining P_1 to T intersects V again at a point which we denote by Q_1'. The line q_1' in X intersects both p_2 and p_3, since P_1 and T are both in the tangent spaces at P_2 and P_3. The lines p_2, p_3, t, q_1', and p_4 in X are related as follows: t and q_1' are skew and intersect both p_2 and p_3; t also intersects p_4, which is not in the space of p_2 and p_3. Denote the intersections of t with p_2 and p_3 by A_1 and A_2 respectively, and the intersections of q_1' with p_2 and p_3 by A_4 and A_3; select the unit point in R so that K is $1, 1, 0, 0, 0$, and select A_5 on p_4. Then S_3 takes the form 5.

Suppose S_3 intersects V in the conic C and in no other point. Any point of the plane ρ of the conic, not on V, determines the three-space R and the five-space Σ. Let R be $x_1 = 0$. If P_1 and P_2 are chosen on C and P_3 is the pole of P_1P_2 with respect to C, a frame of reference in X can be chosen with A_1 arbitrary, not in R, so that ρ is $0, 0, 0, 0, k, m, 0, 0, m, l$. There is a point in S_3, not on ρ, with coordinates $a_1, a_2, a_3, a_4, 0, a_6, a_7, a_8, 0, 0$. If A_1 is replaced by $A_1' = 1, a, b, c, d$ and the other A's are left unchanged, this point has new coordinates $a_1', \cdots, a_{10}'$. The numbers a, b, c, d can be selected so that $a_6' = a_9'$ and $a_7' = 0$. S_3 contains a point

$$P_4 = a_1, a_2, a_3, a_4, 0, 0, 0, a_8, 0, 0$$

(dropping the accents). Since P_4 is not on V, $a_8 \neq 0$, and not both a_1 and a_4 are zero. Any point in S_3 is

$$P = a_1n, a_2n, a_3n, a_4n, k, m, 0, a_8n, m, l.$$

For P we have

$$B_1 = a_3kn - a_2mn + a_1a_8n^2,$$

$$B_2 = a_4kn + a_1mn,$$

$$B_3 = a_1ln + a_4mn,$$

$$B_4 = a_2ln - a_3mn + a_4a_8n^2,$$

$$B_5 = kl - m^2.$$

The conditions that P be on V are (1) $n = 0$, $kl - m^2 = 0$, which gives C, or (2) $n \neq 0$,

$$\begin{aligned} a_3k \quad - a_2m + a_1a_8n &= 0 \\ a_4k \quad + a_1m \qquad &= 0 \\ a_1l + a_4m \qquad &= 0 \\ a_2l - a_3m + a_4a_8n &= 0. \end{aligned}$$

The last three equations have a solution k, l, m, n not all zeros. $n \neq 0$ requires either $a_1a_3 + a_2a_4 \neq 0$ or $a_4 = 0$. Suppose $a_1a_3 + a_2a_4 \neq 0$, $a_4 = 0$; then $k, l, m, n = a_1a_8, 0, 0, -a_3$, and $a_3 \neq 0$. This is a solution of the four equations, and hence gives a point on V not on C. This is not possible with this S_3. Suppose $a_1a_3 + a_2a_4 \neq 0$, $a_4 \neq 0$. Then if $a_1 \neq 0$, the solution of the last three equations is a_1, a_4^2/a_1, $-a_4$, $-(a_1a_3 + a_2a_4)/a_1a_8$ which also satisfies the first equation. If $a_1 = 0$, the solution of the last three has $n \neq 0$ and satisfies the first, and hence is not suitable. Then suppose $a_1a_3 + a_2a_4 = 0$, $a_4 = 0$. Since P_4 is not on V, $a_1 \neq 0$ and hence $a_3 = 0$. A solution of the last three equations is k, 0, 0, 1, k arbitrary, and this does not satisfy the first. Hence in this case

$$P_4 = a_1, a_2, 0, 0, 0, 0, 0, a_8, 0, 0, \qquad a_1a_8 \neq 0.$$

R_4 intersects R in the plane $x_1 = x_5 = 0$. We note also that S_3 contains a τ-plane $P_1P_3P_4$ tangent to V at the point P_1 which is on C. If a_2 is not zero, it may be made so by moving A_2 to $A_2' = A_2 + a_2A_3/a_1$, and A_5 to $A_5' = -a_2A_4/a_1 + A_5$. Then proper choice of the unit point puts S_3 in the form 6.

There remains the possibility that $a_1a_3 + a_2a_4 = 0$, $a_4 \neq 0$. We show that this is not different from the space just considered, showing first that it contains a plane tangent to V at a point of C.

The space tangent to V at the point $B = b_1, \cdots, b_{10}$ is[10]

$$b_8x_1 - b_6x_2 + b_5x_3 + b_3x_5 - b_2x_6 + b_1x_8 = 0,$$

$$b_9x_1 - b_7x_2 + b_5x_4 + b_4x_5 - b_2x_7 + b_1x_9 = 0,$$

[10] We call attention to this, for we shall have frequent use for this space in what follows.

$$b_{10}x_1 - b_7x_3 + b_6x_4 + b_4x_6 - b_3x_7 + b_1x_{10} = 0,$$
$$b_{10}x_2 - b_9x_3 + b_8x_4 + b_4x_8 - b_3x_9 + b_2x_{10} = 0,$$
$$b_{10}x_5 - b_9x_6 + b_8x_7 + b_7x_8 - b_6x_9 + b_5x_{10} = 0.$$

Its intersection with S_3 is

$$\begin{aligned}
b_3k \qquad\quad - b_2m + (a_8b_1 + a_3b_5 - a_2b_6 + a_1b_8)n &= 0,\\
b_4k \qquad\quad + b_1m + (a_4b_5 - a_2b_7 + a_1b_9)n &= 0,\\
b_1l + b_4m + (a_4b_6 - a_3b_7 + a_1b_{10})n &= 0,\\
b_2l - b_3m + (a_8b_4 + a_4b_8 - a_3b_9 + a_2b_{10})n &= 0,\\
b_{10}k + b_5l - (b_6 + b_9)m + a_8b_7n &= 0.
\end{aligned}$$

If this intersection is a plane, the rank of the matrix of coefficients must be 1. This requires that $b_1 = b_2 = b_3 = b_4 = 0$. If the plane is not $n = 0$, then the coefficients of n in the first four equations are zero. This gives four linear equations in $b_6, \cdots, b_{10}$. Two obvious solutions are

$$a_2, a_3, a_4, 0, 0, 0 \quad \text{and} \quad 0, 0, a_1, 0, a_2, a_3.$$

On the line joining them is $-a_1a_2, -a_1a_3, 0, 0, a_2a_4, a_3a_4$ which is also a solution. The point is in the plane ρ since $a_1a_3 + a_2a_4 = 0$, and is also on C. The τ-plane, which is given by the last equation above, passes through P_4, since b_7 is zero. If now this point of C is selected for P_1 and coordinates are determined as before, P_4 will have $a_3 = a_4 = 0$ since P_4 is in the tangent space at P_1. This completes the consideration of S_3's with a nondegenerate conic on V.

(iii) *Three-spaces with a plane on V.*

7. $k, l, m, 0, n, 0, 0, 0, 0, 0.$
8. $k, l, m, 0, 0, 0, n, 0, 0, 0.$
9. $k, l, m, 0, n, 0, 0, 0, 0, n.$
9′. $k, l, m, n, n, 0, 0, 0, 0, 0.$[11]
10. $k, l, 0, 0, m, 0, 0, 0, 0, n.$
11. $k, l, 0, 0, m, n, 0, 0, n, 0.$

The planes of V are of two types: (1) planes whose points represent the lines of a plane in X, and (2) planes whose points represent the lines of a bundle. In the first four spaces above, the plane $n = 0$ is of the second type; in the other two the only plane on V is of the first type. Space 7 has two planes on V; space 8 has a plane and a line; spaces 9 and 9′ intersect V only in a plane. Space 9′ is in the space tangent to V at each point of P_1P_2; space 9 is not a τ-space. Space 11 is a τ-space, and 10 is not.

[11] Spaces 9′, and later 20′ and 20″, were missing from the paper cited earlier; it is desired to keep the numbering of the earlier paper for the other spaces.

Suppose S_3 contains a plane ρ of the second type. The points of ρ represent the lines of a bundle in X; these lines lie in a three-space R. The vertex of the bundle may be taken to be A_1, and A_2, A_3, A_4 may be taken on any three independent lines of the bundle. Then ρ will take the form of $n = 0$ in 7, 8, 9, 9′. If S_3 contained another plane of the second type, their line of intersection would represent the lines of a pencil common to the two bundles, and so the two bundles would have the same vertex and S_3 would lie on V. This possibility has been dealt with. So a second plane on V must be of the first type. This second plane intersects ρ in a line, and hence its points represent the lines of a plane in X which lies in R and passes through A_1. S_3 is therefore in the Σ determined by R. The plane in R may be taken to be $A_1A_2A_3$. If A_5 is selected to be any point not in R, S_3 takes the form 7. This is a Σ-space; the two planes constitute the degenerate quadric in which S_3 intersects V.

Suppose S_3 contains ρ and a point P_4 on V and not on ρ. The line p_4 is not in R, for if it were, S_3 would be a Σ-space and would intersect V in a quadric consisting of two planes since it contains ρ and an additional point. Hence, p_4 intersects R in a point. The point cannot be A_1, for then S_3 would lie wholly on R. The point may be taken to be A_2. A_5 may be taken on p_4, and then S_3 has the form 8. This space intersects V in the plane ρ and the line $l = m = 0$.

Any other S_3 which contains ρ can have no further point on V. Let S_3 contain ρ and a point P_4 not on V. P_4 determines a three-space R_4 in X. R_4 and R cannot coincide, for then S_3 would be a Σ-space intersecting V in a quadric consisting of the plane counted twice, and P_4 would be in each space tangent to V at a point of ρ. There is no such point not on V. Therefore R_4 intersects R in a plane σ. If σ does not pass through A_1, the plane π on V whose points represent the lines of σ does not intersect ρ. The polar of P_4 with respect to V intersects π in a line. If Q_4 is selected in π not on the polar of P_4, then the line P_4Q_4 will intersect V in a second point Q_4'. q_4 lies in σ, and q_4', which lies in R_4, intersects σ. A_2 and A_3 may be taken on q_4, A_4 on q_4' and σ, and A_5 on q_4'. Then S_3 will take the form 9.

Next, suppose the plane σ of the last paragraph passes through A_1. The planes ρ and π will intersect in a line λ_1. The polar of P_4 intersects π in a line λ_2. Suppose λ_1 and λ_2 coincide. Then the point P_4 is in each tangent space to V at a point of λ_1 which we may take to be P_1P_2. It then follows that $P_4 = 0, 0, 0, a_4, a_5, 0, 0, 0, 0, 0$. A choice of the unit point puts S_3 in the form 9′. This space is then tangent to V at every point of P_1P_2.

Space 9 is not in the space tangent to V at any point of V; however, to show that 9 and 9′ are different, we need only to note that in 9 the point P_4 is not in the space tangent to V at any point of ρ.

There is one further possibility to consider. If the lines λ_1 and λ_2 in the plane π do not coincide, they intersect in a point which we may take to be P_1. P_4 would be in the space tangent to V at P_1; hence P_4 =

$0, 0, 0, a_4, a_5, a_6, a_7, 0, 0, 0$. By examining S_3 for points on V, it is found that unless $a_7 = 0$ there is a point $k, l, m, n = 0, a_4a_5, a_4a_6, a_7$ on V and not on ρ. In that case the space is 8 with an additional line on V; if $a_7 = 0$ and $a_6 \neq 0$, P_4 can be selected so that λ_1 and λ_2 coincide, showing S_3 to be 9′.

An S_3 which contains a plane on V and is not one of the foregoing contains a plane of the first type. A coordinate system can be chosen so that the plane is $n = 0$ of 10 and 11, A_1, A_2, and A_3 being arbitrary independent points of the plane σ in X whose lines are imaged on the plane ρ in S_3. If P_4 is a point of S_3 on V and not in ρ, then the line p_4 in X may or may not intersect σ. If it intersects σ, we have S_3 in a five-space Σ given by a three-space in X; S_3 then intersects V in two planes, giving 7, or else lies wholly on V. If p_4 does not intersect σ, A_4 and A_5 may be selected on p_4 and we have 10.

Finally, suppose S_3 contains the plane ρ of the last paragraph and no other point of V. Let P_4 be a point of S_3 not on ρ. The three-space R_4 intersects σ in a line, for if σ were in R_4, S_3 would be in the five-space determined by R_4 and would be 7. This line of intersection of σ and R is imaged on V in a point of ρ which is such that the line joining it to P_4 is tangent to V, being in a Σ and having no other point on V. If Q_4 is selected in R_4 so that q_4 intersects the above line, then q_4' will intersect the above line also. These intersections may be taken to be A_2 and A_3, respectively. If then A_4 is taken on q_4 and A_5 on q_4', S_3 will have the form 11.

We have considered all the possibilities for S_3 with a plane on V.

(iv) *Three-spaces containing at least two rulings but no plane of V.*

12. $k, l, 0, 0, 0, m, 0, 0, 0, n$.
13. $k, l, 0, n, 0, m, n, 0, 0, 0$.
14. $k, l, 0, 0, 0, m, 0, 0, n, n$.
15. $k, l, 0, n, 0, m, 0, n, 0, 0$.
16. $k, l, 0, n, 0, m, n, n, 0, 0$.
17. $k, l, 0, 0, n, m, 0, 0, 0, n$.
18. $k, l, 0, n, n, m, 0, 0, 0, 0$.

The first two have three rulings on V; in 13 the rulings pass through a point; in 12 they do not. In each of the rest there are two intersecting rulings; 14 contains one additional point and the others none. 15 and 18 are τ-spaces; 16 and 17 are not. 18 is in the space tangent to V at a point of intersection with V; 15 is in the space tangent to V at a point not in S_3. The distinction between 16 and 17 is more difficult; it is shown at the end of this section.

If S_3 contains rulings of V but no planes or nondegenerate conics, the number of rulings cannot be greater than three since otherwise S_3 would contain planes with four or more discrete points on V. If S_3 contains three rulings of V, each ruling must intersect another for otherwise S_3 would contain planes on one ruling intersecting V in two additional points and no such planes exist.

Suppose S_3 contains three rulings which do not pass through a point. Denote the rulings by l_1, l_2, and l_3, and let l_1 and l_2 intersect. l_1 and l_2 are images of two pencils of lines in X whose planes have a line in common. The planes of the pencils lie in a three-space R. An obvious choice of the coordinate system in X gives the plane l_1l_2 the form of the plane $n = 0$ in the spaces above. The line l_3 intersects one of the lines l_1 and l_2; we may assume the intersection to be P_3. Any point on l_3 is in the space tangent to V at P_3, and hence its coordinates satisfy $a_2 = a_4 = a_9 = 0$; and since it is a point of V,

$$a_1a_8 + a_3a_5 = 0, \qquad a_1a_{10} - a_3a_7 = 0, \qquad a_5a_{10} + a_7a_8 = 0.$$

Hence, $a_3/a_1 = a_{10}/a_7 = -a_8/a_5 = r$. The line in X which is imaged on this point is $\begin{cases} a_1, 0, -a_5, 0, -a_7 \\ 0, 1, 0, r, 0 \end{cases}$. The lines p_1, p_2, p_3 are

$$p_1 = \begin{cases} 1, 0, 0, 0, 0 \\ 0, 1, 0, 0, 0 \end{cases}, \qquad p_2 = \begin{cases} 1, 0, 0, 0, 0 \\ 0, 0, 1, 0, 0 \end{cases}, \qquad p_3 = \begin{cases} 0, 1, 0, 0, 0 \\ 0, 0, 0, 1, 0 \end{cases}.$$

A change of coordinates: $A_i' = A_i$, $i = 1, 2, 3$, $A_4' = (1/r)A_2 + A_4$, $A_5' = a_1A_1 - a_5A_3 - a_7A_5$ leaves P_1, P_2, P_3 unchanged, but makes the point on l_3 take the form $P_4 = 0, 0, 0, 0, 0, 0, 0, 0, 0, 1$. S_3 has the form 12.

When S_3 contains three rulings which pass through a point, P_1, P_2, P_3 may be taken as above, and the third ruling passes through P_1. For any point P_4 on this ruling, we have $a_8 = a_9 = a_{10} = 0$, and

$$a_2a_6 - a_3a_5 = 0, \qquad a_2a_7 - a_4a_5 = 0, \qquad a_3a_7 - a_4a_6 = 0.$$

From this $a_5/a_2 = a_6/a_3 = a_7/a_4 = r$. $P_4 = \begin{cases} 1, r, 0, 0, 0 \\ 0, 0, a_2, a_3, a_4 \end{cases}$. A change of coordinates: $A_i' = A_i$, $i = 1, \cdots, 4$, $A_5' = a_2A_3 + a_3A_4 + a_4A_5$ and a proper selection of the unit point give the form 13.

A three-space S_3 containing two skew lines l_1 and l_2 which are rulings of V has three or more lines which are rulings of V. The lines l_1 and l_2 determine two pencils of lines in X lying in two planes σ_1 and σ_2. If the planes intersect in a line, they lie in a three-space, and S_3 is a Σ-space. S_3 is of the form 1 and intersects V in a nondegenerate ruled quadric. If σ_1 and σ_2 intersect in a point, that point cannot be the vertex of either pencil, for otherwise one line of one pencil would intersect every line of the other and S_3 would contain a plane and a line of V; it would be 7 or 8. The remaining possibility allows us to take the pencils in the planes $A_1A_2A_4$ and $A_1A_3A_5$ with vertices at A_2 and A_3. Then S_3 is $k, l, 0, 0, 0, m, 0, 0, n, 0$ which has three rulings and is 12.

The remaining S_3's in this section contain two rulings, and the two rulings intersect. The plane of S_3 containing the rulings is the plane $n = 0$ above. We designate this Σ-plane by ρ and the corresponding three-space in X by R.

Let S_3 contain an additional point P_4 on V. The line p_4 intersects R in a point. This point is not in either of the planes determined by the lines of ρ on V, for if p were the line of the pencil through that point, PP_4 would be a pencil of points on V representing the pencil of lines pp_4, and S_3 would contain a third line of V. A line may be taken through the intersection of p_4 and R intersecting the planes of the two pencils in points which may be taken for A_3 and A_4 without changing the coordinates of P_1, P_2, or P_3. Then A_5 may be selected on p_4 not in R. S_3 becomes 14.

Any other S_3 which contains two rulings of V contains the plane $n = 0$ above and a point

$$P_4 = 0, 0, a_3, a_4, a_5, 0, a_7, a_8, a_9, a_{10}.$$

To this S_3 we apply transformation T_{15} (page 658). This transforms ρ into itself, and transforms P_4 into $P_4' = a_1', \cdots, a_{10}'$. There is in S_3 a point for which $a_1' = a_2' = a_6' = 0$, and

$$a_3' = a_3 - a_4h - a_8a + a_9ah + a_{10}(e - af), \qquad a_4' = a_4 - a_9a - a_{10}c,$$
$$a_5' = a_5 - a_7g + a_8d + a_9(f - dh) + a_{10}dg, \qquad a_7' = a_7 - a_9b - a_{10}a,$$
$$a_8' = a_8 - a_9h + a_{10}g, \qquad a_9' = a_9, \qquad a_{10}' = a_{10}.$$

(a) Suppose $a_9a_{10} \neq 0$. Then since b appears only in a_7', c only in a_4', d in a_5', and e in a_3', we may make $a_3' = a_4' = a_5' = a_7' = a_8' = 0$ by selecting a, f, g, h, to satisfy $a_8 - a_9h + a_{10}g = 0$ and solving for b, c, d, and e. This gives 14 again.

(b) Suppose $a_9 = 0$, $a_{10} \neq 0$. a and g can be selected to make $a_7' = a_8' = 0$; then if $g \neq 0$, d, c, e can be selected to make $a_5' = a_4' = a_3' = 0$. This is 12 again. If $g = 0$, we get $P_4 = 0, 0, 0, 0, 1, 0, 0, 0, 0, 1$, which is 17.

(c) Suppose $a_9 \neq 0$, $a_{10} = 0$. b and h can be selected to make $a_7' = a_8' = 0$; then f and a can be selected to make $a_5' = a_4' = 0$. a_3' is then determined; it cannot be zero since P_4 is not on V, but a choice of the unit point will make $a_3' = a_9'$. We shall postpone the identification of $S_3 : k, l, n, 0, 0, m, 0, 0, n, 0$.

(d) Suppose $a_9 = a_{10} = 0$.

(i) $a_7a_8 \neq 0$. a_3' and a_5' can be made zero. a_4', a_7', a_8' cannot be changed. We have the possibilities:

$$P_4 = 0, 0, 0, a_4, 0, 0, a_7, a_8, 0, 0. \qquad \text{This is space 16.}$$

$$P_4 = 0, 0, 0, 0, 0, 0, a_7, a_8, 0, 0.$$

(ii) $a_7 = 0$, $a_8 \neq 0$. d and a can be selected to make $a_5' = a_3' = 0$.

$$P_4 = 0, 0, 0, 1, 0, 0, 0, 1, 0, 0.$$

(iii) $a_7 \neq 0$, $a_8 = 0$. g can be selected to make $a_5' = 0$. Then if $a_4 \neq 0$, a_3' can be made zero, and P_4 is on V. Hence $a_4 = 0$ and

$$P_4 = 0, 0, 1, 0, 0, 0, 1, 0, 0, 0.$$

(iv) $a_7 = a_8 = 0$. Then $a_4 \neq 0$ since R_4 is not R. $a_5 \neq 0$, since P_4 is not on V. a_3' can be made zero. S_3 is 18.

The transformation T_{15} is the most general collineation of X that leaves the form of ρ unchanged and also leaves A_1 and A_2 unchanged. A collineation which interchanges A_1 and A_2, and of course interchanges the pencils with vertices at A_1 and A_2, leaves ρ unchanged. If A_3 and A_4 are interchanged as well as A_1 and A_2, the pencils will be interchanged. This transformation puts the space of (c) above into that of (b); it puts the second space of (d, i) into the space of (d, ii), which is 15; and it puts the space of (d, iii) into that of (d, iv).

