Higher homotopy associativity of power maps on finite H-spaces

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Abstract Let p be an odd prime, and let $\lambda \in \mathbb{Z}$. Consider the loop space $Y_t = S^{2t-1}_{(p)}$ for $t \geq 1$ with $t|(p-1)$. Then we first determine the condition for the power map Φ_λ on Y_t to be an A_p-map. We next assume that X is a simply connected $\mathbb{F}_p$-finite A_p-space and that λ is a primitive $(p-1)$st root of unity mod p. Our results show that if the reduced power operations $\{\mathscr{P}^i\}_{i\geq 1}$ act trivially on the indecomposable module $QH^*(X;\mathbb{F}_p)$ and the power map Φ_λ on X is an A_n-map with $n > (p-1)/2$, then X is $\mathbb{F}_p$-acyclic.

1. Introduction

A grouplike space is a homotopy associative H-space with a homotopy inverse. Let X be a grouplike space. We denote the multiplication and the homotopy inverse of X by $\mu\colon X^2 \to X$ and $\iota\colon X \to X$, respectively. Consider the power maps $\{\Phi_\lambda\colon X \to X\}_{\lambda\in\mathbb{Z}}$ on X given as follows. If $\lambda \geq 0$, then Φ_λ is inductively defined by $\Phi_0(x) = e$ and

$$\Phi_\lambda(x) = \mu\big(\Phi_{\lambda-1}(x), x\big) \quad \text{for } \lambda > 0, \tag{1.1}$$

where $e \in X$ is the base point of X. In the case of $\lambda < 0$, we can define Φ_λ by $\Phi_\lambda(x) = \iota(\Phi_{-\lambda}(x))$ with (1.1). From the definition, the multiplication of X is homotopy commutative if and only if all the power maps $\{\Phi_\lambda\}_{\lambda\in\mathbb{Z}}$ on X are H-maps. On the other hand, if X is a double loop space, then we see that $\{\Phi_\lambda\}_{\lambda\in\mathbb{Z}}$ are loop maps.

Consider the p-localization $S^{2t-1}_{(p)}$ of the $(2t-1)$-dimensional sphere for an odd prime p and $t \geq 1$. Then $S^{2t-1}_{(p)}$ is a loop space if and only if $t|(p-1)$ by Sullivan [31, pp. 103–105] (see also [14, p. 172, Theorem A]). We denote the loop space $S^{2t-1}_{(p)}$ by Y_t. The loop multiplication of Y_t is assumed to be strictly associative (cf. [14, p. 45]).

From the result of Arkowitz, Ewing, and Schiffman [2, Theorem 2.4], we have the following (see also [20, Theorem 2(d)]).

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THEOREM 1.1 ([2])
Let p be an odd prime. Then the power map Φ_λ on Y_{p-1} is an H-map if and only if $\lambda(\lambda-1)\equiv 0 \bmod p$.

When $t \neq p-1$, all the power maps $\{\Phi_\lambda\}_{\lambda\in\mathbb{Z}}$ on Y_t are H-maps since the multiplication of Y_t is homotopy commutative by [2, Theorem 0.1(1)]. Although p-completed spheres are considered in [2], their results are also valid for p-localized spheres (see [2, p. 296]). We note that Theorem 1.1 is generalized to the case of several p-localized finite loop spaces by McGibbon [20, Section 4] and Theriault [32, p. 85].

On the other hand, the condition for the power map Φ_λ on Y_t to be a loop map is determined by Lin [18, Theorem 1.3].

THEOREM 1.2 ([18])
Let p be an odd prime, and let $t \geq 1$ with $t|(p-1)$. Then the power map Φ_λ on Y_t is a loop map if and only if $\lambda = \alpha^t$ for some p-adic integer $\alpha \in \mathbb{Z}_p^\wedge$.

REMARK 1.3
When $\lambda \not\equiv 0 \bmod p$, the above result was proved by Arkowitz, Ewing, and Schiffman [2, Theorem 4.4 and Corollary 4.5] (see also Rector [24, Section 3]). As is noted in [18, p. 740], Theorem 1.2 can also be derived from the result of Wojtkowiak [35, Theorem 1] or Møller [23, Theorem 1.2].