To distinguish between spaces 16 and 17 we note that any point k, l, m, n of 16 determines in X the three-space

$$n^2x_1 - n^2x_2 + mnx_3 + lnx_4 + (kn - lm)x_5 = 0;$$

any point of 17 determines the three-space

$$n^2x_1 - lnx_2 + knx_3 - lmx_5 = 0.$$

All the spaces in X determined by points of 16 pass through 1, 1, 0, 0, 0, which is a point of the special line, the line in both pencils determined by the intersections of S_3 and V. All the spaces in X determined by points of 17 pass through 0, 0, 0, 1, 0, which is not on the special line.

(v) *Three-spaces containing one ruling of V.*

19. $m, 0, 0, k, 0, 0, l, n, k, l.$
20. $k, l, 0, 0, 0, n, 0, 0, n, m.$
20′. $k, l, 0, n, n, 0, 0, 0, 0, m.$
20″. $k, l, 0, n, n, n, 0, 0, 0, m.$
21. $k, l, m, n, 0, rn, m, 0, 0, 0.$
21′. $k, l, m, n, m, n, 0, 0, 0, 0.$
22. $k, l, 0, 0, 0, n, m, m, rn, 0.$
23. $k, l, m, 0, m, 0, n, n, 0, 0.$
24. $k, l, 0, 0, n, 0, m, m, 0, -rn.$
25. $k, l, m, 0, n, 0, 0, 0, m, n.$
26. $k, l, n, 0, 0, 0, m, m, n, 0.$
27. $k, l, m, n, 0, 0, m, n, 0, 0.$

Each r above is a not-square.

Space 19 has a ruling and two points on V. Only spaces 20, 20′, and 20″ have the ruling and one additional point on V; space 20 contains a line tangent to V at P_3; the other two do not have such a tangent; space 20′ contains the Σ-plane $P_1P_2P_4$; space 20″ contains no Σ-plane. Spaces 21 and 21′ are in the space tangent to V at P_1, 21′ having a plane tangent to V at P_2, 21 having no such plane; space 22 is not in a space tangent to V at a point of P_1P_2; it is in the space tangent to V at a point not in it; none of the others is a τ-space. To distinguish among the remaining five we state some geometric facts that are obviously sufficient, and then to show how these facts may be established we carry out in detail the argument for space 23.

In space 23 every plane on P_1 is a τ-plane, and every line in $P_1P_2P_3$ is tangent to V at its intersection with P_1P_2. The space 24 contains a τ-plane $n = 0$, which is in the space tangent to V at 0, 0, 0, 0, 1, 0, 0, 0, 0, 0; it contains a pencil of τ-planes on P_3P_4; P_3P_4 is a Σ-line which does not intersect the ruling. All the τ-planes pass through P_3, but not every plane on P_3 is a τ-plane. Space 25 contains a pencil of τ-planes on P_2P_3 and no others. Space 26 contains a pencil of τ-planes on P_1P_2; the plane $m = 0$ is tangent to V at P_2; there is no plane tangent at any other point of P_1P_2. Space 27 contains a pencil of τ-planes on P_1P_2; $m = 0$ is tangent to V at P_2, and $n = 0$ at P_1.

We now establish the facts stated for space 23. Let $B = b_1, b_2, \cdots, b_{10}$ be a point of V. The space tangent to V at B is given on page 664. Substituting in these equations the coordinates of a point in space 23, we get five linear equations in k, l, m, n with the following matrix of coefficients:

$$M = \begin{bmatrix} b_8 & -b_6 & b_3 + b_5 & b_1 \\ b_9 & -b_7 & b_4 & -b_2 \\ b_{10} & 0 & -b_7 & -b_3 \\ 0 & b_{10} & -b_9 & b_4 \\ 0 & 0 & b_{10} & b_7 + b_8 \end{bmatrix}.$$

If S_3 were a τ-space, it would be possible to select B so that the rank of M would be zero; this would require all the b_i's to be zero. Hence, S_3 is not a τ-space. If the rank of M is one, the space tangent to V at the point B will intersect S_3 in a plane. This requires $b_4 = b_7 = b_8 = b_9 = b_{10} = 0$; hence all the τ-planes pass through P_1. In addition, we should have either (a) $b_6 = 0$ and $b_3 + b_5 = 0$, or (b) $b_2 = b_3 = 0$. In case (a) the τ-plane is $n = 0$; it is a Σ-plane. In case (b) the τ-plane is $-b_6 l + b_5 m + b_1 n = 0$. Since b_1, b_5, b_6 are arbitrary, every plane on P_1 is a τ-plane.

We now show that the above are the only three-spaces meeting V only in one ruling and possibly some isolated points. S_3 can have no more than two isolated points on V, for if it had three, the plane on them would intersect the ruling or contain it, and no such plane exists. If S_3 contains two points of V besides the ruling, the line joining the two points must be skew to the ruling.

Let S_3 contain the ruling P_1P_2 and the two points P_3 and P_4 on V. Then in X the lines p_3 and p_4 are skew to each other, and both are skew to the plane of the pencil p_1p_2. The lines p_3 and p_4 determine a three-space R which intersects the plane p_1p_2 in a line λ. The line λ may belong to the pencil p_1p_2, or it may not. If λ does not belong to the pencil, it intersects the two lines p_1 and p_2 in two distinct points, O_1 and O_2 respectively. The plane p_3O_1 intersects p_4 in a point we take to be A_3, and A_3O_1 intersects p_3 in a point we take to be A_1. By means of O_2 we determine A_4 on p_4 and A_2 on p_3. O_1 may be taken to be $A_1 + A_3$, and O_2 to be $A_2 + A_4$. A_5 may be taken to be the vertex of the pencil p_1p_2. Then S_3 will be 19.

If λ belongs to the pencil p_1p_2, we may suppose that it coincides with p_1. We may take O_1 to be the vertex of the pencil and O_2 any other point on p_1.

We may proceed as above and finally take A_5 to be a point of p_2. Then S_3 will have the form

$$k+m, 0, k, l, -k, 0, 0, k+n, l, 0.$$

It is easy to verify that this S_3 has a conic and a line on V and hence is space 4.

We consider an S_3 which intersects V only in the line P_1P_2 and the additional point P_3. In X the plane of the pencil p_1p_2 is skew to the line p_3. The plane of the pencil may be taken as $A_1A_2A_3$, with A_1 the vertex of the pencil, and A_4 and A_5 may be taken on p_3. The plane $P_1P_2P_3$ will then have the form $k, l, 0, 0, 0, 0, 0, 0, 0, m$. In S_3 there is the point

$$P_4 = 0, 0, a_3, a_4, a_5, a_6, a_7, a_8, a_9, 0.$$

The space tangent to V at P_3 does not intersect the line P_1P_2, so its intersection with S_3 will be the line P_3P_4 if $a_5 = 0$, or will be P_3 alone if $a_5 \neq 0$. The space tangent to V at a point of P_1P_2 does not contain P_3, and hence its intersection with S_3 will be at most the plane $P_1P_2P_4$, but may be only the line P_1P_2.

Now suppose $a_5 = 0$, so that P_3P_4 is a tangent. Conditions that $P_1P_2P_4$ be tangent to V at the point $aP_1 + bP_2$ are

$$a_8a - a_6b = 0, \qquad a_9a - a_7b = 0.$$

If a and b exist so that these equations are satisfied we must have $a_6a_9 - a_7a_8 = 0$. In that case S_3 has an additional point[12] on V. So an S_3 with only a line and a point on V, with a line tangent to V at P_3, has P_4 with $a_6a_9 - a_7a_8 \neq 0$. The three-space R_4, determined by P_4, does not contain A_1. Hence the intersection of R_4 and p_1p_2, which is a line, may be taken to be A_2A_3; we denote the line by q_4. The corresponding point Q_4 on V is such that P_4Q_4 intersects V in a second point unless it is a tangent. Suppose P_4Q_4 is a tangent. Then since $Q_4 = 0, 0, 0, 0, 1, 0, 0, 0, 0, 0$, P_4 must have $a_3 = a_4 = a_{10} = 0$. Since $a_6a_9 - a_7a_8 \neq 0$, we may select

$$A_4' = a_6A_4 + a_7A_5 \quad \text{and} \quad A_5' = a_8A_4 + a_9A_5.$$

Then P_4 becomes 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, and S_3 is the space 20.

The final supposition, that led to space 20, was that P_4Q_4 is tangent to V. If this were not so, there would exist a t such that $Q_4 + tP_4$ would be on V. The B_5 for this point is $(a_6a_9 - a_7a_8)t^2$, which requires $t = 0$. We have thus shown that the only S_3 with a line and a point on V and with a line tangent to V at the isolated point is 20; and space 20 has no plane tangent to V at a point of the line on V.

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[12] The additional point is

$$a_6, a_8, a_3, a_4, 0, a_6, a_7, a_8, a_9, (a_3a_7 - a_4a_6)/a_6, \qquad \text{if } a_6 \neq 0;$$

$$a_7, a_9, a_3, a_4, 0, 0, a_7, 0, a_9, a_3, \qquad \text{if } a_6 = 0,\ a_7 \neq 0;$$

$$0, 1, a_3, a_4, 0, 0, 0, a_8, a_9, (a_3a_9 - a_4a_8) \qquad \text{if } a_6 = a_7 = 0.$$

We now consider S_3 with a point and a line on V which contains a plane tangent to V at every point of the line; such a plane is a Σ-plane in the five-space Σ determined by any point in the plane not on V. For the point P_4, which is in the Σ-plane, has $a_6 = a_7 = a_8 = a_9 = 0$. S_3 contains no tangent line at P_3, so $a_5 \neq 0$. $P_4 = 0, 0, a_3, a_4, a_5, 0, 0, 0, 0, 0$. A_5 can be moved along p_3 so that S_3 becomes 20′.

Suppose S_3 contains a plane tangent to V at one and only one point of P_1P_2. The point may be taken to be P_1. Then P_4 has $a_8 = a_9 = 0$, $a_5 \neq 0$. $P_4 = 0, 0, a_3, a_4, a_5, a_6, a_7, 0, 0, 0$. R_4 intersects p_3 at $0, 0, 0, a_3, a_4$; using this point for A_5 we reduce a_3 to zero. We can now move A_4 to $0, 0, 0, a_6, a_7$ and remove a_7, if $a_6 \neq 0$. In that case S_3 is 20″.

If $a_6 = 0$ just above, S_3 intersects V in another point, namely,

$$0,\ a_4a_5/a_7,\ 0,\ a_4,\ a_5,\ 0,\ a_7,\ 0,\ 0,\ 0.$$

We will now show that this list of S_3's containing a point and a line on V, and no other point on V, is complete by showing that such an S_3 having no plane tangent to V at a point of the line is 20. As shown above, the fact that S_3 contains no plane tangent to V at a point of P_1P_2 requires P_4 to be such that $a_6a_9 - a_7a_8 \neq 0$. Now, making use of T_9, the point $P_4 = 0, 0, a_3, a_4, a_5, a_6, a_7, a_8, a_9, 0$ is changed to $P_4' = a_1', a_2', \cdots, a_{10}'$,

$$a_1' = a_5(ad - c), \qquad a_2' = a_5(a - bc),$$
$$a_3' = a_3 + a_4f + a_6a + a_7af + a_8c + a_9cf,$$
$$a_4' = a_3e + a_4 + a_6ae + a_7a + a_8ce + a_9c,$$
$$a_5' = a_5(1 - bd), \qquad a_6' = a_6 + a_7f + a_8d + a_9df,$$
$$a_7' = a_6e + a_7 + a_8de + a_9d, \qquad a_8' = a_6b + a_7bf + a_8 + a_9f,$$
$$a_9' = a_6be + a_7b + a_8e + a_9, \qquad a_{10}' = 0.$$

Since $a_6a_9 - a_7a_8 \neq 0$, a and c in T_9 can be found to make $a_3' = a_4' = 0$. Then the change of A_4 and A_5 on p_3 that gave 20, and if necessary a change of P_4' in the plane $P_1P_2P_4'$ to make $a_1' = a_2' = 0$, will give $P_4 = 0, 0, 0, 0, a_5, 1, 0, 0, 1, 0$. Examining S_3 for points on V, we find the additional point $k = l = a_5m - n = 0$, which is one too many points unless $a_5 = 0$.

We consider a space S_3 which contains the ruling P_1P_2 and no other point of V, and which lies in the space tangent to V at P_1; no S_3 with only one line on V could lie in more than one such tangent space. Let P_3 and P_4 be two points of S_3 which are on a line skew to P_1P_2. The plane $P_1P_3P_4$ is tangent to V since every line in it through P_1 is a tangent. P_3 and P_4 determine two three-spaces R_3 and R_4 in X. R_3 and R_4 may or may not be distinct, but both certainly contain the line p_1. If R_3 and R_4 coincide, then $P_1P_3P_4$ is a plane in a five-space Σ, and it has one point on V. This is plane 6 of the list of planes. Coordinates can be selected so that the plane is $k, 0, m, n, 0, rn, m, 0, 0, 0$, r not a square. The space $R_3 = R_4$ is $x_5 = 0$.

The line p_1 is in R_3, and consequently the vertex of the pencil p_1p_2 is in R_3. We now show that coordinates can be selected so that P_1, P_3, and P_4 have the above form and at the same time A_1 is at the vertex of the pencil p_1p_2. Let σ be an arbitrary plane in R_3 on the line p_1, and let π be the image on V of σ. The polar spaces of P_3 and P_4 with respect to V cut π in two distinct lines which intersect at P_1; let the lines be respectively λ_3 and λ_4. Q_3 and Q_4 may be selected respectively on λ_4 and λ_3 to give the above form of the coordinates of P_1, P_3, and P_4. The point A_1 is the intersection of p_1, q_3, and q_4'; A_2 is the intersection of p_1, q_3', and q_4. Since A_1 and A_2 enter symmetrically, if either is the vertex of the pencil p_1p_2, we may take it to be A_1. If neither is the vertex of the pencil, we may move P_3 along P_3P_4. The line λ_3 then swings in π about P_1, and the intersection of q_3 and p_1 moves along p_1. Thus we may move A_1 to the vertex of the pencil p_1p_2.

Now, the plane of the pencil p_1p_2 is not in R_3, for otherwise S_3 would be in the space tangent to V at each point of P_1P_2. Therefore the line p_2 intersects R_3 only at A_1, and any other point on it may be taken for A_3. S_3 is thus seen to be 21.

If there were any other S_3 intersecting V only in a ruling and tangent to V at a point of it, then for no selection of P_3 and P_4 would R_3 and R_4 coincide. For any selection of P_3 and P_4 the line p_1 would be in both R_3 and R_4. Coordinates can be selected so that

$$P_1 = 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, \qquad P_3 = 1, 0, 0, 0, 0, 0, 0, 1, 0, 0,$$
$$P_4 = 0, 1, 0, 0, 0, 0, 1, 0, 0, 0.$$

P_2 is in the space tangent to V at P_1 and hence has $a_3 = a_4 = a_{10} = 0$. We may suppose that the vertex of the pencil p_1p_2 is at A_2 (see, for example, the change in A_3 in deriving T_3). p_2 will be a line joining A_2 to a point of $A_1A_3A_4A_5$. Hence,

$$P_2 = a_1, 0, 0, 0, a_5, a_6, a_7, 0, 0, 0.$$

We may move P_2 along the line P_1P_2, and so we may assume $a_5 = 0$. Then any point in S_3 is

$$P = m + a_1l, n, 0, 0, k, a_6l, n + a_7l, m, 0, 0.$$

For this point we have

$$\begin{aligned} B_1 &= m^2 + a_1lm - a_6ln, \\ B_2 &= -n(n + a_7l), \\ B_3 &= 0, \\ B_4 &= 0, \\ B_5 &= m(n + a_7l). \end{aligned}$$

$n + a_7l = 0$ gives a plane every point of which determines the three-space $x_5 = 0$, which is R_3. Thus PP_3 is a Σ-line which does not intersect P_1P_2 for arbitrary m unless $a_7 = 0$. If $a_7 = 0$, $a_1 \neq 0$, S_3 has another point on V, and hence is no new space. If $a_1 = a_7 = 0$, an obvious change of coordinates puts S_3 in the form $21'$.

We now consider an S_3 which intersects V only in the line P_1P_2, which is in the space tangent to V at a point but not in the space tangent to V at a point of P_1P_2. We may select points P_3 and P_4 in S_3 so that P_3P_4 is skew to P_1P_2 and such that R_3 and R_4 are distinct. This follows from the fact that since S_3 is not in the tangent space at P_1, it can contain at most a plane which is in that tangent space, and the plane contains P_2. P_3 can be selected so that P_1P_3 is not a tangent, and then P_1 will not be in the five-space Σ_3 determined by P_3. So if P_3 and P_4 determine the same three-spaces in X, then $P_1 + P_3$ and P_4 will determine distinct three-spaces.

The line P_3P_4 determines a point M on V such that MP_3P_4 is tangent to V at M; P_1P_2 is in the space tangent to V at M. P_1P_2 does not pass through M, since S_3 is not in the space tangent to V at a point of P_1P_2. The plane P_1P_2M lies wholly on V. Two possibilities arise: (a) the lines p_1, p_2, and m lie in a plane; or (b) the vertex of the pencil p_1p_2 is on m.

In case (a) the plane of intersection of R_3 and R_4 and the plane of the pencil p_1p_2 intersect in the line m. We may take the vertex of the pencil to be A_1, and we may take A_2 and A_3 to be respectively the intersections of m with p_1 and p_2. Then P_3 and P_4 will be in the space tangent to V at $M = 0, 0, 0, 0, 1, 0, 0, 0, 0, 0$; hence, for each we have $a_3 = a_4 = a_{10} = 0$. Now, we may determine two other points for P_3 and P_4, each of the form $0, 0, 0, 0, a_5, a_6, a_7, a_8, a_9, 0$. The new line P_3P_4 is a Σ-line; the corresponding three-space in X is $x_1 = 0$. Since coordinates of P_3 and P_4 can be put in canonical form by transformations in the space $x_1 = 0$, and since P_1 and P_2 are arbitrary points of P_1P_2, S_3 becomes 22.

In case (b) the vertex of the pencil p_1p_2 is on m. We may take the vertex to be A_1, the plane of the pencil to be $A_1A_2A_3$, and the line m to be A_1A_4. Then P_3 and P_4, being in the space tangent to V at M, will each have $a_5 = a_7 = a_9 = 0$; moreover, for each we may take $a_1 = a_2 = 0$, since each may be moved in the plane determined by it and the line P_1P_2 without affecting the relations in consideration. Then on the line joining P_3 and P_4 there will be a point $0, 0, a_3, 0, 0, a_6, 0, a_8, 0, a_{10}$ which is on V. Hence, case (b) gives no S_3 with the required properties.

None of the remaining S_3's with a ruling on V is in the space tangent to V at a point; the largest intersection of S_3 with a tangent space would be a τ-plane. S_3 may have several such planes.

We consider first the possibility that S_3 contains a τ-plane $P_1P_2P_3$, where P_1P_2 is a ruling of V and the plane is tangent to V at every point of P_1P_2; the plane is a Σ-plane. The line joining P_3 to any point of P_1P_2 is tangent to V, and hence the three-space R_3 contains the plane of the pencil p_1p_2. If P_4 is any point of S_3 not in $P_1P_2P_3$, R_4 cannot be R_3, for otherwise P_1P_4 would be a tangent and S_3 would be in the space tangent to V at P_1. Coordinates can be selected so that

$$P_3 = 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, \qquad P_4 = 0, 0, 0, 0, 0, 0, 1, 1, 0, 0.$$

The line P_3P_4 is in the space tangent to V at $M = 0, 0, 0, 0, 0, 1, 0, 0, 0, 0$. The tangent space at M is $a_2 = a_4 = a_9 = 0$; the five-space Σ determined by P_3 is $a_4 = a_7 = a_9 = a_{10} = 0$. Since S_3 is not in the tangent space at M, not both P_1 and P_2 can have $a_2 = 0$; one point of P_1P_2 does have $a_2 = 0$, and we may take it to be P_1. Hence, S_3 contains a τ-plane $P_1P_3P_4$, which is not tangent at P_1 but is in the space tangent to V at M. Therefore, p_1 intersects m; p_2 does not intersect m, for otherwise S_3 would be in the space tangent to V at M. The plane σ of intersection of R_3 and R_4 contains m. The point P_4 can be selected on P_3P_4 so that q_3 passes through the intersection of p_1 and m. The plane p_1p_2 is not σ since p_2 is not in R_4. The line P_1P_3 is a tangent; p_1 intersects q_3 and hence must intersect q_3'. q_3' may be moved in the pencil $q_3'm$ until it passes through the vertex of the pencil p_1p_2; then A_1 may be moved along q_3' to this point. P_1 then becomes $1, 0, 0, 0, 0, 0, 0, 0, 0, 0$. P_3 is in the space tangent to V at P_2. p_2 intersects q_3' and hence must also intersect q_3. Therefore, the intersection of p_1p_2 and σ is q_3, and $P_2 = 0, 1, 0, 0, 0, 0, 0, 0, 0, 0$. So an S_3 containing only a ruling on V, containing a τ-plane tangent at every point of the ruling, and not in the space tangent to V at any point, is 23.

A Σ-plane intersects V in at least one point; any line in the plane which passes through the point on V is tangent to V at the point. Hence if S_3 intersects V in a ruling P_1P_2 and no other point, and if S_3 contains a Σ-plane, the Σ-plane contains P_1P_2, or else S_3 is in the tangent space to V at the point where P_1P_2 intersects the Σ-plane. Therefore, no other S_3 than those already considered contains a ruling and a Σ-plane.

Let us suppose that S_3 contains two τ-planes which intersect in a line skew to P_1P_2. The line of intersection may be taken to be P_3P_4. The line is a Σ-line, since otherwise it could not be in the spaces tangent to V at two points. The two τ-planes intersect P_1P_2 and may be taken to be $P_1P_3P_4$ and $P_2P_3P_4$. Neither p_1 nor p_2 can be in either of the three-spaces R_3 or R_4, for then P_1, P_2, P_3, and P_4 would be in the space tangent to V at P_1 (or P_2). The plane of the pencil p_1p_2 intersects R_3 in a line which is not a line of the pencil. This line may be taken to be q_3; then q_3' is determined, and q_4 and q_4' may be selected so that P_3 and P_4 are in canonical form (for a Σ-line which does not intersect V). The vertex of the pencil p_1p_2 is outside R_3 and may be taken to be A_1. Then S_3 has the form 24.

To help with the remaining cases we prove:

Every S_3 which contains a ruling and no other point of V contains at least $p + 1$ τ-planes.

Unless S_3 contains a pencil of τ-planes on the ruling P_2P_3, it will contain a plane on P_2P_3 which has no other point on V and which is not a τ-plane.

Coordinates may be selected so that this plane is $k, 0, 0, 0, 0, 0, 0, k, l, m$. (This is number 22 of the list of planes.) Then in S_3 we may select the point $P_4 = a_1, a_2, \cdots, a_7, 0, 0, 0$. Any point of S_3 is

$$P = k + a_1n, a_2n, a_3n, \cdots, a_7n, k, l, m.$$

The conditions that $B = b_1, b_2, \cdots, b_{10}$ be a point of V such that the space tangent to V at B intersect S_3 in a plane give a set of five linear congruences in k, l, m, n which has for a matrix of coefficients

$$\begin{bmatrix} b_1 + b_8 & 0 & 0 & a_1b_8 - a_2b_6 + a_3b_5 + a_5b_3 - a_6b_2 \\ b_9 & b_1 & 0 & a_1b_9 - a_2b_7 + a_4b_5 + a_5b_4 - a_7b_2 \\ b_{10} & 0 & b_1 & a_1b_{10} - a_3b_7 + a_4b_6 + a_6b_4 - a_7b_3 \\ b_4 & -b_3 & b_2 & a_2b_{10} - a_3b_9 + a_4b_8 \\ b_7 & -b_6 & b_5 & a_5b_{10} - a_6b_9 + a_7b_8 \end{bmatrix},$$

and it must be possible to select B so that the rank of the matrix is 1. If the matrix has rank 1, $b_1 = 0$; then since B is on V,

$$b_2b_6 - b_3b_5 = 0, \qquad b_2b_7 - b_4b_5 = 0, \qquad b_3b_7 - b_4b_6 = 0.$$

Unless $b_2 = b_3 = b_4 = 0$, we have $b_5 = rb_2$, $b_6 = rb_3$, $b_7 = rb_4$. Under these conditions the rank of the matrix is 1 if the first three elements in the fourth column are zeros. These give

$$\begin{aligned} (a_3r - a_6)b_2 + (a_5 - a_2r)b_3 \qquad\qquad\qquad &= 0, \\ (a_4r - a_7)b_2 \qquad\qquad + (a_5 - a_2r)b_4 &= 0, \\ (a_4r - a_7)b_3 + (a_6 - a_3r)b_4 &= 0. \end{aligned}$$

The determinant of the matrix of coefficients of the b_i's is zero. Hence, for any set of a's there is a τ-plane $b_4k - b_3l + b_2m = 0$, where

$$b_2:b_3:b_4 = a_2r - a_5:a_3r - a_6:a_4r - a_7.$$

These are not all zero since P_4 is not on V. There is one for every r, and hence there are $p + 1$ of them. The τ-planes all pass through the intersection of the planes

$$a_4k - a_3l + a_2m = 0 \quad \text{and} \quad a_7k - a_6l + a_5m = 0$$

and hence constitute a pencil. A necessary and sufficient condition that this line of intersection have a point in common with P_2P_3, the ruling of V, is that $a_2a_6 - a_3a_5 = 0$. When the condition is satisfied, the point of intersection of the axis of the pencil of τ-planes and the ruling is $k, l, m, n = 0, a_2, a_3, 0$. The line in X corresponding to this point is $\begin{cases} 0, 0, a_2, a_3, 0 \\ 0, 0, 0, 0, 1 \end{cases}$. The three-space R_4 in X, determined by P_4, is

$$(-a_3a_7 + a_4a_6)x_3 - (-a_2a_7 + a_4a_5)x_4 = 0.$$

Hence the line in X is in R_4, and the axis of the pencil of τ-planes is a Σ-line with a point on V and is therefore a tangent to V at that point. The axis of the pencil of τ-planes and the ruling lie in a plane tangent to V at their intersection.

Any other S_3 which intersects V in a ruling only will consequently contain a pencil of τ-planes whose axis is either a Σ-line intersecting the ruling or the ruling itself. We consider the first possibility.

Let S_3 contain the ruling P_3P_4 and a pencil of τ-planes on P_1P_3, P_1 not on P_3P_4. P_1P_3 is a Σ-line; $P_1P_3P_4$ is a τ-plane tangent to V at P_3. P_1P_4 is not tangent, for otherwise p_4 would be in R_1 and $P_1P_3P_4$ would be a Σ-plane. Let P_2 be any point of S_3 not in $P_1P_3P_4$. P_2 is not in the tangent space at P_3, for in that case S_3 would be a τ-space and of a type already considered. Since P_2P_3 is not a tangent, the line p_3 is not in R_2. Hence R_1 and R_2 are distinct. Therefore the plane $P_1P_2P_3$ is a τ-plane, since it contains P_1P_3, with the line P_1P_3 tangent to V at P_3. This is number 11 of the list of planes. Coordinates can be selected so that $P_1P_2P_3$ is $k, l, 0, 0, 0, m, l, k, 0, 0$. The point P_4 is on V and is such that p_3 and p_4 intersect. The line p_3 is A_2A_4. The vertex of the pencil p_1p_2 is not A_2, for then S_3 would be in the space tangent to V at $M = 0, 0, 0, 0, 1, 0, 0, 0, 0, 0$. The vertex may be made A_4 by proper choice of Q_1 on the line Q_1M. Hence we have

$$p_4 = \begin{cases} a_3, a_6, a_8, 0, -a_{10} \\ 0, 0, 0, 1, 0, \end{cases} \quad \text{and} \quad P_4 = 0, 0, a_3, 0, 0, a_6, 0, a_8, 0, a_{10}.$$

By moving P_4 along P_3P_4, a_6 may be made to take any value. Now by applying transformation T_3 (page 646), which moves P_2 along P_1P_2, we may keep the plane $P_1P_2P_3$ unchanged and obtain

$$P_4 = 0, 0, a_3 - 2a_{10}a, 0, 0, a_6 + a_8a, 0, a_8, 0, a_{10}.$$

Selecting a to satisfy $a_3 - 2a_{10}a = 0$, and then selecting a_6 so that $a_6 + a_8a = 0$, we have $P_4 = 0, 0, 0, 0, 0, 0, 0, a_8, 0, a_{10}$. Applying T_2 with $k = 0, a_8 - a_{10}l = 0$, we get $P_4 = 0, 0, 0, 0, 0, 0, 0, 0, 0, 1$. Changing coordinates will put S_3 in the form 25.

Every other S_3 which intersects V only in the ruling P_1P_2 contains a pencil of τ-planes on P_1P_2. We observe first that S_3 contains a line P_3P_4 skew to P_1P_2 and not a Σ-line. Suppose $P_3'P_4$ to be a Σ-line skew to P_1P_2; then no point, say P_1, of P_1P_2 can be in the five-space Σ_4, for otherwise $P_1P_3'P_4$ would be a Σ-plane, P_1P_3' and P_1P_4 would be tangents, and S_3 would be in the space tangent to V at P_1. Now since P_1P_3' is not a Σ-line, $P_3 = P_1 + P_3'$ determines in X an R_3 which is different from R_4, and P_3P_4 is skew to P_1P_2.

Two τ-planes on P_1P_2 intersect P_3P_4 in two points which may be taken to be P_3 and P_4. Let ρ be the plane of the pencil p_1p_2; let σ be the plane of intersection of R_3 and R_4; let π be the plane on V whose points represent the lines of σ. The plane π contains a point M such that MP_3P_4 is tangent to V at M. Planes ρ and σ may coincide, may intersect in a line, or may intersect in a point. If ρ and σ coincide, then P_1P_2 is in π, and S_3 is in the space tangent to V at M; S_3 is then either 21 or 22 according as M is on or is not on P_1P_2.

Now suppose ρ and σ do not coincide but intersect in a line l. Let L be the point of π which represents l; every point of P_1P_2 is in the space tangent to V at L. Hence if L coincides with M, S_3 is again a τ-space. So we suppose l and m distinct but intersecting in the point D. If D is the vertex of the pencil p_1p_2, every line of the pencil intersects m, and S_3 is in the space tangent to V at M. We therefore suppose D is not the vertex of the pencil; D then determines a line of the pencil which we may take to be p_1. S_3 contains the τ-plane $P_1P_3P_4$ which has one point on V. This τ-plane must be one of planes 10, 11, and 13 of the preceding list.

Plane 10 is tangent to V at its intersection with V, P_2 is in the space tangent to V at P_1, and hence if $P_1P_3P_4$ were plane 10, S_3 would be a τ-space. We then consider $P_1P_3P_4$ to be plane 11, which contains one line tangent to V at P_1. For the rest of this argument we interchange the roles of P_1P_2 and P_3P_4 so we may use the plane 11 in the given form. Plane 11 is $P_1P_2P_3$

$$k, l, 0, 0, 0, m, l, k, 0, 0;$$

it intersects V at P_3, and contains the tangent line P_1P_3. Now the point P_4 is on V and is in the space tangent to V at P_3; hence for P_4, $a_2 = a_4 = a_9 = 0$, and

$$a_1a_8 + a_3a_5 = 0, \qquad a_1a_{10} - a_3a_7 = 0, \qquad a_5a_{10} + a_7a_8 = 0.$$

Also, since P_4 may be any point on P_3P_4, we may suppose $a_6 = 0$. Unless $a_1 = a_5 = a_7 = 0$, the above conditions give $a_3/a_1 = -a_8/a_5 = a_{10}/a_7 = r$. The conditions that $P_2P_3P_4$ be a τ-plane are the conditions that there exist a $B = b_1, b_2, \cdots, b_{10}$ on V with the plane $P_2P_3P_4$ in the tangent space at B. The requirement leads to the result that all the b's are zero except b_1 and b_5 which satisfy $a_8b_1 + a_3b_5 = a_{10}b_1 = a_{10}b_5 = 0$. Hence $a_{10} = 0$. Then (1) $r = 0$, or (2) $a_7 = 0$. In case (2), the plane $P_1P_3P_4$ is a Σ-plane, and S_3 is space 23. In case (1), $P_4 = a_1, 0, 0, 0, a_5, 0, a_7, 0, 0, 0$. Then S_3 intersects V in the line P_3P_4 and also in the conic: $l + a_7n = 0$, $k^2 + a_1kn + a_7mn = 0$. If the only intersection is P_3P_4, we must have $a_1 = a_7 = 0$, and S_3 is 21; it is in the space tangent to V at P_4. This disposes of plane 11.

Next suppose the plane $P_1P_3P_4$ above is plane 13, and take it in the form $P_1P_2P_3 = k, l, 0, 0, 0, m, l + m, k, 0, 0$. P_4 is on V and is in the space tangent to V at P_3. Hence for P_4, $a_2 = a_3 - a_4 = a_8 - a_9 = 0$. Also, either (1) $a_1 = a_5 = a_7 = 0$, or (2) $a_3/a_1 = -a_8/a_5 = a_{10}/a_7 = r$. The requirement that $P_2P_3P_4$ be a τ-plane leads again to the requirement that $a_{10} = 0$, and hence that $ra_7 = 0$. So we have the possibilities:

$$P_4' = a_1, 0, 0, 0, a_5, 0, a_7, 0, 0, 0,$$
$$P_4'' = a_1, 0, ra_1, ra_1, a_5, 0, 0, -ra_5, -ra_5, 0.$$

P_4' gives an S_3 with additional points on V, unless $a_1 = a_7 = 0$, in which case S_3 is in the space tangent to V at P_4'. If $r = 0$, P_4'' is P_4'; if $r \neq 0$, P_4'' gives an S_3 whose τ-planes all pass through P_2P_3 and hence is 25. This completes consideration of plane 13; it also proves that no new S_3 is obtained by supposing that ρ and σ intersect in a line.

We therefore suppose that ρ and σ intersect in a point D. The pencil of lines in σ on D maps into a line d in π. If M is on d, then at least one of the points of the ruling P_1P_2, say P_1, is in the space tangent to V at M, and $P_1P_3P_4$ is plane 10, 11, or 13. The argument just completed still holds. Hence for a new S_3, M is not on d. Two new spaces, 26 and 27, are obtained according as D is or is not the vertex of the pencil p_1p_2.

Since d does not pass through M it intersects the polars of P_3 and P_4 in two distinct points which may be taken to be Q_4 and Q_3 respectively. The point D is the intersection of q_3 and q_4. Coordinates may be selected so that D is $A_1 = 1, 0, 0, 0, 0$, and

$$P_3 = 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, \qquad P_4 = 0, 1, 0, 0, 0, 0, 1, 0, 0, 0.$$

If D is the vertex of the pencil p_1p_2, the line of intersection of ρ with each of R_3 and R_4 is a line of the pencil since it contains D. These lines can be taken to be p_1 and p_2 respectively. p_1 then passes through A_1 and a point of $A_2A_3A_4$, which cannot be on A_2A_3 since p_1 is not in R_4. By moving A_4 on the line A_3A_4 (which can be done without changing the form of P_3 or P_4), the line p_1 may be made to intersect A_2A_4. Hence,

$$P_2 = a, 0, 1, 0, 0, 0, 0, 0, 0, 0.$$

But since p_1 is in R_3, P_1P_3 is tangent to V at P_1. Hence, $a = 0$. By the same considerations we may select A_5 on p_2, and have

$$P_2 = 0, 0, 0, 1, 0, 0, 0, 0, 0, 0.$$

An interchange of names of vertices of the frame of reference in X changes this into space 27.

If D is not the vertex of the pencil p_1p_2, the plane ρ meets R_3 in a line of the pencil, say p_1, but meets R_4 in a line not of the pencil. Coordinates can be chosen so that P_1, P_3, and P_4 are as above and the vertex of the pencil is A_4. The intersection of ρ with R_4 is a line joining A_1 to a point of $A_2A_3A_5$ which cannot be on A_2A_3 and hence can be taken to be on A_3A_5. Thus $P_2 = 0, 0, 0, 0, 0, 0, 0, a, 0, 1$. In order for $P_1P_2P_4$ to be a τ-plane, it is required that $a = 0$. This is space 26. We have completed the determination of all the spaces which contain one and only one ruling of V.

(vi) *Three-spaces with at least three points but no plane curve on V.*

28. $k + n, k, 0, 0, 0, n, l, m, n, 0$.
29. $k, k, n, -n, -n, 0, l, m, 0, n$.
30. $k, k, n, -n, 0, 0, l, m, 0, n$.

31. $k, k, n, n, 0, 0, l, m + n, n, n.$
32. $k, k, n, - n, n, 0, l, m, 0, n.$
33. $k, k, 0, n, n, 0, l, m, 0, n.$
34. $k, k, 0, n, 0, n, l, m, 0, 0.$
35. $k, k, n, n, n, n, l, m, 0, n.$

Spaces 28 and 29 intersect V respectively in a twisted cubic curve and in five points; spaces 30 and 31 have four points on V, the first with a line tangent to V at one of the points and the second with no such line; the others intersect V in three points. In all the spaces the plane $n = 0$ contains Σ-lines joining pairs of P_1, P_2, P_3; space 35 contains no other Σ-line, space 34 contains one other which is tangent to V, and space 32 contains one other which does not intersect V.

Suppose S_3 contains three points of V and does not intersect V in a line or a conic. The three points can be taken to be P_1, P_2, and P_3, and coordinates can be selected so that $P_1P_2P_3$ is

$$k, k, 0, 0, 0, 0, l, m, 0, 0.$$

If S_3 contains two more points of V, the line joining them cannot intersect any of the lines P_1P_2, P_1P_3, or P_2P_3, for otherwise S_3 would contain a plane with four points on V and hence would intersect V in a line or a conic. This line intersects the plane of $P_1P_2P_3$ in a point P which can be taken to be the unit point in the plane; furthermore the line is a Σ-line and contains a point uniquely defined as the conjugate of P with respect to V. Let this conjugate of P be $P_4 = a_1, a_2, \cdots, a_{10}$. The fact that P_4 is conjugate to

$$P = 1, 1, 0, 0, 0, 0, 1, 1, 0, 0$$

gives

$$a_1 - a_6 + a_8 = 0, \qquad a_2 + a_7 - a_9 = 0,$$

$$-a_3 + a_{10} = 0, \qquad a_4 + a_{10} = 0, \qquad a_7 + a_8 = 0.$$

These relations hold not only when S_3 has five points on V, but also whenever S_3 has three points on V and the line PP_4 is a Σ-line. We note that all or none of a_3, a_4, a_{10} are zero.

The transformation T_5 (page 653) leaves each of the points P_1, P_2, P_3, P unchanged, but changes P_4 to P_4' with

$$\begin{aligned} a_1' &= a_1 - a_4c, & a_6' &= -a_3a + a_6 + a_{10}c, \\ a_2' &= a_2 - a_3b, & a_7' &= -a_4a + a_7, \\ a_3' &= a_3, & a_8' &= -a_3a + a_8, \\ a_4' &= a_4, & a_9' &= -a_4a + a_9 - a_{10}b, \\ a_5' &= a_1a - a_2b + a_3ab - a_4ac & a_{10}' &= a_{10}. \\ &\quad + a_5 - a_6b + a_9c - a_{10}bc, & & \end{aligned}$$

In the case where $a_3 = a_4 = a_{10} = 0$, a, b, and c can be selected to make $a_5' = 0$. We then have $P_4' = a_1', a_2', 0, 0, 0, a_6', a_7', a_8', a_9', 0$. S_3 contains the point $P_4'' = a_1'', 0, 0, 0, 0, a_6'', 0, 0, a_9'', 0$. It may be verified that if $a_1''a_6''a_9'' = 0$, S_3 intersects V in a line or a conic. An obvious change of the unit point in X changes the a's to 1's. The space is thus shown to be 28. It will be useful to consider this space more closely.

The B's for a point in S_3 are

$$B_1 = km + mn - kn,$$

$$B_2 = kn + n^2 - kl,$$

$$B_3 = 0,$$

$$B_4 = 0,$$

$$B_5 = lm - n^2.$$

Setting the B's equal to zero we get three cones with vertices at P_1, P_2, and P_3. Each pair of the cones has a common ruling, and the remainder of the intersection is a cubic curve; the ruling is not on the third cone, but the cubic curve is. S_3 thus intersects V in the cubic curve; of course S_3 contains a line tangent to the curve at each of its points.

In the case where $a_3a_4a_{10} \neq 0$ we may select a, b, and c in T_5 so that $a_1' = a_2' = a_7' = 0$. Taking account of the fact that P_4' is conjugate to P and making the proper selection of the unit point in X, we obtain

$$P_4' = 0, 0, 1, -1, -r, 0, 0, 0, 0, 1.$$

Changing the unit point in X to $1, d, d, 1, 1$ changes r in P_4' to rd^2. Hence the possibilities are: r is 0, 1, or a particular not-square. If $r = 1$, S_3 has five points on V and is 29. Conversely, if S_3 has five points on V, $r = 1$.

If $r = 0$, then P_4' is on V, PP_4' is tangent to V, and S_3 is 30. Conversely, if S_3 has just four points on V and contains a line tangent to V at one of them, the above argument holds, and we obtain P_4' with $r = 0$.

If r is a not-square, then S_3 has only three points on V. The line PP_4' is a Σ-line not in the plane $P_1P_2P_3$ and with no point on V. This is space 32 and is defined by these properties.

There is no other S_3 intersecting V in a curve or in five points. If there is an S_3 other than 30 with just four points on V, it can have no line tangent to V at any of the four points. Let the four points on V be P_1, P_2, P_3, P_4 where $P_1P_2P_3$ is as above and P_4 is $a_1, a_2, \cdots, a_{10}$. Any point in S_3 is

$$k + a_1n,\ k + a_2n,\ a_3n,\ \cdots,\ l + a_7n,\ m + a_8n,\ a_9n,\ a_{10}n.$$

If $a_3 = 0$, the space tangent to V at P_2 intersects V in the line $k + a_2n = m + a_8n = 0$; likewise if $a_4 = 0$, the space tangent to V at P_3 intersects S_3 in a line. We may therefore suppose that $a_3a_4 \neq 0$. Then a, b, c in T_5 may be selected so that $P_4 = 0, 0, a_3, a_4, a_5, a_6, 0, a_8, a_9, a_{10}$. Since P_4 is on V, we have $a_3a_5 = a_4a_5 = a_4a_6 = -a_3a_9 + a_4a_8 = a_5a_{10} - a_6a_9 = 0$. Thus, $a_5 = a_6 = 0$, $a_4 = ra_3$, $a_9 = ra_8$. It may be verified that if $a_{10} = 0$, S_3 has a line tangent to V at P_1, and if $a_8 = 0$ it contains a line tangent at P_4. The unit point in X can be selected to make $r = a_3 = a_8 = a_{10} = 1$. The space is 31.

Any other space of this set will have just three points on V; if it has a Σ-line not in the plane of the three points, one of the three points may be on it; it does not intersect the triangle $P_1P_2P_3$ elsewhere since no Σ-plane intersects V in two points. We suppose that S_3 has a Σ-line tangent to V at P_2; we take $P_1P_2P_3$ as above and P_4 an arbitrary point, not P_2, on the Σ-line. Then $P_4 = a_1, 0, 0, a_4, a_5, a_6, 0, 0, a_9, a_{10}$. We have the following possibilities:

(1) $a_4 = a_{10} = 0$. P_4 is not on V and hence $a_9 \neq 0$. We may determine c in T_5 to make $a_5' = 0$. The unit point in X may be selected to make $a_1 = a_6 = a_9$. This S_3 is 28.