Using some results from number theory, Theorem 1.2 implies the following corollary (cf. [2, Lemma 4.3]).

COROLLARY 1.4
Let p and t be as in Theorem 1.2. Put $m = (p-1)/t$. Assume that $\lambda \neq 0$, and write $\lambda = p^a b$ with $a \geq 0$ and $b \not\equiv 0 \bmod p$. Then the power map Φ_λ on Y_t is a loop map if and only if $t|a$ and $b^m \equiv 1 \bmod p$.

According to Sugawara [30, Section 2], we have a condition for an H-map between topological monoids (strictly associative H-spaces) to be homotopic to a loop map. His condition is called *strongly homotopy-multiplicativity*. Generalizing the condition, Stasheff [27, II, Definition 4.4] introduced the concept of A_n-maps between topological monoids.

From the definition, an A_2-map is just an H-map. On the other hand, a map $f\colon X \to Y$ is an A_∞-map if and only if we have the induced map $Bf\colon BX \to BY$ with $f \simeq \Omega(Bf)$ by Stasheff [27, II, Theorem 4.5], where BX and BY denote the classifying spaces of X and Y, respectively. Hence, A_n-maps can be regarded as intermediate stages between an H-map and a loop map.

McGibbon [21] considered a condition for the power map on a topological monoid to be an A_3-map. Applying the main result [21, Theorem 9] to the case of $Y_{(p-1)/2}$, he proved the following result.

THEOREM 1.5 ([21, THEOREM 10(III)])

Let $p > 3$. Then the power map Φ_λ on $Y_{(p-1)/2}$ is an A_3-map if and only if $\lambda(\lambda^2 - 1) \equiv 0 \bmod p$.

We first generalize Theorems 1.1 and 1.5 as follows.

THEOREM A

Let p be an odd prime, and let $t \geq 1$ with $t|(p-1)$. Put $m = (p-1)/t$. Then the power maps $\{\Phi_\lambda\}_{\lambda \in \mathbb{Z}}$ on Y_t satisfy the following.

(1) *Φ_λ is an A_m-map for any $\lambda \in \mathbb{Z}$.*
(2) *Φ_λ is an A_{m+1}-map if and only if $\lambda(\lambda^m - 1) \equiv 0 \bmod p$.*

When $\lambda \not\equiv 0 \bmod p$, the power map Φ_λ on Y_t is an A_{m+1}-map if and only if it is a loop map by Theorem A(2) and Corollary 1.4.

In the case of $\lambda \equiv 0 \bmod p$, we have the following.

THEOREM B

Let p, t, and m be as in Theorem A. Assume that $\lambda \equiv 0 \bmod p$ and $2 \leq j \leq t$. Then the power maps $\{\Phi_\lambda\}_{\lambda \in \mathbb{Z}}$ on Y_t satisfy the following.

(1) *If Φ_λ is an $A_{(j-1)m+1}$-map, then it is also an A_{jm}-map.*
(2) *Φ_λ is an A_{jm+1}-map if and only if $\lambda \equiv 0 \bmod p^j$.*

From Theorems A(2) and B(2) and Corollary 1.4, we have the following corollary.

COROLLARY 1.6

Let p, t, and m be as in Theorem A. Then the power map Φ_λ on Y_t is an A_p-map if and only if $\lambda \equiv 0 \bmod p^t$ or $\lambda^m \equiv 1 \bmod p$.

Sugawara [29, Theorem 1.1] also gave a criterion for an H-space to be of the homotopy type of a topological monoid. His criterion is higher homotopy associativity for multiplication. Later Stasheff [27, I, Section 2] expanded the criterion into the concept of A_n-spaces (see Section 2).

From the definition, an A_n-space is an H-space whose multiplication is homotopy associative of the nth order. In particular, an A_2-space and an A_3-space are an H-space and a homotopy associative H-space, respectively. Moreover, a space is an A_∞-space if and only if it is of the homotopy type of a topological monoid by Stasheff [28, I, Theorem 5] (see Remark 2.1).

According to Iwase and Mimura [11, Section 3], Stasheff's definition of A_n-maps between topological monoids is also generalized to the case of maps between A_n-spaces (see Section 2).