(2) $a_4a_{10} \neq 0$. Then c in T_5 can be selected to make $a_1' = 0$. If $a_1a_{10} + a_4a_6 = 0$, then a_6' is also zero. a and b can be selected to make $a_5' = 0$. Proper choice of the unit point gives 33. If $a_1a_{10} + a_4a_6 \neq 0$, selection of c to make $a_1' = 0$ makes $a_6' \neq 0$. Then b can be selected to make $a_5' = 0$ and a to make $a_9' = 0$. In this case S_3 has a fourth point on V, namely,

$$k, l, m, n = -a_4a_6, 0, a_6a_{10}, a_{10}.$$

(3) $a_4 = 0$, $a_{10} \neq 0$. c and b in T_5 can be selected to make $a_6' = a_9' = 0$. If $a_1 = 0$, the plane $k = 0$ intersects V in a conic; if $a_1 \neq 0$, a can be selected to make $a_5' = 0$. Hence we need consider here only

$$k + n, k, 0, 0, 0, 0, l, m, 0, n.$$

(4) $a_4 \neq 0$, $a_{10} = 0$. T_5 can be selected to make $a_1' = a_9' = 0$, and if $a_6 \neq 0$ to make $a_5' = 0$ also. If $a_6 = 0$, the plane $m = 0$ intersects V in a conic. Hence we have $k, k, 0, n, 0, n, l, m, 0, 0$.

Each of (2), (3), (4) gives an S_3 with three points on V and a line tangent to V at P_2. We examine their intersections with the spaces tangent to V at P_1 and P_3 also. In the respective cases, the tangent spaces are

Case (2)	at P_3: $k = l = n = 0$,	at P_1: $l = m = n = 0$,
Case (3)	at P_3: $k + n = l = 0$,	at P_1: $l = m = n = 0$,
Case (4)	at P_3: $k = l = n = 0$,	at P_1: $m - n = l = 0$.

Hence S_3 in case (2) differs from the other two which are alike, as may be shown by interchanging the roles of P_1 and P_3. Case (4) is 34.

Finally, any other S_3 with just three points on V contains no line tangent to V at any of the points. In P_4 none of a_3, a_4, a_{10} is zero. T_5 can be selected to make $a_1' = a_2' = a_4' = 0$. Then P_4' can be changed in S_3 to make $a_7' = a_8' = 0$. If either of a_5' or a_6' is zero. there is a fourth point on V. This S_3 is 35.

(vii) *Three-spaces with two points on* V.

36. $k+n, l, n, 0, 0, m, l, k+rm, m, 0, \quad x^3+rx-1$ irreducible.
37. $k, l, 0, -n, n, 0, l, m, 0, n$.
38. $k, l, 0, 0, n, 0, l, m, 0, n$.
39. $k, l, 0, n, rn, 0, l, m, 0, 0$.
40. $k, l, 0, n, 0, n, l, m, n, 0$.
41. $k, l, 0, n, 0, n, l, m, 0, 0$.
42. $k, l, n, n, n, 0, l, m, 0, 0$.

The τ-plane $n = 0$ in 36 has no point on V; every τ-plane in each of the others has at least one point on V. Spaces 37 and 38 have three τ-planes; in 37 one of the τ-planes contains both points of V; in 38 two of the τ-planes contain both points of V. All of the planes on P_3 in 39 are τ-planes, and so also is $P_1P_2P_4$. Space 40 contains two τ-planes. Spaces 41 and 42 have pencils of τ-planes on the two points of V, and in each the plane $m = 0$ is a τ-plane; the difference between them is harder to describe and will be left to the end of this section.

We consider a three-space S_3 with two points, O_1 and O_2, on V. The line O_1O_2 is obviously a Σ-line. S_3 contains planes with no points on V; such planes are of three types: 7, 8, and 9 of the preceding list. We shall show first that there is just one type of S_3 which contains a τ-plane with no point on V; then we shall show that every other S_3 on O_1 and O_2 contains a τ-plane on O_1O_2.

Let S_3 contain the τ-plane which has no point on V:

$$k, l, 0, 0, 0, m, l, k + rm, m, 0.$$

In considering transformation T_{14} it was shown that P_1 could be chosen arbitrarily and then P_2 and P_3 determined so that the plane has this form. Hence we may assume that O_1O_2 passes through P_1 and that O_1 is

$$P_4 = a_1, a_2, a_3, 0, a_5, a_6, 0, a_8, 0, 0,$$

where $a_1a_8 - a_2a_6 + a_3a_5 = 0$, and since P_1P_4 intersects V in two points $a_1 + a_8 \neq 0$. Transformation T_2 leaves P_1 and P_2 unchanged; it changes P_3 and P_4 to

$$P_3' = 0, 0, 0, 0, rk, 1, 0, r, 1, 0,$$

$$P_4' = a_1 - a_3k, a_2, a_3, 0, -a_1k - a_2l + a_3k^2 + a_5 + a_8k, -a_3l + a_6, 0, a_3k + a_8, 0, 0.$$[13]

Transformation T_1 then changes P_3' and P_4' to

$$P_3'' = 0, 0, 0, 0, rk - a + b, 1, 0, r, 1, 0, \qquad P_4'' = a_1'', a_2'', \cdots, a_{10}'',$$

[13] It is to be noted that the k and l here are the parameters of transformation T_2.

where

$$a_1'' = a_1 - a_3k, \qquad a_6'' = -a_3l + a_6,$$
$$a_2'' = a_2 - a_3a, \qquad a_7'' = 0,$$
$$a_3'' = a_3, \qquad a_8'' = a_3k + a_8,$$
$$a_4'' = 0, \qquad a_9'' = 0,$$
$$a_5'' = -a_1k - a_2l + a_3k^2 + a_5 + a_8k - (-a_3l + a_6)a, \qquad a_{10}'' = 0.$$

We select a, b, k, and l to satisfy

$$a_3k + a_8 = 0, \qquad -a_3l + a_6 = 0, \qquad a_2 - a_3a = 0, \qquad rk - a + b = 0.$$

Then

$$P_3'' = 0, 0, 0, 0, 0, 1, 0, r, 1, 0, \qquad P_4'' = a_1'', 0, a_3'', 0, a_5'', 0, 0, 0, 0, 0.$$

Since P_4'' is on V, $a_3''a_5'' = 0$. If $a_3'' = 0$, S_3 would be in the space tangent to V at 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, and in particular S_3 would contain a τ-plane on O_1O_2. If $a_1'' = 0$, P_1P_4'' has only one point on V. An obvious choice of the unit point in X changes P_4'' to 1, 0, 1, 0, 0, 0, 0, 0, 0, 0. S_3 is space 36. We have thus shown that an S_3 with two points on V and a τ-plane which does not intersect V either is 36 or else contains a τ-plane which has two points on V.[14]

Suppose S_3 contains plane 8, which has no point on V but has a Σ-line. The plane is k, l, m, 0, $-rm$, 0, l, k, 0, 0, r not a square. P_1P_3 is the Σ-line; P_1P_2 is any line in the plane except the Σ-line. The line O_1O_2 intersects this plane in a point which cannot be on P_1P_3, for then the plane $O_1P_1P_3$ would be a Σ-plane and would intersect V in more than two points. The intersection can be taken to be P_2. R_2 is $x_4 = 0$. Hence O_1 is

$$P_4 = a_1, a_2, 0, a_4, a_5, 0, a_7, 0, a_9, 0, \qquad a_1a_9 - a_2a_7 + a_4a_5 = 0, \quad a_2 + a_7 \neq 0.$$

Transformation T_{13} puts P_4 into

$$P_4' = a_1 + a_5a, a_2, 0, a_4 - a_9a, a_5 - a_1a, 0, a_7, 0, a_4a + a_9, 0.$$

If $a_9 = 0$, the plane $k = 0$ is a τ-plane on P_2P_4; if $a_4 = 0$, $m = 0$ is a τ-plane on P_2P_4; if $a_9 \neq 0$, then T_{13} may be selected to make $a_4' = 0$. Hence in any case S_3 contains a τ-plane on O_1O_2.

Any other S_3 contains a plane with no point on V which is not a τ-plane and which contains no Σ-line. This plane is 9:

$$k, l, 0, 0, m, 0, l, k, 0, m.$$

The line O_1O_2 intersects this plane; we examine S_3 according to the location of the intersection with respect to the conic C: $m^2 - 2kl = 0$. If the intersection

[14] It will appear later that this second possible S_3 does not exist.

is on C it may be taken to be P_1; if outside C, let it be P_3; if inside C, then let it be $P_1 + P_2 = 1, 1, 0, 0, 0, 0, 1, 1, 0, 0$.[15]

(a) The intersection is P_1. We may take O_1 to be

$$P_4 = a_1, a_2, a_3, 0, a_5, a_6, 0, a_8, 0, 0,$$

where $a_1a_8 - a_2a_6 + a_3a_5 = 0$, $a_1 + a_8 \neq 0$. The plane $l + a_2n = 0$ is in the space tangent to V at the point $0, 0, 0, 0, 0, 1, 0, 0, 0, 0$. If $a_2 = 0$, this τ-plane contains both O_1 and O_2; if $a_2 \neq 0$, the τ-plane contains neither. So S_3 either is 36 or else contains a τ-plane on O_1O_2.

(b) The intersection is P_3. The three-space R_3 in X is $x_1 = 0$. Let O_1 be

$$P_4 = 0, 0, 0, 0, a_5, a_6, a_7, a_8, a_9, a_{10},$$

where $a_5a_{10} - a_6a_9 + a_7a_8 = 0$, $a_5 + a_{10} \neq 0$. A τ-plane intersects $P_1P_2P_3$ in a line and hence is in the space tangent to V at the point

$$bc^2, ac^2, b^2c, -a^2c, (2ab - c^2)c, b^3, -a(ab + c^2), -b(ab + c^2), a^3, -abc.$$

a, b, c must be such that the matrix

$$\begin{bmatrix} b^2 & a_8bc^2 - a_6ac^2 + a_5b^2c \\ -a^2 & a_9bc^2 - a_7ac^2 - a_5a^2c \\ bc & a_{10}bc^2 - a_7b^2c - a_6a^2c \\ ac & a_{10}ac^2 - a_9b^2c - a_8a^2c \\ ab+c^2 & a_{10}(2ab+c^2)c - a_9b^3 - a_8a(ab+c^2) - a_7b(ab+c^2) - a_6a^3 - a_5abc \end{bmatrix}$$

has rank 1. The space tangent to V at the above point meets $P_1P_2P_3$ in the line $ak + bl - cm = 0$. If $c = 0$, the rank of the matrix is 1 for a and b satisfying $a_6a^3 + a_8a^2b + a_7ab^2 + a_9b^3 = 0$. If this polynomial is reducible, S_3 has a τ-plane on P_3P_4. So at this time we need consider only the case where the polynomial is irreducible. Then a τ-plane would be given only by $a = b = 0$. The τ-plane would be $m + a_{10}n = 0$. It would pass through $O_1 = P_4$ only if $a_{10} = 0$, in which case $a_6a_9 - a_7a_8 = 0$ and the polynomial is reducible. The τ-plane exists and either it contains O_1 and O_2, or S_3 is 36.

(c) The intersection is $P_1 + P_2$. O_1 and O_2 represent lines in the three-space R determined by $1, 1, 0, 0, 0, 0, 1, 1, 0, 0$. We take O_1 to be

$$P_4 = a_1, a_2, a_3, -a_3, a_5, a_6, a_1 - a_6, a_8, a_2 - a_8, a_3$$

with $a_1a_8 - a_2a_6 + a_3a_5 = 0$. An argument about τ-planes similar to that in (b), with $a = b$ and $c = 0$, shows that $k + l + (a_1 + a_2)n = 0$ is a τ-plane. If neither O_1 nor O_2 is in this plane, then S_3 is 36. If one of O_1 and O_2 is in the plane, we may suppose the one is O_1, and then $a_1 + a_2 = 0$. If $a_1 + a_2 = 0$, then $a = a_3$, $b = -a_3$, $c = -a_1$ gives the τ-plane $a_3k - a_3l + a_1m = 0$ which contains both O_1 and O_2. This settles the question unless $a_1 = a_2 = a_3 = 0$, and in this case $m = 0$ is a τ-plane which contains both

[15] We recall that these forms are for $p = 7$; -1 is not a square.

O_1 and O_2. This completes the proof that S_3 with just two points on V either is 36 or else contains a τ-plane on O_1 and O_2.

We now investigate S's containing a τ-plane on two points of V, and we take the plane in the form

$$k, l, 0, 0, 0, 0, l, m, 0, 0.$$

S_3 will contain the point $P_4 = 0, 0, a_3, a_4, a_5, a_6, a_7, 0, a_9, a_{10}$. Transformation T_6 leaves P_1, P_2, P_3 unchanged and puts P_4 into P_4' where

$$\begin{aligned} a_1' &= -a_4c, & a_6' &= -a_3a + a_6 + a_{10}c, \\ a_2' &= -a_3b + a_4a, & a_7' &= -a_4a + a_7, \\ a_3' &= a_3, & a_8' &= -a_{10}a, \\ a_4' &= a_4, & a_9' &= a_9 - a_{10}b, \\ a_5' &= a_3ab - a_4a + a_5 - a_6b + a_7a + a_9c - a_{10}bc, & a_{10}' &= a_{10}. \end{aligned}$$

We shall sort the S_3's according to the zeros among a_3, a_4, and a_{10}.

(1) Suppose $a_3a_4a_{10} \neq 0$. Then b in T_6 can be selected to make $a_9' = 0$, a to satisfy $a_3b - 2a_4a + a_7 = 0$ making $a_2' = a_7'$, then c to make $a_6' = 0$. In S_3 there is the point $P_4'' = 0, 0, a_3, a_4, a_5', 0, 0, 0, 0, a_{10}$. Transformation T_7, which leaves P_1 and P_3 fixed and moves P_2 along the Σ-line P_1P_2, can be applied with $b = 0$ and $a_3 + a_{10}c = 0$; this changes P_4'' to

$$0, 0, 0, a_4, a_5, 0, 0, 0, 0, a_{10}.$$

A change of the unit point[16] gives $P_4'' = 0, 0, 0, -1, 1, 0, 0, 0, 0, 1$, and S_3 is 37.

We have shown that a coordinate system can be selected so that the particular S_3 we have been studying takes the form 37. We seek information about it that is independent of the coordinate system to help distinguish among S_3's given in different coordinate systems. We examine 37 for τ-planes. The space tangent to V at $B = b_1, b_2, \cdots, b_{10}$ intersects S_3 in a plane if the matrix

$$\begin{bmatrix} b_8 & -b_6 & b_1 & b_3 \\ b_9 & -(b_2 + b_7) & 0 & b_4 - b_5 \\ b_{10} & -b_3 & 0 & b_1 - b_6 \\ 0 & b_{10} & b_4 & b_2 - b_8 \\ 0 & b_8 & b_7 & b_5 + b_{10} \end{bmatrix}$$

[16] There is getting to be less freedom in the change of the unit point, and we should perhaps point out the details here. If in X the point $1, d_1, d_2, d_3, d_4$ is taken for the new unit point, the unit point in S is changed to

$$d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4.$$

In order to keep the plane $P_1P_2P_3$ in the canonical form, it is necessary only to require that $d_2 = d_1d_4$. In order to get P_4'' into the desired form, we must have $a_5d_1d_2 = a_{10}d_3d_4 = -a_4d_4$. These requirements can be satisfied since a_4a_5 is not a square; if a_4a_5 were a square, S_3 would have three points on V.

has rank 1. The only such points B and the corresponding τ-planes are

$$0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \quad \text{with plane } k = 0;$$

$$1, 0, 0, 0, 0, 1, 0, 0, 0, 0, \quad \text{with plane } l - m = 0;$$

$$0, 0, 0, 0, 1, 0, 0, 0, 0, 0, \quad \text{with plane } n = 0.$$

Thus S_3 contains just three τ-planes, and only one of them, $n = 0$, is on both O_1 and O_2.

(2) Suppose $a_3 = a_4 = a_{10} = 0$. Then S_3 contains the point

$$P_4 = 0, 0, 0, 0, a_5, a_6, a_7, 0, a_9, 0.$$

Since P_4 is not on V, $a_6a_9 \neq 0$. Then T_6 can be selected so that $a_5' = 0$, and the unit point can be selected so that $P_4' = 0, 0, 0, 0, 0, 1, r, 0, 1, 0$. If $r \neq 0$, the line $k = l = 0$ has two points on V, and hence S_3 has at least three; if $r = 0$, S_3 intersects V in a cubic curve.

(3) Suppose $a_3 = a_4 = 0$, $a_{10} \neq 0$. In T_6 we may select b to make $a_9' = 0$, c to make $a_6' = 0$. Then $a_5' = a_5 - a_6b + a_7a$. Hence if $a_7 \neq 0$, we may select a to make $a_5' = 0$, but in that case P_4' is on V. Hence with proper choice of the unit point we have $P_4' = 0, 0, 0, 0, 1, 0, 0, 0, 0, 1$, and S_3 is 38. It is readily verified that S_3 contains the three τ-planes: $k = 0; l = 0; n = 0$. Each of the last two is on O_1O_2, and therefore 37 and 38 are different.

(4) Suppose $a_3 = a_{10} = 0$, $a_4 \neq 0$. In T_6 we may select a to make $a_2' = a_7'$, then $a_5' = -a_4a + a_5 - a_6b + a_7a + a_9c$. We can select b and c to make $a_5' = 0$ unless $a_6 = a_9 = 0$.

If $a_6 = a_9 = 0$, S_3 is 39. The points of V whose tangent spaces intersect S_3 in planes, and the planes, are:

$$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \quad \text{with } m = 0;$$

$$0, b_2, 0, 0, b_5, 0, 0, 0, b_9, 0, \quad \text{with } b_9k - b_2l + b_5n = 0.$$

Thus every plane on P_3 is a τ-plane.

Suppose now that not both a_6 and a_9 are zero. Then

$$P_4 = 0, 0, 0, a_4, 0, a_6, 0, 0, a_9, 0.$$

Since P_4 is not on V, $a_6 \neq 0$. If $a_9 \neq 0$, S_3 is space 40; it contains only two τ-planes: $m = 0$, and $n = 0$. The plane $m = 0$ does not pass through O_2.

If $a_9 = 0$, then S_3 is 41. S_3 contains the τ-plane $m = 0$ tangent to V at O_1, and the pencil of τ-planes $b_6l + b_2n = 0$ each in the space tangent to V at $0, b_2, b_3, 0, b_5, b_6, 0, 0, 0, 0$, where the b's satisfy $b_2b_6 - b_3b_5 = 0$, $b_2^2 + b_5b_6 = 0$.

(5) Suppose $a_4 = a_{10} = 0$, $a_3 \neq 0$. In T_6, b can be selected to make $a_2' = a_7'$, and a to make $a_6' = 0$. Then $a_5' = a_3ab + a_5 - a_6b + a_7a + a_9c$ which can be made zero if $a_9 \neq 0$. If $a_9 = 0$, S_3 intersects V in a conic. Thus we have only to consider $S_3 = k, l, n, 0, 0, 0, l, m, n, 0$. It has three τ-planes, two on O_1O_2; it is the same as 38 with the two τ-planes on O_1O_2 interchanged.

(6) Suppose $a_3 = 0$, $a_4a_{10} \neq 0$. In T_6, b can be selected to make $a_9' = 0$, a to make $a_2' = a_7'$, and c to make $a_6' = 0$. If $a_5' \neq 0$, this is 37; if $a_5' = 0$, P_4' is on V.

(7) Suppose $a_4 = 0$, $a_3a_{10} \neq 0$. In T_6, b can be selected to make $a_9' = 0$, c to make $a_6' = 0$, and a to make $a_2' = a_7'$. a_5' cannot be zero since P_4' is not on V. This S_3 has three τ-planes, two on O_1O_2. Transformation T_7 can be used to change it into 38.

(8) Suppose $a_{10} = 0$, $a_3a_4 \neq 0$. In T_6, a can be selected to make $a_6' = 0$, b to make $a_2' = a_7'$, and then if $a_9 \neq 0$, c can be selected to make $a_5' = 0$. This S_3 has a third point on V. Hence $a_9 = 0$ and S_3 is 42. It contains the τ-plane $m = 0$ tangent to V at P_1; it contains also the pencil of τ-planes $b_3l - b_2n = 0$ each in the space tangent to V at $0, b_2, b_3, 0, b_5, b_6, 0, 0, 0, 0$, where $b_2b_6 - b_3b_5 = 0$, $b_3^2 - b_6^2 + b_3b_5 = 0$.

We have shown that any S_3 with just two points on V is one of spaces 36 to 42. We have still to show that 41 and 42 differ other than by a choice of coordinate system. In either space any plane on O_1O_2 could be taken for $P_1P_2P_3$, and it is necessary to show that no such choice could turn one into the other.

We examine further the space

$$k, l, n, n, n, 0, l, m, 0, 0.$$

For any point P the B's are

$$B_1 = km + n^2,$$

$$B_2 = -l^2 + n^2,$$

$$B_3 = -ln,$$

$$B_4 = mn,$$

$$B_5 = lm.$$

The three-space R in X determined by P is

$$lmx_1 - mnx_2 - lnx_3 + (l^2 - n^2)x_4 + (km + n^2)x_5 = 0.$$

If we suppose a set x_1, x_2, x_3, x_4, x_5 given, the above relation defines a quadric surface in S_3. Every point P, excepting P_1 and P_3, determines a three-space in X; on the other hand, every point A in X, without exception, determines a quadric Q in S_3. If A is in the space R determined by P, then P is on the quadric Q determined by A. The points of S_3 which are on V do not determine R's, but these points are on every Q determined by a point of X. These relations do not depend on any particular choice of the coordinate system. A change of coordinate system changes the B's but does not change the four-parameter system of quadrics in S_3.

Now S_3 has two points, P_1 and P_3, on V. Each of these points is the image of a line in X. The points of a line in X determine the quadrics of a pencil in S_3. Consequently, the set of quadrics in S_3 determined by the points of X contains two pencils uniquely defined by the relation of S_3 to V. The pencil determined by p_1 is $lmx_1 - mnx_2 = 0$; the pencil determined by P_3 is $lnx_3 - (l^2 - n^2)x_4 = 0$. Every quadric of the first pencil consists of a pair of planes one of which is $m = 0$; likewise, every quadric of the second pencil is a pair of planes also, since $x_3^2 + 4x_4^2$ is irreducible.

For S_3 of type 42 the corresponding system of quadrics is

$$lmx_1 - mnx_2 + n^2x_3 + l^2x_4 + (km - ln)x_5 = 0.$$

The special pencils are

$$lmx_1 - mnx_2 = 0, \quad \text{given by } P_1;$$

$$n^2x_3 + l^2x_4 = 0, \quad \text{given by } P_3.$$

The latter pencil contains the two quadrics $l^2 = 0$ and $n^2 = 0$, each consisting of two coincident planes. Thus by no change of coordinate system can 41 be changed into 42.

(viii) *Three-spaces with one point on* V.

43. $k, l, 0, 0, n, m, l, k + rm, m, 0,\quad x^3 + rx - 1$ irreducible.
44. $k, l, 0, n, m, n, l, k, 0, 0.$
45. $k, l, n, 0, n, m, l, k, 0, 0.$
46. $k, l, 0, -n, n, m, l, k, 0, 0.$
47. $k, l, n, 0, 0, m, l, k, n, 0.$
48. $k + n, l, m, 0, 0, rn, l, k, n, 0,\quad x^3 + x^2 - r^2$ irreducible.
49. $k, l, n, 0, 0, m, l + m, k, n, 0.$
50. $k, l, 0, -n, n, m, l + m, k, 0, 0.$
51. $k, l, n, -n, n, 0, l, k, 0, m.$
52. $k, l, n, n, -n, 2n, l, k, 0, m.$

Space 43 is tangent to V at P_4 which is on V; none of the others has this property. Spaces 44 and 45 contain one plane each tangent to V at O, the point of S_3 on V; in 45 this tangent plane is a Σ-plane; in 44 it is not. Spaces 46, 47, 48 intersect the space tangent to V at O in a line; 46 contains two τ-planes; 47 and 48 each contains only one; in 47 the τ-plane passes through O; in 48 it does not. The space tangent to V at O intersects none of the other spaces anywhere except at O; space 49 contains a single τ-plane; space 50 contains two. Spaces 51 and 52 contain no τ-planes; space 51 contains three special lines which will be described later; space 52 contains only one special line.

In examining the three-spaces with one point O on V we shall make what use we can of the point O and the space tangent to V at O.

There is one obvious S_3 lying in the space tangent to V at O. Any plane in it not on O is a τ-plane with no point on V, and hence is $k, l, 0, 0, 0, m, l, k + rm, m, 0$; it is in the space tangent to V at

0, 0, 0, 0, 1, 0, 0, 0, 0, 0. The space determined by the plane and the point is 43; it may be readily verified that there is no other point on V.

There is no S_3 with just one point on V which is the space tangent to V at a point not in S_3. Such an S_3 would contain $P_1P_2P_3$ above and the point $O = P_4$ for which $a_3 = a_4 = a_{10} = 0$, and

$$a_1a_8 - a_2a_6 = 0, \qquad a_1a_9 - a_2a_7 = 0, \qquad a_5a_{10} - a_6a_9 = 0.$$

The point k, l, m, n whose coordinates satisfy $k + a_1n = l + a_7n = m + a_6n = 0$ is also a point of V. This point is different from O unless $a_1 = a_6 = a_7 = 0$. If they are zero, then $l = m + a_9n = k + rm + a_8n = 0$ is on V and is different from O unless $a_8 = a_9 = 0$ also. The only nonzero coordinate of P_4 is thus seen to be a_5, and the space is 43.

We consider next S_3's which contain O and a plane tangent to V at O. This plane is $k, l, 0, 0, m, 0, l, k, 0, 0$. It contains no Σ-line except the lines through O. P_1 and P_2 can be selected arbitrarily in the plane except that P_1P_2 does not pass through P_3. S_3 will contain the point $P_4 = a_1, a_2, a_3, a_4, 0, a_6, 0, 0, a_9, a_{10}$. Not all of a_3, a_4, a_{10} are zero. We consider first those S_3's for which $a_4 \neq 0$. T_3 can be applied to make $a_{10}' = 0$; T_2 can be applied to make $a_2' = a_7'$, $a_9' = 0$; and then T_1 can be applied to make $a_1' = a_2' = 0$. We then have

$$P_4 = 0, 0, r, 1, 0, 1, 0, 0, 0, 0.$$

The point $k, l, m, n = 0, r, 1, r^2$ is on V. Hence S_3 has more than one point on V unless $r = 0$. If $r = 0$, S_3 is 44. The τ-planes in S_3 are $b_1k - b_2n = 0$, each in the space tangent to V at $b_1, b_2, 0, 0, b_5, 0, b_7, 0, b_9, 0$ which must be on V; they constitute a pencil on P_2P_3.

Those S_3's which contain $P_1P_2P_3$ above and a P_4 which has $a_4 = 0$ give nothing new. The interchange of P_1 and P_2 interchanges a_3 and a_4 in P_4, and hence it changes S_3 into one we have just considered unless $a_3 = a_4 = 0$, and in that case S_3 has at least two points on V.

In any other S_3 with just one point on V and a τ-plane tangent to V at O, the τ-plane must be a Σ-plane. Any other plane on O contains a Σ-line necessarily tangent to V at O. If such other plane is a τ-plane, it can be taken to be

$$k, l, 0, 0, 0, m, l, k, 0, 0.$$

The line P_1P_3 is the tangent line; P_2 is any point in the plane not on P_1P_3. P_4 can be selected in the Σ-plane $P_1P_3P_4$. R_4 is then R_1 which is $x_5 = 0$. Therefore $P_4 = a_1, a_2, a_3, 0, a_5, a_6, 0, a_8, 0, 0$. Since P_4 is in the space tangent to V at P_3, $a_2 = 0$; also, P_4 can be moved along the line P_1P_4 to make $a_8 = 0$ and along the line P_3P_4 to make $a_6 = 0$. Hence S_3 contains the point $P_4' = a_1, 0, a_3, 0, a_5, 0, 0, 0, 0, 0$. If $a_1 \neq 0$, T_2 can be applied to change it to zero. Then S_3 is 45. It contains the τ-planes $b_1k - b_6l + b_5n = 0$, each in the space tangent to V at $b_1, 0, 0, 0, b_5, b_6, 0, 0, 0, 0$.

Any other S_3 which contains a Σ-plane on O can contain no τ-plane on O

except that one. Hence any other plane on O is not a τ-plane but contains a line tangent to V at O; it is $k, l, m, 0, 0, 0, l, k, 0, 0$. The tangent line is $l = 0$; it contains P_1 and is in the Σ-plane. If P_4 is selected in the Σ-plane, then $R_4 = R_1$, and P_4 has $a_4 = a_7 = a_9 = a_{10} = 0$. Since P_3P_4 is tangent to V, $a_5 = 0$. It is easy to verify that S_3 contains a second point on V: viz., $k = m = l + a_2 n = 0$, if $a_2 \neq 0$, or another point on P_1P_4 if $a_2 = 0$.

For all other S_3's with just one point on V the space tangent to V at O can intersect S_3 in at most a line. We consider now the possibility that S_3 contains a line tangent to V at O and contains a τ-plane on that line. The τ-plane can be taken to be $k, l, 0, 0, 0, m, l, k, 0, 0$. If S_3 contains any other Σ-line, the Σ-line does not cut P_1P_3, for then S_3 would contain a Σ-plane. Since P_2 is arbitrary in $P_1P_2P_3$, we may assume the Σ-line is P_2P_4 where $P_4 = a_1, a_2, 0, a_4, a_5, 0, 0, 0, a_9, 0$. If $a_4 = 0$, the line P_1P_4 contains a point of V. Since $a_4 \neq 0$, T_2 can be applied to remove a_2 and a_9, and then T_1 to remove a_1. S_3 is 46; it contains only the two τ-planes $k = 0$ and $n = 0$.

We now consider an S_3 with a line tangent to V at O, with a τ-plane on that tangent line, but with no Σ-line except the tangent line. The τ-plane is $k, l, 0, 0, 0, m, l, k, 0, 0$. The line tangent to V at $O = P_3$ is $l = 0$. S_3 contains the point $P_4 = a_1, a_2, a_3, a_4, a_5, 0, 0, 0, a_9, a_{10}$. Not all of a_3, a_4, a_{10} are zero, for otherwise S_3 would be 43.

(a) Suppose $a_4 \neq 0$. T_3 can be used to remove a_{10}; T_2 can be used to make $a_2 = a_9 = 0$; T_1 can be used to remove a_1; T_{10} can be used to remove a_3. S_3 is 46.

(b) Suppose $a_4 = 0$, $a_{10} \neq 0$. T_3 will make $a_3 = 0$, and T_2 will make $a_1 = a_2 = 0$. Then $P_4' = 0, 0, 0, 0, a_5, 0, 0, 0, a_9, a_{10}$. If $a_9 \neq 0$, the line P_3P_4' contains two points of V. If $a_9 = 0$, S_3 is readily seen to contain a pencil of τ-planes and to be 44.

(c) Suppose $a_4 = a_{10} = 0$. Then $P_4 = a_1, a_2, a_3, 0, a_5, 0, 0, 0, a_9, 0$. $a_9 \neq 0$, for otherwise P_1P_4 would be a Σ-line. T_{10} can be selected to make $a_1' = a_8'$, $a_2' = a_7'$. Hence S_3 contains $P_4' = 0, 0, a_3, 0, a_5, 0, 0, 0, a_9, 0$. Then T_1 with $a = 0$ can be selected to make $a_5 = 0$. S_3 is 47; it contains only one τ-plane.

We have so far determined all the S_3's with one point O on V which contain a line tangent to V at O and a τ-plane on the tangent line. Any other S_3 with a line tangent to V at O will contain the plane

$$k, l, m, 0, 0, 0, l, k, 0, 0$$

which is not a τ-plane, but which contains the tangent line $l = 0$. S_3 contains the point $P_4 = a_1, a_2, 0, a_4, a_5, a_6, 0, 0, a_9, a_{10}$. We now apply transformation T_8, which leaves P_1, P_2, and P_3 unchanged.

(a) If $a_9 \neq 0$, T_8 will remove a_4 and a_{10}. In this case $m = 0$ is a τ-plane not on O; such an S_3 is different from any we have obtained previously.

(b) If $a_9 = 0$, $a_5 \neq 0$, T_8 will remove a_1 and a_6. $P_4 = 0, a_2, 0, a_4, a_5, 0, 0, 0, 0, a_{10}$.

(c) If $a_5 = a_9 = 0$, then $P_4 = a_1, a_2, 0, a_4, 0, a_6, 0, 0, 0, a_{10}$. Here the plane $l = 0$ is tangent to V at P_3. Hence, we need consider cases (a) and (b) only.

Case (a). T_3 will remove a_2, and T_1 will remove a_5. Then, $P_4' = a_1, 0, 0, 0, 0, a_6, 0, 0, a_9, 0$. The unit point in X can be chosen to make $a_1 = a_9$, if $a_1 \neq 0$, but a_6 cannot at the same time be made equal to a_9 unless $a_1^3 = a_6^2 a_9$. If $a_1 = a_6 = a_9$, or if $a_1 = 0$, S_3 has a second point on V. If $a_6 = 0$, S_3 contains the Σ-line P_2P_4. Hence, S_3 is

$$k + n, l, m, 0, 0, rn, l, k, n, 0, \quad x^3 + x^2 - r^2 \text{ irreducible.}$$

This is 48; it contains the τ-plane $m = 0$. The irreducibility of $x^3 + x^2 - r^2$ is required for there to be no second point on V.

Case (b). $a_{10} \neq 0$, for otherwise P_2P_4 would be a Σ-line. If $a_4 \neq 0$, T_3 would make $a_{10} = 0$. Hence, $a_4 = 0$. The unit point can be chosen to give P_4 one of the forms

$$(1) \quad 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, \qquad (2) \quad 0, 1, 0, 0, 0, 0, 0, 0, 0, 1,$$

$$(3) \quad 0, 0, 0, 0, 1, 0, 0, 0, 0, 1,$$

depending on the zeros of a_2 and a_5. In cases (1) and (2), S_3 has two points on V; in case (3), the plane $l = 0$ is a τ-plane on the tangent line, and S_3 is 47.

The remaining S_3's with one point on V will contain no line tangent to V at O. Such an S_3 contains the plane $k, l, 0, 0, 0, m, l + m, k, 0, 0$ and a point $P_4 = a_1, a_2, a_3, a_4, a_5, 0, 0, 0, a_9, a_{10}$. Not all of a_3, a_4, and a_{10} are zero, for otherwise S_3 would lie in the space tangent to V at $0, 0, 0, 0, 1, 0, 0, 0, 0, 0$.

(a) Suppose $a_4 = a_{10} = 0$. If $a_9 = 0$, the line P_1P_4 is a Σ-line, and since P_1 is on the Σ-line $l + m = n = 0$, S_3 contains a Σ-plane not on P_3 and hence contains another point of V. Since $a_9 \neq 0$, T_{11} with $a_2 - a_9 a = 0$ removes a_2; T_2 will remove a_1; and T_1 will remove a_5. The unit point can be chosen so that $P_4 = 0, 0, 1, 0, 0, 0, 0, 0, 1, 0$. S_3 is space 49; it contains a single τ-plane and has no line tangent to V.

(b) Suppose $a_4 \neq 0$. If not both a_4 and a_{10} are zero, we may suppose $a_4 \neq 0$. T_{11} can be used to make $a_{10} = 0$; T_2 will remove a_9; and T_1 will remove a_1. Hence, $P_4 = 0, a_2, a_3, a_4, a_5, 0, 0, 0, 0, 0$. S_3 contains two τ-planes: $n = 0$ and $k = 0$. If $a_3 \neq a_4$, S_3 contains no line tangent to V, and hence is different from 46. We may apply T_{12} to remove a_3, and then choose the unit point so that $P_4 = 0, r, 0, -1, 1, 0, 0, 0, 0, 0$. If $r = 0$, S_3 is 50 which is different from any S_3 previously obtained. If $r \neq 0$ and S_3 has no point except P_3 on V, it contains two τ-planes on P_3, and an interchange of the τ-planes will put S_3 into 50. We shall not carry out this change, but will point out the relations that must be considered in doing it.

The space $k, l, 0, -n, n, m, l + m, k, 0, 0$ contains two τ-planes: $k = 0$ and $n = 0$. The line $l + m = n = 0$ is the Σ-line in one of them; the line

P_2P_4 is the Σ-line in the other. The line P_2P_3 is special, the intersection of the two τ-planes. The two Σ-lines in the τ-planes determine two special points on the line P_2P_3, their intersections with P_2P_3. The point P_2 is therefore uniquely determined as the intersection of the line in both τ-planes with the Σ-line in one of them. Every point of a Σ-line determines another point of it, the point conjugate to it with respect to its "imaginary" intersections with V. P_2 and P_4 are conjugate points of the Σ-line in $k = 0$; P_1 and $(0, 1, -1, 0)$ are conjugate points of the Σ-line in $n = 0$, the second point being the intersection of the Σ-line with P_2P_3. Thus the coordinate system in S_3 is determined as soon as we decide in which of the τ-planes to take P_1. In the case above with $r \neq 0$, a change of coordinates required by selecting P_1 in the plane $k = 0$ puts S_3 into 50.

Any S_3 with one point O on V, other than those so far obtained, will have no τ-plane. Any plane on O will be one or the other of types 14 and 15 of the list of planes. We shall show first that S_3 always contains a plane of type 15.

Suppose S_3 contains a plane of type 14: $k, l, m, -m, 0, 0, l, k, 0, 0$. Then S_3 contains the point $P_4 = 0, 0, 0, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}$. Any point in S_3 is

$$k, l, m, -m + a_4n, a_5n, a_6n, l + a_7n, k + a_8n, a_9n, a_{10}n.$$

The points of intersection of S_3 with the space tangent to V at P_3 satisfy $a_5n = k + (a_8 + a_9)n = l + (a_6 + a_7)n = 0$. S_3 has no line tangent to V at P_3 and hence $a_5 \neq 0$. Then a and b in T_{16} can be selected to make $a_1' = a_8'$, $a_2' = a_7'$, and consequently S_3 contains $P_1P_2P_3$ and

$$P_4 = 0, 0, 0, a_4, a_5, a_6, 0, 0, a_9, a_{10}.$$

For a point P in S_3 we have

$$\begin{aligned} B_1 &= k^2 - a_6ln + a_5mn, \\ B_2 &= a_9kn - l^2 - a_5mn + a_4a_5n^2, \\ B_3 &= a_{10}kn - lm - a_6mn + a_4a_6n^2, \\ B_4 &= a_{10}ln - km + a_4kn - a_9mn, \\ B_5 &= kl + (a_5a_{10} - a_6a_9)n^2. \end{aligned}$$

Using the relation $B_5x_1 - B_4x_2 + B_3x_3 - B_2x_4 + B_1x_5 = 0$, a point in X determines a quadric Q in S_3. The point P_3, being on V, determines a line p_3 in X; the points of p_3 determine the quadrics of a special pencil in S_3. The line p_3 is $\begin{cases} 1, 0, 0, 0, 0 \\ 0, 0, 0, 1, -1 \end{cases}$. The corresponding pencil of quadrics is determined by

$$kl + (a_5a_{10} - a_6a_9)n^2 = 0, \qquad k^2 + a_9kn - l^2 - a_6ln + a_4a_5n^2 = 0.$$

The first of these two quadrics intersects each of the planes $k = 0$ and $l = 0$ in a line through P_3. Hence, both planes are of type 15.

We have thus shown that S_3 contains the plane

$$k,\ l,\ 0,\ 0,\ 0,\ 0,\ l,\ k,\ 0,\ m.$$

S_3 contains the point $P_4 = 0, 0, a_3, a_4, a_5, a_6, a_7, a_8, a_9, 0$. Not both a_3 and a_4 are zero, for then S_3 would contain a τ-plane; $a_5 \neq 0$, for otherwise the line P_3P_4 would be tangent to V at P_3. T_4 can be used to make $a_1' = a_8'$ and $a_2' = a_7'$. Hence we may assume $P_4 = 0, 0, a_3, a_4, a_5, a_6, 0, 0, a_9, 0$. This is as far as we can go in reducing P_4 without changing the plane $P_1P_2P_3$. We shall now find a special line in S_3 and making use of it determine a canonical form.

We examine the special pencil of quadrics in S_3 determined by the line p_3 in X. For a point P in S_3 we have

$$\begin{aligned} B_1 &= k^2 - a_6 ln + a_3 a_5 n^2, \\ B_2 &= a_9 kn - l^2 + a_4 a_5 n^2, \\ B_3 &= km - a_3 ln + a_4 a_6 n^2, \\ B_4 &= lm + a_4 kn - a_3 a_9 n^2, \\ B_5 &= kl + a_5 mn - a_6 a_9 n^2. \end{aligned}$$

The line p_3 is $\begin{cases} 0, 0, 0, 1, 0 \\ 0, 0, 0, 0, 1 \end{cases}$. The quadrics of the pencil are

$$k^2 + a_9 \lambda kn - \lambda l^2 - a_6 ln + a_5(a_3 + a_4\lambda)n^2 = 0.$$

These quadrics are all cones with vertex at P_3. The condition that the quadric given by λ be a pair of planes is that

$$\text{(A)} \qquad a_9^2\lambda^3 + 3a_4a_5\lambda^2 + 3a_3a_5\lambda - a_6^2 = 0$$

have a root in GF(p). We shall show that this root exists.

So far we have not used to the full the fact that S_3 intersects V only at P_3. The conditions that P be on V are that $B_i = 0$, $i = 1, \cdots, 5$. From each of the pairs $B_1 = B_2 = 0$ and $B_3 = B_4 = 0$ it follows that $a_4k^2 + a_3l^2 - a_3a_9kn - a_4a_6ln = 0$. Hence if we solve $B_2 = 0$ for k in terms of l and n, use that value of k in $B_1 = 0$, and solve $B_4 = 0$ for m, we will have a set of values of k, l, m, n which satisfy the first four equations. The equation obtained from $B_1 = 0$ is

$$\text{(B)} \qquad l^4 - 2a_4a_5l^2n^2 - a_6a_9^2ln^3 + (a_4^2a_5^2 + a_3a_5a_9^2)n^4 = 0.$$

This is also the condition that k, l, m, n satisfy $B_5 = 0$. The condition that S_3 intersect V only at P_3 is that (B) have no solution in GF(p). Thus (B) must be either an irreducible quartic, or else the product of two irreducible quadratics. In either case the resolvent cubic

$$\text{(C)} \quad t^3 + 2a_4a_5t^2 - 4(a_4^2a_5^2 + a_3a_5a_9^2)t - (a_4^3a_5^3 + a_3a_4a_5^2a_9^2 + a_6^2a_9^4) = 0$$

of (B) has a root in GF(p).[17]

There exists a transformation $t = (a\lambda + b)/(c\lambda + d)$, a, b, c, d in GF(p), which changes (C) into (A).[18] Hence if (B) has no root in GF(p), (A) has a root in GF(p).

The pencil of cones in S_3 determined by the line p_3 therefore contains one member which consists of a pair of planes. The line of vertices of this quadric is the special line we sought. We take P_1 on this line of vertices. Any plane on P_1P_3 which is not a plane of the quadric determined by the root of (A) in question is cut by the pencil of cones determined by p_3 in a pencil of conics one of which is the line P_1P_3 counted twice. Hence, any such plane is of the type of 15 of the list of planes and may therefore be taken to be $P_1P_2P_3$ above. The cone $B_2 = 0$ intersects the plane $n = 0$ in the parabola $l^2 = 0$ which is the line P_1P_3 counted twice. The cone $B_2 = 0$ intersects the plane $l = 0$ in the conic $a_9kn + a_4a_5n^2 = 0$; since this is the parabola $n^2 = 0$, it follows that $a_9 = 0$. With this choice of coordinate system the equation (B) above becomes $(l^2 - a_4a_5n^2)^2 = 0$. Since (B) has no linear factor in GF(p), it follows that a_4a_5 is not a square. Moreover, the quadric $B_2 = 0$ is $l^2 - a_4a_5n^2 = 0$ and consists of two "imaginary" planes; the only points on it are the vertices. Any plane on P_1P_3 will therefore serve for $P_1P_2P_3$ above, but when the plane is chosen, the locations of P_1, P_2, and P_4 are determined.