In this article, all spaces are assumed to be pointed, connected, and of the homotopy type of CW-complexes. Hence, any A_n-space can be regarded as a grouplike space for $n \geq 3$. A space X is called $\mathbb{F}_p$-*finite* if the mod p cohomology

$H^*(X;\mathbb{F}_p)$ is finite-dimensional as a vector space over $\mathbb{F}_p$, and is called $\mathbb{F}_p$-*acyclic* if the reduced mod p cohomology $\widetilde{H}^*(X;\mathbb{F}_p)=0$.

Our result is as follows.

THEOREM C

Let p be an odd prime. Assume that X is a simply connected $\mathbb{F}_p$-finite A_p-space and that λ is a primitive $(p-1)$st root of unity mod p. *If the reduced power operations $\{\mathscr{P}^i\}_{i\geq 1}$ act trivially on the indecomposable module $QH^*(X;\mathbb{F}_p)$ and the power map Φ_λ on X is an A_n-map with $n>(p-1)/2$, then X is $\mathbb{F}_p$-acyclic.*

REMARK 1.7

(1) In Theorem C, the assumption that X is an "A_p-space" cannot be relaxed. In fact, the power maps $\{\Phi_\lambda\}_{\lambda\in\mathbb{Z}}$ on the A_{p-1}-space Z_t in Example 2.2 are A_{p-1}-maps for any $p>3$ and $t\geq 1$.

(2) If λ is not a primitive $(p-1)$st root of unity mod p, then Theorem C does not hold from the following facts.

(i) When $\lambda\equiv 0 \bmod p$, the power map Φ_λ on Y_2 is an $A_{(p+1)/2}$-map by Theorem A(2).

(ii) Assume that $\lambda^k\equiv 1 \bmod p$ for some k with $1\leq k<p-1$ and $k|(p-1)$. Put $t=(p-1)/k>1$. Then the power map Φ_λ on Y_t is a loop map by Corollary 1.4.

(3) Since the power maps $\{\Phi_\lambda\}_{\lambda\in\mathbb{Z}}$ on Y_2 are $A_{(p-1)/2}$-maps by Theorem A(1), the assumption "$n>(p-1)/2$" is necessary.

This article is organized as follows. In Section 2, we outline the higher homotopy associativity of H-spaces and H-maps introduced by Stasheff [27, Section 2] and Iwase and Mimura [11, Section 3], respectively. In Section 3, Theorems A and B are proved by using the Brown–Peterson operations. Section 4 is devoted to the proof of Theorem C. We first recall the modified projective space $R_p(X)$ of an A_p-space X constructed by Hemmi [7, Section 2]. Based on the mod p cohomology of $R_p(X)$, we have an unstable $\mathscr{A}_p$-algebra $T(p)$ (see Theorem 4.2(2)). We next recall the concept of $\mathscr{D}_n$-algebras in Hemmi and Kawamoto [8, Section 2], and we show that if the power map Φ_λ on X is an A_n-map, then $T(p)$ is a $\mathscr{D}_n$-algebra (see Theorem 4.3). Theorem C is proved by using Theorem 4.3 and some results on $\mathscr{D}_n$-algebras from [8, Section 3].

2. Higher homotopy associativity

We first recall associahedra and multiplihedra constructed by Stasheff [27] and Iwase and Mimura [11], respectively.

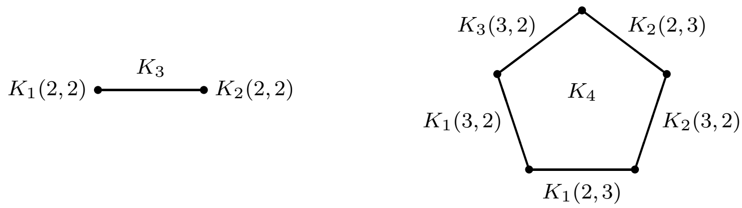

Figure 1. The associahedra K_3 and K_4.