The cones of the special pencil determined by p_3 are

$$k^2 - a_6ln + \lambda l^2 + a_5(a_3 - a_4\lambda)n^2 = 0.$$

The matrix of the conic intersection of the cone with $P_1P_2P_4$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 3a_6 \\ 0 & 3a_6 & a_5(a_3 - a_4\lambda) \end{bmatrix}.$$

Setting the determinant of this matrix equal to zero and solving for λ we obtain the λ's which give quadrics consisting of one or two planes. The rank of the matrix is at least two unless $a_3 = a_6 = 0$, in which case the plane $k = 0$ is a τ-plane. Therefore a_3 and a_6 are not both zero. One of the degenerate

[17] For the irreducible quartic this comes under a theorem by L. E. DICKSON, *Criteria for the irreducibility of functions in a finite field*, Bull. Amer. Math. Soc., vol. 13 (1906), p. 7. The quartic which is the product of two irreducible quadratics defines a GF(p^2) in which the quartic is completely reducible and reducible to quadratic factors in three ways corresponding to the three roots of the resolvent cubic. The roots of the cubic are in GF(p^2), and hence at least one of them is in GF(p).

[18] This is done most easily by transforming both (A) and (C) to the form $x^3 + \alpha x + \beta = 0$ which can be made the same for both.

cones is given by $\lambda = \infty$; the others are given by λ's which satisfy

$$a_4a_5\lambda^2 - a_3a_5\lambda + 2a_6^2 = 0.$$

The discriminant of this quadratic, $a_3^2a_5^2 - a_4a_5a_6^2$, cannot be zero since it is the sum of two squares not both zero.[19] Hence, the quadratic has two distinct roots, both or neither in GF(p). There are two new S_3's corresponding to these two possibilities.

We consider first the case where the special pencil of cones contains three degenerate members. Two of them must each consist of a pair of imaginary planes, for otherwise S_3 would have points on V besides P_3. We may take P_2 to be on the line of vertices of the second degenerate cone. Then the cone $k^2 - a_6ln + a_3a_5n^2 = 0$ cuts the plane $k = 0$ in a parabola, and hence $a_6 = 0$. A choice of the unit point puts S_3 in the form 51.

When the special pencil of cones contains only one degenerate member, the one given by $\lambda = \infty$, the number $a_3^2a_5^2 - a_4a_5a_6^2$ must be a not-square, and hence neither a_3 nor a_6 is zero. A proper selection of the unit point will put P_4 into one of

$$0, 0, 1, 1, -1, r, 0, 0, 0, 0, \qquad 0, 0, 1, -1, 1, r, 0, 0, 0, 0,$$

depending on whether a_3a_5 is not or is a square; in either case $1 + r^2$ is not a square. There are $(p + 1)/2$ possibilities for r, and hence there are $p + 1$ possibilities for P_4. We recall that the plane $P_1P_2P_3$ is arbitrary on the line P_1P_3. There are $p + 1$ planes in S_3 on P_1P_3. For a given S_3, the plane $P_1P_2P_3$ can be selected[20] to give P_4 any one of the $p + 1$ forms listed above. Hence 52 is a canonical form for S_3.

6. Three-spaces with no point on V

53. $k, l, m, 0, m, n, l, k + n, n, 0.$
54. $k, l, 0, 2n, m + 3n, n, l, k, 0, m.$

Space 53 contains the τ-plane $m = 0$ and the Σ-line P_1P_3; space 54 contains no τ-plane and no Σ-line.

We shall prove first that *an S_3 with no point on V contains a τ-plane and a Σ-line, or it contains neither*. In an S_3 with no point on V every plane is of one of the types 7, 8, 9 of the list of planes. If S_3 contains more than one τ-plane, the intersection of two of them is a Σ-line; hence the theorem is true, or else S_3 contains not more than one τ-plane. Likewise, if S_3 contains more than one Σ-line, it contains a τ-plane. To prove this, let $k, l, m, 0, m, 0, l, k, 0, 0$

[19] Again we note that the details are being carried out for p such that -1 is not a square.

[20] The simplest way to verify this is to take S_3 with r arbitrary, change the plane $P_1P_2P_3$ from $n = 0$ to $l - an = 0$, and change the coordinate system so that P_1', P_2', P_3', P_4' are in proper form. It will then appear that for no a except $a = \infty$ is the form of P_4 left unchanged.

be a plane on one Σ-line. For this canonical form P_2 can be any point in the plane not on the Σ-line P_1P_3. Hence if S_3 contains a second Σ-line, it may be taken to pass through P_2. P_4 may be selected on the second Σ-line, and hence $P_4 = a_1, a_2, 0, a_4, a_5, 0, a_7, 0, a_9, 0$. If $a_9 = 0$, then $k = 0$ is a τ-plane. If $a_9 \neq 0$, transformation T_{13} can be used to remove a_4. Then $m + a_3'n = 0$ is a τ-plane. Hence, S_3 contains not more than one τ-plane and not more than one Σ-line, or else it contains both a τ-plane and a Σ-line.

We now show that if S_3 contains a Σ-line it contains a τ-plane. S_3 contains a plane which is not a τ-plane and is not on the Σ-line; it may be taken to be $k, l, 0, 0, m, 0, l, k, 0, m$. This plane contains the uniquely defined conic C: $m^2 - 2kl = 0$. The Σ-line intersects this plane (a) on C, (b) outside C, or (c) inside C.

(a) The Σ-line passes through P_1. Then P_4 on the Σ-line is

$$a_1, a_2, a_3, 0, a_5, a_6, 0, a_8, 0, 0.$$

Then $l + a_2n = 0$ is a τ-plane.

(b) The Σ-line passes through P_3. P_4 has $a_1 = a_2 = a_3 = a_4 = 0$, and $m + a_{10}n = 0$ is a τ-plane.

(c) The Σ-line passes through 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, which is inside C. Then $P_4 = a_1, a_2, a_3, -a_3, a_5, a_6, a_1 - a_6, a_8, a_2 - a_8, a_3$. This is exactly the situation that was discussed in determining the space 36; it was shown there that $k + l + (a_1 + a_2)n = 0$ is a τ-plane. Hence, if S_3 has no point on V and contains a Σ-line, it contains a τ-plane.

Now assume that S_3 contains a τ-plane. S_3 contains the plane $k, l, 0, 0, m, 0, l, k, 0, m$. The τ-plane intersects this plane in a line which is (a) a secant of C, (b) a tangent to C, or (c) a line through P_3 not intersecting C.

(a) Let the τ-plane contain P_1P_2, and select P_4 on it. Then $P_4 = a_1, a_2, 0, 0, a_5, a_6, a_7, a_8, a_9, 0$. The line $k + a_1n = l + a_2n = 0$ is a Σ-line.

(b) Let the τ-plane contain P_1P_3 and select P_4 on it.

$$P_4 = a_1, 0, a_3, 0, a_5, a_6, a_7, a_8, 0, a_{10}.$$

The Σ-line is $l + a_7n = m + a_{10}n = 0$.

(c) Let the τ-plane contain P_3 and 1, -1, 0, 0, 0, 0, -1, 1, 0, 0. The τ-plane is in the space tangent to V at 0, 0, 0, 0, 0, 1, -1, -1, 1, 0. It contains $P_4 = 0, 0, a_3, -a_3, a_5, a_6, a_7, a_8, a_9, 0, \quad a_6 + a_7 + a_8 + a_9 = 0$. For any point P in the τ-plane

$$\begin{aligned}
B_1 &= k^2 + (a_6 + a_8)kn + a_3mn + a_3a_5n^2,\\
B_2 &= a_9kn - k^2 + a_7kn - a_3mn - a_3a_5n^2,\\
B_3 &= km + a_3kn - (a_3a_6 + a_3a_7)n^2,\\
B_4 &= -km - a_3kn - (a_3a_8 + a_3a_9)n^2,\\
B_5 &= m^2 + a_5mn - k^2 + a_7kn - a_8kn + (a_7a_8 - a_6a_9)n^2.
\end{aligned}$$

The three-space in X determined by P is $B_5x_1 - B_4x_2 + B_3x_3 - B_2x_4 + B_1x_5 = 0$. For any k, m, n, $B_1 + B_2 = 0$ and $B_3 + B_4 = 0$. If k, m, n are selected so that $B_3 = 0$ and $B_1 = B_5$, then the three-space will be $x_1 + x_4 + x_5 = 0$, which is the three-space determined by 1, 1, 0, 0, 0, 0, 1, 1, 0, 0. The solution is $k = a_6 + a_7$, $m = 2a_3$, $n = 2$. This completes the proof that if S_3 with no point on V contains a τ-plane, it contains a Σ-line. Also it completes the proof of the theorem in italics above.

We now determine a canonical form for S_3 which has a τ-plane and a Σ-line but has no point on V. Any plane on the Σ-line is $k, l, m, 0, m, 0, l, k, 0, 0$, where P_1 is the intersection of the τ-plane and the Σ-line and P_2 is also in the τ-plane. The τ-plane is in the space tangent to V at 0, 0, 0, 0, 1, 0, 0, 0, 0, 0. If P_4 is in the τ-plane, then $a_3 = a_4 = a_{10} = 0$. Since P_1 and P_2 are in the τ-plane also, we may take

$$P_4 = 0, 0, 0, 0, a_5, a_6, a_7, a_8, a_9, 0.$$

The condition that S_3 have no point on V is that the polynomial $f(x) = a_9x^3 - a_7x^2 + a_8x - a_6$ be irreducible. Every suitable S_3 determines such an irreducible cubic, and every irreducible cubic determines a suitable S_3. We note that $a_9 \neq 0$, and hence T_1 can be used to remove a_5.

By changing the unit point we may transform $f(x)$ as it is transformed by $x = dx'$; by interchanging P_1 and P_2 we may transform $f(x)$ as it is transformed by $x = 1/x'$; by means of T_3, which leaves P_3 unchanged, we may transform $f(x)$ as it is transformed by $x = x' + a$. Therefore any S_3 with a Σ-line but no point on V is space 53.

The three-space

$$k, l, 0, 2n, m + 3n, n, l, k, 0, m$$

has no point on V and has no Σ-line. To prove this directly is rather difficult. The following proof is instructive. For a point P of S_3 we have

$$\begin{aligned} B_1 &= k^2 - ln, \\ B_2 &= -l^2 + 2mn - n^2, \\ B_3 &= km + 2n^2, \\ B_4 &= lm + 2kn, \\ B_5 &= m^2 + kl + 3mn. \end{aligned}$$

The condition that there be a point on V is that there exist k, l, m, n which make the B's zero. If we solve $B_1 = 0$ for l in terms of k and n, $B_3 = 0$ for m in terms of k and n, and use these values in $B_5 = 0$, we obtain the relation $k^5 + kn^4 + 4n^5 = 0$. The polynomial $f(x) = x^5 + x + 4$ is irreducible.[21]

[21] We are dealing with $p = 7$. For the next several pages we shall be more closely tied to $p = 7$ than we have been heretofore. At the end we shall divest the argument of dependence on $p = 7$, but it seems desirable to separate the difficulties of the problem from the difficulties that arise from different properties of different primes.

Hence S_3 has no point on V. If $f(x)$ were reducible, S_3 might still have no point on V, but then $f(x)$ would be the product of an irreducible quadratic and an irreducible cubic. We have seen irreducible cubics before in this discussion, in connection with τ-planes with no point on V. If $f(x)$ were factorable but had no linear factor in GF(p), it is clear that S_3 would be space 53. For if X, S, V, and S_3 were immersed in spaces $\bar{X}$, $\bar{S}$, $\bar{V}$, and $\bar{S}_3$ over GF(p^3), then $\bar{S}_3$ would have three points on $\bar{V}$. When $f(x)$ is irreducible, then $\bar{S}_3$ has no points on $\bar{V}$, and hence S_3 has no τ-plane.

We propose to show that any S_3 which has no point on V and no Σ-line, or, which is the same thing, any S_3 whose quintic polynomial $f(x)$ is irreducible, can be put in the form 54. We cannot distinguish among the points of S_3, among the lines, or among the planes; we cannot distinguish among the points of a line, but we can distinguish among the points of a plane by means of the absolute conic C. In seeking something similar to C which may aid in characterizing S_3 we shall examine some complicated relations between S_3 and X.

For this S_3 the equation $B_5x_1 - B_4x_2 + B_3x_3 - B_2x_4 + B_1x_5 = 0$ is

$$(m^2 + kl + 3mn)x_1 - (lm + 2kn)x_2 + (km + 2n^2)x_3 + (l^2 - 2mn + n^2)x_4 + (k^2 - ln)x_5 = 0.$$

When k, l, m, n are given, this is the three-space R in X determined by P; when an arbitrary point $A = x_1, x_2, x_3, x_4, x_5$ in X is given, it is a quadric surface Q in S_3. The points of Q are the points of S_3 whose three-spaces R in X contain A. No two R's given by different P's are the same, since S_3 contains no Σ-line. The B's are linearly independent polynomials in k, l, m, n. There is thus determined a four-parameter system W of quadrics in S_3. Some of the quadrics of W are degenerate, and thereby a distinction can be made among the points of X. The locus of points in X which give cones in S_3 is

$$J: \quad \begin{vmatrix} x_5 & 4x_1 & 4x_3 & 6x_2 \\ 4x_1 & x_4 & 3x_2 & 3x_5 \\ 4x_3 & 3x_2 & x_1 & 5x_1 - x_4 \\ 6x_2 & 3x_5 & 5x_1 - x_4 & 2x_3 + x_4 \end{vmatrix} = 0.$$

J is a manifold of dimension three and order four in X. A point on J determines a cone in S_3, and the cone has a vertex. It is easy to see that *no cone of the set W has more than one vertex*, and to see that *every point of S_3 is the vertex of one and only one cone of the set W*.

We prove the first statement by showing that if W contains a quadric with a line of vertices, S_3 contains a Σ-line. Let W contain a quadric Q with a line of vertices. Any plane in S_3, in particular a plane on the line of vertices of Q, may be taken to be $k, l, 0, 0, m, 0, l, k, 0, m$. S_3 contains the point $P_4 =$

$0, 0, a_3, a_4, a_5, a_6, a_7, a_8, a_9, 0$. The B's for a point P in S_3 are

$$B_1 = k^2 + a_8kn - a_6ln + a_3mn + a_3a_5n^2,$$
$$B_2 = a_9kn - l^2 - a_7ln + a_4mn + a_4a_5n^2,$$
$$B_3 = km - a_3ln + (a_4a_6 - a_3a_7)n^2,$$
$$B_4 = lm + a_4kn + (a_4a_8 - a_3a_9)n^2,$$
$$B_5 = m^2 + kl + a_7kn + a_8ln + a_5mn + (a_7a_8 - a_6a_9)n^2.$$

The matrix of any quadric of the set W has for the first three columns

$$\begin{bmatrix} x_5 & 4x_1 & 4x_3 \\ 4x_1 & x_4 & 3x_2 \\ 4x_3 & 3x_2 & x_1 \\ 4(a_7x_1 - a_4x_2 - a_9x_4 + a_8x_5) & 4(a_8x_1 - a_3x_3 + a_7x_4 - a_6x_5) & 4(a_5x_1 - a_4x_4 + a_3x_5) \end{bmatrix}$$

Now the line of vertices of Q in the plane $n = 0$ has one of three positions: (1) it is tangent to C and may be taken to be P_1P_3; (2) it intersects C in two points and may be taken to be P_1P_2; or (3) it passes through P_3 and does not intersect C; it may be taken to be $\begin{cases} 1, -1, 0, 0 \\ 0, 0, 1, 0 \end{cases}$. In case (1) the quadric is given by the point $x_1, x_2, x_3, x_4, x_5 = 0, 0, 0, 1, 0$. Its equation is $B_2 = 0$, and since it consists of two planes we have $a_4 = a_9 = 0$. If this is so, the line $l + a_7n = 0, m = 0$ is a Σ-line. Cases (2) and (3) would require $x_1 = \cdots = x_5 = 0$. Hence W contains no quadric with more than one vertex.

That an arbitrary point $P = k, l, m, n$ of S_3 be the vertex of some cone of the set W requires that it be possible to select $x_1, \cdots, x_5$ so that k, l, m, n are the constants of dependence of the columns of the matrix of which three columns are given just above. This gives four linear equations in the x's with coefficients linear in k, l, m, and n. Properly signed four-rowed determinants of the matrix of coefficients constitute a solution for the x's, if they are not all zeros. There is at least one solution for every k, l, m, n; there would be more than one if the rank of the matrix of coefficients were less than four. There is not more than one solution, as we shall now prove. Let P be any point in S_3. Any plane on P can be taken to be $k, l, 0, 0, m, 0, l, k, 0, m$. P may be (1) on the conic C, $P = P_1 = 1, 0, 0, 0$; (2) outside C, $P = P_3$; (3) inside C, $P = 1, 1, 0, 0$. If any one of these sets of k, l, m, n is used for constants of dependence of the three columns above, a set of four independent equations in the x's is obtained. Hence, in every case the solution is unique.

Let P_1 and P_2 be arbitrary points on the line l in S_3, and let the three-spaces in X determined by them be R_1 and R_2 respectively. R_1 and R_2 intersect in a plane σ. Every point in σ determines a quadric in S_3 which passes through both P_1 and P_2. There is thus determined in W a net of quadrics on P_1 and

P_2. The line l determines a point M on V, the point such that MP_1P_2 is tangent to V at M. M is the image on V of a line m in σ. Every point on l determines a three-space in X which contains m, and consequently the quadric in S_3 determined by a point of m has the line l for a ruling. Thus the points P_1 and P_2 determine a net of quadrics in S_3, and in that net is a pencil of quadrics each of which has l for a ruling. If A is a point of m, the quadric Q has l for a ruling and hence is a ruled quadric; it is a cone if A is on J.

Now let us consider two lines l_1 and l_2 in S_3. They determine two lines m_1 and m_2 in X. If m_1 and m_2 intersect in a point A, the quadric Q determined by A has both l_1 and l_2 for rulings. If m_1 and m_2 do not intersect, there will be no quadric of the set W which has both l_1 and l_2 for rulings. If m_1 and m_2 intersect, the quadric Q will not be degenerate if l_1 and l_2 do not intersect. If m_1 and m_2 intersect and l_1 and l_2 intersect also, Q will be a cone if A is on J; otherwise it will be nondegenerate, and l_1 and l_2 will belong to different reguli on Q.

To study further the relations of lines and quadrics of S_3 to lines and planes of X, we consider the six-spaces tangent to V along a ruling of V. For this purpose we may take the points of a ruling and the tangent spaces to be

$$M_0 = 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \qquad T_0: \quad x_8 = x_9 = x_{10} = 0,$$

$$M_\infty = 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, \qquad T_\infty: \quad x_6 = x_7 = x_{10} = 0,$$

$$M_\lambda = 1, \lambda, 0, 0, 0, 0, 0, 0, 0, 0, \qquad T_\lambda: \quad \begin{cases} -\lambda x_6 + x_8 = 0 \\ -\lambda x_7 + x_9 = 0, \quad x_{10} = 0. \end{cases}$$

The six-spaces T_λ are all in the eight-space S_8: $x_{10} = 0$; the intersection of two of them is the four-space $S_4 : x_6 = x_7 = x_8 = x_9 = x_{10} = 0$. Any point in S_8 on the hyperquadric Q_7: $x_6x_9 - x_7x_8 = 0$, $x_{10} = 0$ is in some T_λ. Any point in two T_λ's is in S_4. Any line in S_4 contains a point on V.

Now let S_3 be a three-space in S with no point on V and no Σ-line. Either S_3 lies wholly in S_8 or intersects it in a plane. The points of Q_7 lie in the hyperquadric in S determined by $a_5a_{10} - a_6a_9 + a_7a_8 = 0$, and hence its intersection with S_3 is one of the quadrics of the set W. The intersection of S_3 and S_8 therefore cannot be a plane. S_3 can have no more than one point in S_4, since S_3 has no point on V. Q_7 intersects S_3 in a quadric Q. If one T_λ intersects Q in a line, then every T_λ intersects it in a line. If two lines in distinct T_λ's intersect, the intersection is in S_4 and hence is on each of the rulings of Q, and Q is a cone. If Q has no point in S_4, then the rulings of Q cut out by the T_λ's do not intersect, and Q is not degenerate.

Now let us consider the cone Q_1 in S_3 with vertex at an arbitrary point P_1. The rulings of Q_1 are in the tangent spaces at points of a ruling of V, and these points on V represent the lines of a pencil in X. Thus a point P_1 in S_3 determines a plane σ in X. Every point in σ determines a quadric on P_1 in S_3; the vertex A_1 of the pencil determines Q_1, and A_1 is on J. Any other point A_2 in

σ determines a quadric in S_3 which has a ruling in common with Q_1. All the quadrics of the set W which contain a particular ruling of Q_1 have been shown to belong to a pencil and hence are given by a particular line in σ on A_1. Consequently all the quadrics of the set W that intersect Q_1 in a ruling belong to the net determined by the points of σ.

Let A_2 be a second point on the intersection of σ and J. Then A_2 determines a cone Q_2 in S_3; let the vertex of Q_2 be P_2. The cone Q_2 determines a plane σ' in X. The line P_1P_2 is a ruling of both Q_1 and Q_2; it determines the line A_1A_2 in X, and hence A_1A_2 is in both σ and σ'. We shall show that the two planes coincide. Consider a plane ρ on P_1P_2 and not tangent to Q_1 or Q_2. This plane cuts out rulings l_1 and l_2, not P_1P_2, on Q_1 and Q_2 respectively; let the intersection of l_1 and l_2 be P. P, P_1, and P_2 determine the three-spaces R, R_1, and R_2 in X. The intersection of R and R_1 is the plane whose points give all the quadrics of the set W which pass through P and P_1. It contains the line AA_1, which is a line of σ corresponding to the ruling PP_1 of Q_1, and, since P is on Q_2, the point A_2. The plane of intersection of R and R_1 is therefore σ which is not dependent on the choice of ρ and hence not dependent on R. From this it follows that σ and σ' are the same, and that σ is the intersection of R_1 and R_2.

The plane σ was determined as the plane of the pencil of lines in X determined by the rulings of the cone Q_1; σ has been shown to have the same relation to Q_2. There are thus determined two pencils of lines in σ with vertices at A_1 and A_2 respectively. The plane ρ in S_3 on P_1 and P_2 contains a ruling of Q_1 and a ruling of Q_2, and hence determines lines in σ on A_1 and A_2 respectively. The pencil of planes on P_1 and P_2 thus sets up a projectivity between the two pencils of lines in σ. The line A_1A_2, which is in both pencils, is not self-corresponding in the projectivity unless the cones Q_1 and Q_2 have a common tangent plane. Corresponding lines of the two projective pencils in σ intersect in a conic if Q_1 and Q_2 do not have a common tangent plane; otherwise they intersect in a line.

Let the intersection of two corresponding lines of the pencils on A_1 and A_2 be A. A determines a quadric Q in S_3. Q has each of the lines l_1 and l_2 in ρ as a ruling; these rulings intersect, and therefore Q is a cone with vertex at P. Hence, if Q_1 and Q_2 do not have a common tangent plane, the points of σ which are on J are points of a conic, and the corresponding cones in S_3 have vertices on the cubic curve of intersection of Q_1 and Q_2. The quadrics determined by the points of σ all contain this cubic curve.

Any line in σ is imaged in S_3 on a point of V which is such that the space tangent to V there intersects S_3 in a line. If A' is any point of such a line and Q' is the corresponding quadric, the rulings of Q' in common with Q_1 and Q_2 respectively belong to the same regulus of Q', the rulings of this regulus determine the lines in σ on A', and one of those lines is the one in question.

If the projective pencils of lines on A_1 and A_2 in σ were perspective, then σ would contain a line each of whose points would determine a cone in S_3, and the vertices of the cones would lie on a line l not P_1P_2. Then the cone Q_1 would contain the plane P_1l. This is not possible since W contains no quadric with a plane on it.

Also, there is no cone Q_1 of the set W whose plane σ contains no second point of J. Let A be a point of σ; then A determines a quadric Q with a ruling in common with Q_1. Let ρ be a plane in S_3 on the common ruling of Q_1 and Q, and let ρ cut Q_1 in a second ruling, which intersects Q at a point P. Through P there is a ruling of Q of the regulus to which the common ruling of Q_1 and Q belongs. The two rulings, one of Q_1 and one of Q, determine two lines on A_1 and A respectively. The intersection of these two lines determines a quadric with two rulings of the same regulus which intersect; this quadric is therefore a cone, and it is distinct from Q_1.

Hence, we have shown

If σ is a plane in X determined by a cone of the set W, it intersects J in a conic which is not degenerate.

If Q_1 and Q_2 are two cones of the set W and if they have a common ruling, they determine in S_3 a net of quadrics each of which has one and only one ruling in common with each other; the cones of the net are $p + 1$ in number and have vertices on the cubic curve of intersection of Q_1 and Q_2.

We have also shown the following theorem about J:

Every point of J determines a unique plane in X which intersects J in a nondegenerate conic.

These planes are the double tangent planes of J. Each of them contains $p + 1$ points of J, and no two have a point of J in common. Their number is thus shown to be $p^2 + 1$. Since two planes of X intersect in at least one point, two double tangent planes of J intersect in a point A which is not on J, and the quadric Q determined by A is a nondegenerate quadric with rulings. The second set of rulings on Q determines a plane σ in X which contains A. Incidentally, we cannot distinguish one point of J from another.

We note that the points of S_3 lie on $p^2 + 1$ cubic curves each of which is the intersection of a net of quadrics of the set W, and no two of the cubics intersect.

We note also that not every point of X is on a double tangent plane. A point not on such a plane determines a quadric Q which has no rulings. Such a point is 0, 0, 0, 1, 1; Q is $k^2 - ln + l^2 - 2mn + n^2 = 0$. Q contains the point $k, l, m, n = 0, 0, 1, 0$; the plane tangent to Q at that point is $n = 0$. Points of intersection of the plane and Q satisfy $k^2 + l^2 = 0$, and hence the only point is the point of tangency.

Every cone of the set W has on it a single one K of the cubic curves. Every nondegenerate ruled quadric of the set W has on it two of the cubic curves, K and K'. It is clear that if Q is a cone with the vertex P determined by the point A in X, each ruling of Q intersects K in P and one other point, excepting the ruling determined by the tangent to the conic intersection of J and the double tangent plane in which A lies. This ruling is the line tangent to K at the point P. If Q is a nondegenerate quadric determined by a point A outside the conic of intersection C of σ and J, then a line of the pencil in σ on A intersects C in one, two, or no points; thus the rulings of Q of the set corresponding to lines on A in σ meet K in one, two, or no points. If A is inside C, then each of these rulings meets K in two or no points. The same situation holds with respect to the other set of rulings of Q and the cubic K'. The situation is different, however, with respect to the rulings of Q determined by the pencil of lines on A in σ and the points of the cubic K'. The curve K' is on Q, it has $p + 1$ points, and no two points of K' are on the same ruling of the set determined by the lines in σ. Hence there is one point of K' on each of these rulings.

We now investigate the space 54 in the light of these relations.[22] The vertices of the frame of reference in the space 54 lie on the quadric $Q_2 : lm + 2kn = 0$, which is given by the point $A_2 = 0, 1, 0, 0, 0$ in X; the edges P_1P_2 and P_3P_4 are rulings of one regulus on Q_2, and P_1P_3 and P_2P_4 are rulings of the other. The planes in X determined by these reguli are respectively $\sigma_2 = A_2A_3A_5$ and $\sigma_1 = A_1A_2A_4$. The plane σ_2 intersects J in the conic $C_2 : x_2^2 + 3x_3x_5 = 0$; A_3 and A_5 are on C_2, and A_2 is the pole with respect to C_2 of the line joining them. The plane σ_1 intersects J in the conic $C_1 : x_1^2 + 4x_2^2 + 3x_1x_4 = 0$; A_4 is on C_1; the tangent to C_1 at A_4 passes through A_2. A_1 is on the polar of A_2 with respect to C_1; the other intersection of this polar with C_1 is 1, 0, 0, 2, 0.

The vertices of the cones in S_3 determined by the points of C_2 lie on the cubic curve K_2 through P_2 and P_3, the vertices of the cones determined respectively by A_3 and A_5. The vertices of the cones in S_3 determined by the points of C_1 lie on the cubic K_1 through P_1; K_1 intersects the line P_3P_4 at 0, 0, 1, 2. This point determines the space $x_3 = 0$ in X.

Let us designate the point 0, 0, 1, 2 by P_4'.

$$P_4' = 0, 0, 0, 4, 0, 2, 0, 0, 0, 1.$$

It is on the line joining the two points of V: 0, 0, 0, 4, 0, 0, 0, 0, 0, 1 and 0, 0, 0, 0, 0, 2, 0, 0, 0, 0. These points represent respectively the lines $\begin{cases} 1, 0, 0, 2, 0 \\ 0, 0, 0, 0, 4 \end{cases}$ and $\begin{cases} 0, 1, 0, 0, 0 \\ 0, 0, 0, 2, 0 \end{cases}$ in X. The points 1, 0, 0, 2, 0 and 0, 0, 0, 1, 0 are the points of C_1 to which tangents to C_1 can be drawn from A_2. A_1A_2 determines the ruling P_2P_4 of Q_2. The line $\begin{cases} 1, 0, 0, 2, 0 \\ 0, 1, 0, 0, 0 \end{cases}$ determines the ruling of the same set which passes through P_4'.

[22] It is to be noted that in the above argument there is no dependence on p being 7. We used that assumption when we exhibited the quadric with no rulings, but that fact is not important for our purposes and as will be seen later can be proved easily without any assumption about p.

We note that the relations described so far are completely determined by the choice of A_2. For any A_2 planes σ_1 and σ_2 are uniquely determined, as well as conics C_1 and C_2, and the polars of A_2 with respect to C_1 and C_2. A_2 must be outside both C_1 and C_2. The ruling P_2P_4 is determined by P_2, and the point P_4' by the tangent to C_1 through A_2.

We may look upon A_2 as being determined by the quadric $lm + 2kn = 0$ of the set W. Any nondegenerate ruled quadric of W in any S_3 which has no point on V and no Σ-line determines a point A in X, two planes σ_1 and σ_2, containing conics C_1 and C_2 and intersecting in A. If A is outside both C_1 and C_2, then the polars of A with respect to C_1 and C_2 respectively intersect C_1 and C_2 in two points each. Each of these four points, on C_1 and C_2, determines a cone with vertex on Q. If P is the vertex of one of these cones, the two rulings of Q through P determine two lines in X, both through A, one in σ_1 and one in σ_2. There are thus distinguished four lines on A in each of the planes σ_1 and σ_2. Now, for the space $k, l, 0, 2n, m + 3n, n, l, k, 0, m$ and the quadric $lm + 2kn = 0$ given by A_2 above, these two sets of four lines reduce in one plane to two and in the other to three. The vertices of the cones determined by 1, 0, 0, 2, 0 and 0, 0, 0, 1, 0 lie on the rulings determined by A_2A_3 and A_2A_5, and the vertices of the cones determined by A_3 and A_5 lie on rulings of Q determined by A_2A_4 and A_1A_2, the latter having no point on $C_1 : x_1^2 + 4x_2^2 + 3x_1x_4 = 0$.

The configuration in X just described characterizes

$$k, l, 0, 2n, m + 3n, n, l, k, 0, m$$

in the sense that any S_3, with no point on V and no Σ-line, whose set W contains a quadric Q which provides the above configuration, is conjugate to $k, l, 0, 2n, m + 3n, n, l, k, 0, m$ under a collineation of X. A proof will be given by showing how to select a coordinate system in X so that S_3 takes the given form; this will be done by going backwards from the configuration through the steps by which it was determined. We shall use primed letters P_1', Q_1', A_1', etc. until we can see that the accents may be dropped and the letters have the same significance as above.

Denote by σ_2' the plane in which the four lines combine into two, and by σ_1' the other. Denote by A_2' the intersection of σ_1' and σ_2'; denote by C_i' the intersection of J with σ_i'. Denote by A_3' and A_5' the intersection of C_2' and the polar of A_2' with respect to C_2', with A_3' the one whose cone in S_3 has vertex P_2' on the ruling of Q_2' determined by the line on A_2' in σ_1' which does not intersect C_1'. Denote by A_4' the point of C_1' which gives in S_3 the cone with vertex on the ruling of Q_2' given by $A_2'A_5'$, and denote by A_1' the intersection of the polar of A_2' with the third line in σ_1', which is not tangent to C_1'. Denote by P_1' the vertex of the cone determined by A_4', and by P_3' the vertex of the cone determined by A_5'.

The plane $P_1'P_2'P_3'$ is completely determined by the configuration. The plane is of type 9 of the list of planes, and we shall now show that P_1', P_2', P_3'

will serve as P_1, P_2, P_3 of that canonical form. The points P_2' and P_3' are on the cubic curve K_2' determined by the vertices of the cones given by points of C_2'. The cubic K_2' lies on the cone with vertex at P_3'. $P_1'P_2'$ is a ruling of Q_2' determined by the line $A_2'A_3'$, which is tangent to C_2'. Hence, $P_1'P_2'$ is tangent to K_2' at P_2', and hence $P_1'P_2'P_3'$ is tangent to the cone with vertex at P_3' and therefore intersects the cone in a single line. The absolute conic of the plane $P_1'P_2'P_3'$ is therefore tangent to the line $P_2'P_3'$ at P_2'. The cone with vertex at P_1' is also tangent to the plane $P_1'P_2'P_3'$, which we proceed to show. The cone with vertex at P_1', given by A_4', has its vertex on the cubic K_1'. K_1' has one point besides P_1' on each of the rulings of the cone with vertex at P_1' except the ruling $P_1'P_3'$ which is determined by the tangent to C_1' at A_4'. Every point of K_1' is on Q_2'. The points common to Q_2' and $P_1'P_2'P_3'$ are the points of $P_1'P_2'$ and $P_1'P_3'$. We have just noted that $P_1'P_3'$ has no second point on K_1'; $P_1'P_2'$ is a ruling of Q_2' determined by a line in σ_2' and has no point except P_1' on K_1'. Hence, the plane $P_1'P_2'P_3'$ is tangent to the cone with vertex at P_1', and the absolute conic in it is tangent to $P_1'P_3'$ at P_1'. P_3' is therefore the pole of $P_1'P_2'$ with respect to the absolute conic, and P_1' and P_2' are on the conic. The vertices of the frame of reference in X can be selected, and in only one way when P_1, P_2, P_3 are given, so that $P_1'P_2'P_3'$ is in canonical form. Then for this S_3 the A_i''s have the coordinates of the A_i's for the space 54.

The points P_1', P_2', P_3' are now P_1, P_2, P_3 with the proper coordinates. To complete the canonical form it is necessary to determine the coordinates of P_4. P_4 is determined as the intersection of two rulings of Q_2. One ruling is determined by A_1A_2, and the other by A_2A_5. The corresponding points on V are

$$A_1A_2 \rightarrow 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \qquad A_2A_5 \rightarrow 0, 0, 0, 0, 0, 0, 1, 0, 0, 0.$$

The respective spaces tangent to V are $a_8 = a_9 = a_{10} = 0$ and $a_2 = a_3 = a_8 = 0$. Hence, $P_4 = a_1, 0, 0, a_4, a_5, a_6, a_7, 0, 0, 0$. There are further conditions that the a's must satisfy. So far we have required of C_2 only that it pass through A_3 and A_5 and that A_2 and A_3A_5 be pole and polar with respect to it; also it has been required of C_1 only that it pass through A_4 and that A_2 and A_1A_4 be pole and polar.

Any point of S_3 is

$$k + a_1n,\ l,\ 0,\ a_4n,\ m + a_5n,\ a_6n,\ l + a_7n,\ k,\ 0,\ m.$$

For this point

$$\begin{aligned} B_1 &= k^2 + a_1kn - a_6ln, \\ B_2 &= -l^2 - a_7ln + a_4mn + a_4a_5n^2, \\ B_3 &= km + a_1mn + a_4a_6n^2, \\ B_4 &= lm + a_4kn, \\ B_5 &= m^2 + kl + a_7kn + a_5mn. \end{aligned}$$

If we take the point of intersection of C_1 and A_1A_4 to be 1, 0, 0, 2, 0, this requires Q', the quadric determined by it, to be a cone. The result is that $a_5 + 2a_4 = 0$. a_4 cannot be zero since S_3 contains no τ-plane. Hence, we may take $a_4 = 2$ and $a_5 = 3$. If we take 1, 1, 0, 3, 0 to be on C_1, this will give $a_7 = 0$. The cone with vertex at P_2 is tangent to P_3P_4 at P_3; this requires $a_1 = 0$. It requires one more point to fix C_2; let it be 0, 1, 1, 0, 2; then $a_6 = 1$. Hence, the point P_4 is 0, 0, 0, 2, 3, 1, 0, 0, 0, 0, and the space S_3 is space 54.

This configuration in X which has just been shown to characterize the space 54 can be described by elements in S_3. We give a representation of the

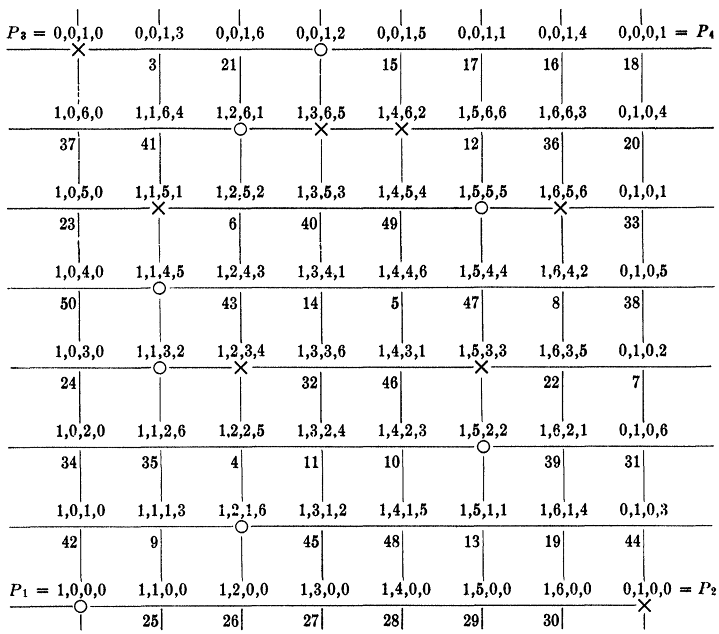

DIAGRAM 1

quadric $Q_2 : lm + 2kn = 0$ in Diagram 1. This is a diagram of points and lines on Q_2. The horizontal lines are the rulings of Q_2 determined by the pencil of lines on A_2 in σ_2; the vertical lines are rulings of the other set and are determined by the pencil on A_2 in σ_1. The cubic K_2 passes through the points marked with a cross (×); the cubic K_1 passes through the points marked with a circle (○). Each horizontal line contains one circle, and may contain two, one, or no crosses. The line P_2P_4 contains no circle and so is determined by a line in σ_1 which does not intersect C_1; it is not a ruling of any cone with vertex on K_1. The two vertical lines each containing just one circle are determined by the two tangents to C_1 from A_2; likewise the two horizontal lines each containing one cross are determined by the two tangents to C_2 from A_2. The horizontal line through P_2, since it has no other cross on it, is determined by the tangent to C_2 at the point which determines the cone with vertex at P_2; it is the ruling of the cone which is tangent to K_2 at P_2. This line contains P_1 which is on K_1. The vertical line through P_1 has no other circle on it; it is a ruling of Q_2 and of the cone with vertex at P_1; it is tangent to K_1 at P_1. P_3 is on this line. The horizontal line through P_3 contains no other cross; it is a ruling of the cone with vertex at P_3 and hence is tangent to K_2 at P_3. Whenever for a given S_3 the set W contains a quadric on which the two cubic curves have the above relations, then the configuration in X of the preceding pages exists, and the S_3 is conjugate to 54 under a collineation of X.

In the diagram above each point of Q_2 is given by its coordinates k, l, m, n, and each, excepting the points of K_1 and K_2, has a number written underneath it. Each of these numbers 3, 4, $\cdots$, 50 is the number of the cubic on which the point lies. The numbers were assigned arbitrarily to the cubics; they are included here for future reference.

We have seen that every nondegenerate ruled quadric of the set W has on it two cubics. No other cubic can have more than one point on Q, since two points P_α and P_β would determine two three-spaces R_α and R_β in X, and their plane of intersection would determine a third set of rulings of Q. The number of points of Q is $(p + 1)^2$; there are $p + 1$ points on each of K_1 and K_2; there remain $p^2 - 1$ points of Q, which is the number of cubics besides K_1 and K_2. Thus the diagram accounts for all the cubics in S_3.

We have given two equivalent, and closely related, ways of characterizing the space 54 in geometric terms which are independent of any coordinate system. An attempt to apply these criteria to an arbitrary S_3 with no point on V and no Σ-line leads to a long series of computations. The goal is to show that any such S_3 is the one we have been studying, and hence that any S_3 whose quintic polynomial $f(x)$ is irreducible is conjugate to 54. The application of this last criterion, namely, the irreducibility of $f(x)$, is relatively a simple matter; the application of the former is likely to require months of work. Although it will be possible to show that the necessary condition, the irreducibility of $f(x)$, is sufficient to ensure that S_3 is 54, the determination of the transformation which puts one such S_3 into another will require essentially determination of the above configuration in X.

It is clear that one is dealing with pairs of cubics when one undertakes to determine the configuration in X for a given S_3. The number of pairs of cubics is large; one finds immediately that not every pair is a canonical pair, and then right away that not every cubic can be one of a canonical pair. A closer look at individual cubics is therefore indicated. So far one cubic is like another. When we consider planes which osculate the cubics, then differences appear.

Each point P of S_3 is on one and only one cubic K. The cubic has an osculating plane at P. The osculating plane is tangent to the cone Q with vertex at P along the ruling of Q which is tangent to K, the ruling which contains no other point of K. For example, the plane $P_1P_2P_3$ osculates the cubic

K_1 at P_1 (page 707). The osculating plane ρ, like every other plane in S_3, contains an absolute conic C determined by the relation of ρ to V. The equation $B_5x_1 - B_4x_2 + B_3x_3 - B_2x_4 + B_1x_5 = 0$ is used to determine both the conic C and the set W of quadrics, and hence also the cubics. The points of X which give conics in ρ which consist of a single line counted twice must give cones in S_3, since the conic in ρ is the intersection of ρ with the quadric. The only degenerate parabolas in ρ, determined by points in X, are the tangents to C. A plane which passes through two points P_1 and P_2 of a cubic K, unless it is tangent to one of the cones with vertices at P_1 and P_2, meets K in a third point, viz., the intersection of the two rulings aside from P_1P_2 in which it meets the cones. If ρ is tangent to the cone with vertex at P_1, along the ruling P_1P_2, then it is tangent to K at P_2. Hence,

Any plane in S_3 is tangent to those cubics which pass through the points of the absolute conic C and to no others; the points of tangency are the points of C.

If ρ is the plane which osculates the cubic K at the point P, then ρ is tangent to p other cubics. Some of these cubics may osculate ρ. The number of cubics which osculate a given plane is a projective invariant. If the $p + 1$ planes which osculate a given cubic are examined, a set of numbers is obtained which enables us to distinguish among the cubics.

We shall say that a cubic is of type a_1, a_2, a_3, a_4 if the osculating plane at each of a_i points osculates i cubics. (We are dealing here, of course, with space 54.) $a_1 + a_2 + a_3 + a_4 = p + 1 = 8$. The distribution of the cubics into types is given by the following table:

Type	*Names of cubics*
2, 3, 2, 1	1, 14, 19, 26, 39
2, 1, 4, 1	2, 16, 27, 35, 37
2, 5, 0, 1	6, 28, 34, 45, 46
4, 3, 0, 1	10, 18, 23, 36, 50
2, 4, 2, 0	3, 7, 8, 13, 33
3, 2, 3, 0	4, 5, 31, 32, 42
3, 4, 1, 0	9, 11, 12, 24, 43; 17, 22, 40, 48, 49
5, 2, 1, 0	15, 20, 21, 30, 38; 25, 29, 41, 44, 47.

This table records only a small selection of the information about S_3 that must be sought out. There is not enough here to distinguish between two sets of five cubics of each of the last two types; there is enough information to enable us to go on to the determination of canonical pairs of cubics.

Each of the twenty cubics of the first four types in the above list has an osculating plane which osculates four cubics. There are therefore five planes in S_3 each of which osculates four cubics. The planes and the cubics which they osculate are

$$\begin{aligned} k + 3l + 2m + 4n &= 0, &\quad& 1, 2, 46, 50,\\ k + 5l + 2m \phantom{{}+ 4n} &= 0, && 6, 18, 26, 35,\\ k + 5l + 5m + n &= 0, && 14, 23, 27, 28,\\ k + 3l + 6m + 6n &= 0, && 10, 16, 19, 34,\\ k + 2l + 3m + 6n &= 0, && 36, 37, 39, 45. \end{aligned}$$

From this list and the preceding table one reads immediately that if there are any canonical pairs besides 1, 2, they are 26, 35; 14, 27; 19, 16; and 39, 37. If two cubics are a canonical pair, they must be of types 2, 3, 2, 1 and 2, 1, 4, 1, and they must have a common osculating plane which osculates four cubics.

A proof that each of the given pairs is a canonical pair could be given by finding the quadric on which the two cubics lie and then noting that we have the configuration which characterizes $lm + 2kn = 0$. We shall do this for one pair and then exhibit the collineation which transforms the pair in question into 1, 2; the collineation is of period five, hence there are five canonical pairs, which could only be these.

The four points 1, 4, 1, 1; 1, 6, 0, 3; 1, 3, 2, 3; 1, 3, 2, 6, two on each of cubics 16 and 19, determine four three-spaces in X which intersect in $A_2' =$ 1, 5, 1, 4, 2. The quadric Q_2' of the set W determined by A_2' is

$$2k^2 + kl + km + 4kn + 4l^2 + 2lm + 5ln + m^2 + 2mn + 6n^2 = 0.$$

It is represented in Diagram 2. The points of cubic 19 are marked with

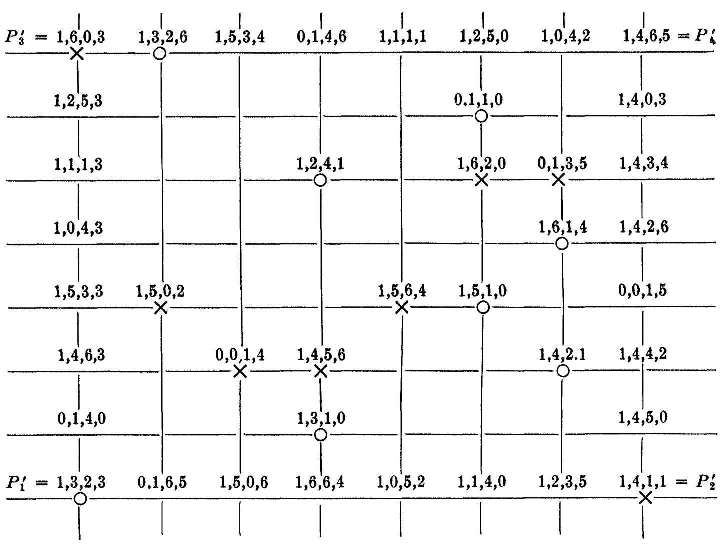

DIAGRAM 2

circles, those of cubic 16 with crosses. The horizontal and vertical lines are rulings of Q_2'. This diagram has the same arrangement of vertices of cones and rulings of Q_2' tangent to cubics 19 and 16 as characterized the quadric $Q_2 : lm + 2kn = 0$ and cubics 1 and 2. Hence if P_1', P_2', P_3' are given the coordinates of P_1, P_2, P_3 in the earlier diagram, S_3 will appear in the form 54. Thus cubics 19 and 16 are shown to be a canonical pair.

The transformation which puts X into itself, S_3 into itself, and cubics 19 and 16 into cubics 1 and 2 respectively is

$$T = \begin{bmatrix} 1 & 2 & 3 & 6 & 6 \\ 6 & 0 & 2 & 6 & 3 \\ 3 & 1 & 5 & 4 & 2 \\ 4 & 1 & 0 & 6 & 2 \\ 3 & 2 & 5 & 3 & 2 \end{bmatrix}.$$

The induced transformation in S_3 is

$$T = \begin{bmatrix} 2 & 5 & 5 & 2 & 4 & 5 & 6 & 6 & 4 & 3 \\ 2 & 3 & 0 & 5 & 0 & 2 & 5 & 3 & 4 & 2 \\ 0 & 2 & 3 & 6 & 4 & 6 & 5 & 4 & 6 & 4 \\ 3 & 3 & 6 & 5 & 4 & 1 & 6 & 0 & 4 & 1 \\ 6 & 3 & 6 & 3 & 5 & 1 & 4 & 6 & 3 & 0 \\ 6 & 6 & 5 & 0 & 5 & 1 & 4 & 5 & 4 & 1 \\ 5 & 3 & 0 & 3 & 3 & 2 & 1 & 4 & 3 & 3 \\ 6 & 1 & 2 & 5 & 2 & 2 & 0 & 2 & 3 & 3 \\ 3 & 0 & 4 & 0 & 2 & 2 & 5 & 2 & 0 & 2 \\ 5 & 6 & 1 & 2 & 5 & 5 & 5 & 5 & 4 & 6 \end{bmatrix}.$$

It may be verified that points of S_3 are transformed as follows:

$$(1, 3, 2, 3)T = (1, 3, 0, 6, 4, 3, 3, 1, 0, 2)T = 1, 0, 0, 0, 0, 0, 0, 1, 0, 0,$$

$$(1, 4, 1, 1)T = (1, 4, 0, 2, 4, 1, 4, 1, 0, 1)T = 0, 1, 0, 0, 0, 0, 1, 0, 0, 0,$$

$$(1, 6, 0, 3)T = (1, 6, 0, 6, 2, 3, 6, 1, 0, 0)T = 0, 0, 0, 0, 1, 0, 0, 0, 0, 1,$$

$$(1, 4, 6, 5)T = (1, 4, 0, 3, 0, 5, 4, 1, 0, 6)T = 0, 0, 0, 2, 3, 1, 0, 0, 0, 0.$$

This verifies that T transforms S_3 into itself by putting P_i' into P_i, $i = 1, 2, 3, 4$. Moreover, noting that in X the points A_1 ($= 1, 0, 0, 0, 0$), A_1T, A_1T^2, A_1T^3, A_1T^4 are linearly independent, and that $A_1T^5 = A_1$, we have the result that T is of period 5.

That the collineation group of X contains a transformation of period 5 that puts S_3 into itself was to be expected. S_3 determines the irreducible polynomial congruence $f(x) = 0$ for the value of k/n which would make $B_1 = \cdots = B_5 = 0$, and determine a point of V. If X, V, S, and S_3 were immersed in spaces $\bar{X}$, $\bar{V}$, $\bar{S}$, and $\bar{S}_3$ over GF(p^5), then the congruence would remain unchanged but would be completely solvable. The Galois group of GF(p^5)

relative to GF(p) is of order 5. This group interchanges the points of intersection of $\bar{V}$ and $\bar{S}_3$ cyclically; it puts $\bar{X}$ into itself, X into itself, S_3 into itself. It is not identity in S_3, for then it would be identity in $\bar{S}_3$. Since the only possible canonical pairs of cubics are the five given above, the Galois group must interchange them.

The collineation of order 5 just described exists for any p, but the fact that the collineation and its powers are the only collineations of X which put S_3 into itself depends on our knowledge of the particular space with $p = 7$. We note that we cannot expect to find any simple short procedure to determine a transformation of X into itself which puts an arbitrary S_3 with an irreducible $f(x)$ into the particular one we have been studying. If it can be done at all, it can be done in only five ways, and doing it requires essentially the finding of a canonical pair of cubics.

We proceed to examine an arbitrary S_3 which has no point on V and no Σ-line. In S_3 we select an arbitrary point P_1 and take for $P_1P_2P_3$ the plane which osculates the cubic through P_1. A coordinate system can be selected so that the plane is $k, l, 0, 0, m, 0, l, k, 0, m$. S_3 contains

$$P_4 = 0, 0, a_3, a_4, a_5, a_6, a_7, a_8, a_9, 0.$$

For any point P in S_3 the B's are

$$\begin{aligned}
B_1 &= k^2 + a_8kn - a_6ln + a_3mn + a_3a_5n^2,\\
B_2 &= a_9kn - l^2 - a_7ln + a_4mn + a_4a_5n^2,\\
B_3 &= km - a_3ln - (a_3a_7 - a_4a_6)n^2,\\
B_4 &= lm + a_4kn - (a_3a_9 - a_4a_8)n^2,\\
B_5 &= m^2 + a_5mn + kl + a_7kn + a_8ln - (a_6a_9 - a_7a_8)n^2.
\end{aligned}$$

The cone of the set W with vertex at P_1 is $a_9B_4 - a_4B_2 = 0$. Now transformation T_{17} changes $P_1P_2P_3$ into itself leaving P_1 fixed, and in it c can be chosen to make $a_4 = 0$ if $a_9 \neq 0$. This transformation moves P_2 along the conic C in $P_1P_2P_3$, so we may assume $a_4 = 0$ if $a_9 \neq 0$. If $a_4 = 0$, the cone with vertex at P_1 is $B_4 = lm - a_3a_9n^2 = 0$, which intersects $P_1P_2P_3$ in the two lines $l = 0$ and $m = 0$. But since the plane osculates the cubic through P_1, it must be tangent to the cone, and hence the choice of P_1 and the plane brings with it the result that a_9 in P_4 is zero. Since $a_9 = 0$, it follows that $a_4 \neq 0$, for otherwise $B_4 = 0$ and $B_2 = 0$ would be two cones with vertices at P_1. Since $a_9 = 0$ and $a_4 \neq 0$, T_{17} can be selected to reduce a_7 to zero.

We now solve $B_3 = 0$ for m in terms of k, l, n; we use this value of m in $B_1 = 0$ to solve for l in terms of k and n; we use this value of l to get m in terms of k and n; and we use the values of l and m in one of $B_2 = 0$, $B_4 = 0$, $B_5 = 0$. We obtain the equation

$$k^5 + \alpha k^4n + \beta k^3n^2 + \gamma k^2n^3 + \delta kn^4 + \varepsilon n^2 = 0,$$

where

$$\alpha = 2a_8, \qquad \beta = 2a_3a_5 + a_8^2, \qquad \gamma = 2a_3a_5a_8 + 4a_3a_4a_6,$$

$$\delta = a_3^2a_5^2 + 4a_3a_4a_6a_8 + a_3^3a_4 - a_4a_5a_6^2, \qquad \varepsilon = a_3^3a_4a_8 + a_4^2a_6^3 - a_3^2a_4a_5a_6.$$

The polynomial $f(x) = x^5 + \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon$ is irreducible. The possible S_3's are those such that the a's of P_4 will give $\alpha, \beta, \gamma, \delta, \varepsilon$ of an $f(x)$ which is irreducible.

We note that multiplication of the coordinates of P_4 by $t \neq 0$ in GF(p) changes $f(x) = 0$ to the equation whose roots are t times those of $f(x) = 0$. This would allow us to restrict attention to P_4's with an arbitrary nonzero coordinate equal to 1, or to one $f(x)$ of the set obtained from one by multiplying its roots by $t \neq 0$. Making use of a change of the unit point in X we can do both of these things. The change of coordinates in X carried out by the diagonal matrix with 1, d, $1/d$, d^2, $1/d^2$ down the main diagonal does not change the coordinates of P_1, P_2, P_3 but does change $f(x) = 0$ to the equation whose roots are d times its roots.

We may therefore look for possible S_3's by separating them into classes: (1) those with $a_3 = 0$, and (2) those with $a_3 = 1$.

(1) If $a_3 = 0$, then $\gamma = 0$ and $\beta - 2\alpha^2 = 0$. By taking account of the fact that changing the unit point in X and changing the coordinates of P_4 by multiplication by $t \neq 0$ do not change S_3, it will be found that there are 14 distinct S_3's for which $a_3 = 0$.

(2) When $a_3 \neq 0$, it may be made 1, and at the same time α may be made 1 if it is not zero, or if $\alpha = 0$, β may be made 1 if it is a square, or a particular not-square if it is not a square. If $a_3 = 1$, α determines a_8; then β determines a_5, and γ determines a_4a_6. With a_8, a_5, and a_4a_6 determined, δ and ε give two linear congruences to determine a_4 and a_6. These determine a_4 and a_6 uniquely when they are independent, and when they are not, the value of a_4a_6 determines a_4 and a_6. There are 66 S_3's so obtained.[23]

The final step in the solution of the problem is now simple. In any three-space in S_3 which has no point on V and no Σ-line, an arbitrary point P_1 may be selected and then a coordinate system in X so that

$$P_1 = 1, 0, 0, 0, 0, 0, 0, 1, 0, 0,$$

the osculating plane of the cubic through P_1, is k, l, 0, 0, m, 0, l, k, 0, m, and $P_4 = 0, 0, a_3, a_4, a_5, a_6, a_7, a_8, a_9, 0$. There are 80 sets of $a_3, \cdots, a_9$ such that $f(x)$ is irreducible and no two of the $f(x)$'s can be obtained one from the

[23] These results are obtained by examining a list of irreducible quintic polynomials; actually only 560 of the total 3360 need be considered. The list would require a lot of space; the preparation of a list to check the above statements is a long process. In *Irreducible quintic congruences*, Thesis, University of Illinois, Urbana, 1952, Dr. C. B. Hanneken gives a straightforward method of determining them. His contribution is a direct and relatively simple way to find one of each set of conjugate quintics under the linear fractional group in GF(p).

other by replacing x with tx. These 80 possibilities may all be realized by proper choice of P_1 in space 54. The group of collineations of X which transform space 54 into itself distributes the 400 points into 80 sets of conjugates. Two P_1's selected from two different sets give different $f(x)$'s since $f(x)$ determines P_4 uniquely.

7. Removal of dependence on the value of p

Some of the argument of the preceding pages depended on p being 7, but most of it did not. The final result is independent of the value of p, and we now divest the argument of dependence on p.

In the treatment of lines, planes, and the first 53 (+ 4) three-spaces any dependence on $p = 7$ comes from the selection of particular polynomials having certain required properties, generally an irreducible quadratic, or cubic, or quartic. The existence of such polynomials does not depend on p. We confine our attention to space 54, i.e., to S_3 with no point on V and no Σ-line. The locus J in X exists, the four-parameter set W of quadrics in S_3 exists, no quadric in the set W has more than one vertex, and no two cones in the set W have the same vertex. S_3 contains $p^2 + 1$ nonintersecting rational cubic curves. The Galois group Γ of GF(p^5) relative to GF(p) transforms X into X, V into V, and S_3 into S_3. Though the final result is the same for all p, there are different geometric situations for different types of the prime.

When $p = 5t + 1$, both $p + 1$ and $p^2 + 1$ are congruent to 2, mod 5. Hence, Γ must transform two cubics, K_1 and K_2, each into itself, and on each of the invariant cubics it leaves two points fixed. Let the fixed points in S_3 be P_1 and P_4 on K_1, and P_2 and P_3 on K_2. Γ must then leave fixed the four points, on J, in X, which give cones with vertices at these points, and also Γ must leave fixed the point A_2 in X, not on J, which gives the nondegenerate ruled quadric Q on which K_1 and K_2 lie. Having these special elements in X and S_3, it is comparatively easy by the methods that have been used to show that a coordinate system can be selected so that S_3 is

$$k,\ l,\ 0,\ n,\ m,\ rn,\ l,\ k,\ 0,\ m,$$

where r is not a fifth power, mod p, but is otherwise arbitrary.

The situation is quite different from the case where $p = 7$ and there are no invariant cubics, no fixed cones, no fixed nondegenerate ruled quadric. When $p = 5t + 1$, the point A_2 is the intersection of the fixed planes σ_1 and σ_2 determined by the cubics K_1 and K_2. In each of the planes σ_1 and σ_2 the four lines on A_2 which determine the rulings of Q through the vertices of the four fixed cones reduce to two. The point A_2 is outside both conics C_1 and C_2 in the planes σ_1 and σ_2.

When $p = 5t - 1$, then $p + 1$ is divisible by 5, $p^2 + 1$ is congruent to 2, mod 5. Hence, in this case there are two fixed cubics, but the cubics have no fixed points. The fixed cubics determine the planes σ_1 and σ_2 in X and a fixed quadric Q of the set W. The intersection A_2 of planes σ_1 and σ_2 is inside both conics C_1 and C_2 in the fixed planes. The polars of A_2 with respect to conics C_1 and C_2 are fixed, under Γ, and they determine two fixed lines in S_3.

When $p = 5t \pm 2$, then there is no cubic in S_3 left fixed by Γ, and hence there are no fixed points. The number of quadrics in W is congruent to 1, mod 5, and hence there is a fixed quadric Q'. Neither Q' nor the point A' in X which determines it came forward to help in characterizing space 54 for $p = 7$. Q' is nondegenerate and has no rulings; the number of points on Q' is $p^2 + 1$, one on each cubic.

Our first step in identifying the space 54, with $p = 7$, was to show that an S_3 containing a quadric in the set W on which the two cubics were properly related to each other could be put in the canonical form in which 54 appears. When $p = 5t + 1$, the group Γ picks out a quadric with two cubics on it and gives all the necessary information to determine a canonical form. With $p = 7$ we started with a configuration we could not be sure was in every S_3, but in this case there is no uncertainty.

Let A_2 be the point in X which determines the nondegenerate ruled quadric Q left fixed by Γ.[24] On A_2 are fixed planes σ_1 and σ_2 containing fixed conics C_1 and C_2. On C_1 are fixed points A_3 and A_5 which determine in S_3 fixed cones with vertices at P_2 and P_3 respectively; P_2 and P_3 are points of the cubic K_1. On C_2 are fixed points A_1 and A_4 which determine cones with vertices at P_4 and P_1 on K_2.

The lines P_1P_2 and P_3P_4 are rulings of Q, they are rulings of the cones with vertices on K_1 at P_2 and P_3, and they are the lines tangent to K_1 at P_2 and P_3. Similarly, lines P_1P_3 and P_2P_4 are rulings of Q, they are rulings of the cones with vertices at P_1 and P_4 on K_2, and they are tangents to K_2 at P_1 and P_4.

The plane $P_1P_2P_3$ osculates K_2 at P_1, since the plane is tangent to K_2 at P_1 and has no other point on K_2; it is tangent to K_1 at P_2. The cone with vertex at P_3 is tangent to $P_1P_2P_3$ along P_2P_3. Hence, the points P_1, P_2, P_3 have the proper relations so that the plane takes the form

$$k,\ l,\ 0,\ 0,\ m,\ 0,\ l,\ k,\ 0,\ m.$$

It is necessary only to determine coordinates of P_4, which is located by rulings of Q through P_2 and P_3. We still have at our disposal the coordinates of one point on C_1 and of one point on C_2. These can be selected so that $P_4 = 0, 0, 0, 1, 0, a_6, 0, 0, 0, 0$, where $f(x) = x^5 + a_6^3$ is irreducible, i.e., where a_6 is not a fifth power. A change of the unit point will change $f(x)$ into $x^5 + d^5a_6^3$, which says that without changing the choice of P_1 the constant term in $f(x)$ can be made to take any value in one coset of the nonzero numbers in GF(p) with respect to the subgroup of fifth powers. The points P_1, P_2,

[24] In the earlier argument we used primed letters, A', Q', P', etc. to denote points, etc. until we found that accents could be dropped and the letters have their usual meanings. As soon as things are named, it will be seen that they are named properly, so we dispense with accents here.

P_3, P_4 enter indistinguishably, i.e., any one of them can be taken for P_1 in the above determination of coordinates of P_4. By changing P_1 the constant term in $f(x)$ may be made any number in GF(p) which is not a fifth power. Therefore, *when* $p = 5t + 1$ *and* S_3 *has no point on* V *and no* Σ*-line, a coordinate system can be selected so that* S_3 *is* $k, l, 0, n, m, rn, l, k, 0, m$, *where* r *is an arbitrary number not a fifth power in* GF(p).

In the foregoing consideration of S_3 for $p = 5t + 1$, attention was directed to the value of p at only two places: (1) $p + 1$ and $p^2 + 1$ were both congruent to 2, mod 5, which ensured two cubics fixed under Γ and two fixed points on each cubic; and (2) $p - 1 = 0$, mod 5, which permits the existence of the polynomial $x^5 + a_6^3$, irreducible in GF(p). For other primes we do not have the convenient P_1, P_2, P_3, P_4 to work with, and neither can we get the simple canonical form.

We can retain the argument and get a canonical form in the following manner. For $p = 5t - 1$, $p^2 = 1$, mod 5, so that $p^2 + 1$ and $(p^2)^2 + 1$ are both congruent to 2, mod 5. For $p = 5t \pm 2$, $p^4 + 1$ and $(p^4)^2 + 1$ are both congruent to 2, mod 5. Thus, if we immerse X, S, and S_3 in spaces $\bar{X}$, $\bar{S}$, and $\bar{S}_3$ over GF(p^2) and GF(p^4) respectively in the two cases, we recover the two fixed cubics and the two fixed points on each; $f(x)$ is still irreducible in the extended fields. The argument goes unchanged to give a canonical form for $\bar{S}_3$, but now a_6 is a number in GF(p^2) or GF(p^4).

A canonical form for $\bar{S}_3$ determines a canonical form for S_3, and vice versa. For p's not of the form $5t + 1$ we can not use the elements fixed under Γ so directly to get a canonical form that will be useful for the groups. However, knowing that one S_3 which gives an irreducible quintic is related to V in the same way as any other, we may take any such S_3 for the canonical form.

To determine that two S_3's are conjugate under a collineation of X, it is necessary only to see that the polynomials $f(x)$ for both are irreducible. To determine the collineation is a direct and reasonably simple problem when $p = 5t + 1$; it is not so simple when $p = 5t \pm 2$. Even at this late stage, when the essentials of the problem and its solution are quite clear, the characterization of S_3 by means of the geometric configuration we did use or by any other looks fortuitous. If we incline to think that now the somewhat tentative method used for $p = 7$ can be replaced by the direct method used for $p = 5t + 1$, we are given pause when we recognize that the work must be carried out in spaces over GF(p^{20}).

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