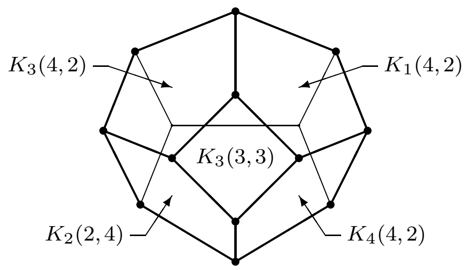

Figure 2. The associahedron K_5.

Stasheff [27, I, Section 6] constructed the associahedra $\{K_n\}_{n\geq 1}$ in order to define A_n-spaces. From the construction, the associahedron K_n is an $(n-2)$-dimensional polytope whose boundary ∂K_n is given by

$$\partial K_n = \bigcup_{(r,s,k)\in\mathbb{K}_n} K_k(r,s) \quad \text{for } n \geq 2,$$

where (see Figures 1 and 2)

$$\mathbb{K}_n = \big\{(r,s,k)\in\mathbb{N}^3 \mid r,s\geq 2 \text{ with } r+s=n+1 \text{ and } k\leq r\big\}.$$

The facet (codimension 1 face) $K_k(r,s)$ is isomorphic (affinely homeomorphic) to the product $K_r\times K_s$ via a face operator

$$\partial_k(r,s)\colon K_r\times K_s \to K_k(r,s) \quad \text{for } (r,s,k)\in\mathbb{K}_n,$$

and there is a family $\{\sigma_j\colon K_n\to K_{n-1}\}_{1\leq j\leq n}$ of degeneracy operators (see [27, I, Section 2]). For convenience, we also put $K_1=\{*\}$. Note that the associahedra $\{K_n\}_{n\geq 1}$ are also used to define A_n-maps from A_n-spaces to topological monoids by Stasheff [28, Definition 11.9].

Iwase and Mimura [11, Section 2] introduced another family $\{J_n\}_{n\geq 1}$ of special complexes when defining A_n-maps between A_n-spaces. Later Forcey [3, Theorem 3] reconstructed J_n as the convex hull of a finite set of points in $\mathbb{R}^n$ (see also [10, Appendix E]). The polytopes $\{J_n\}_{n\geq 1}$ are called multiplihedra.

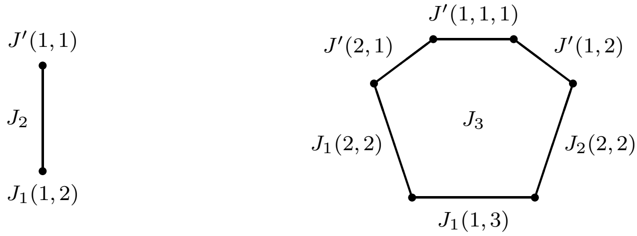

Figure 3. The multiplihedra J_2 and J_3.

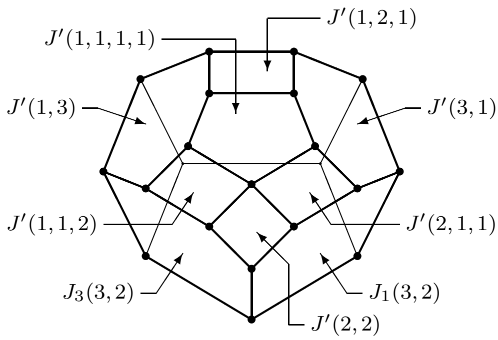

Figure 4. The multiplihedron J_4.

From their constructions, the multiplihedron J_n is an $(n-1)$-dimensional polytope whose boundary ∂J_n is given by

$$\partial J_n = \bigcup_{(r,s,k)\in\mathbb{J}_n} J_k(r,s) \cup \bigcup_{(t_1,\ldots,t_m)\in\mathbb{J}'_n} J'(t_1,\ldots,t_m) \quad \text{for } n \geq 1,$$

where (see Figures 3 and 4)

$$\mathbb{J}_n = \bigl\{(r,s,k)\in\mathbb{N}^3 \mid s\geq 2 \text{ with } r+s=n+1 \text{ and } k\leq r\bigr\}$$

and

$$\mathbb{J}'_n = \bigl\{(t_1,\ldots,t_m)\in\mathbb{N}^m \mid m\geq 2 \text{ and } t_1+\cdots+t_m=n\bigr\}.$$

Moreover, we have face operators

$$\bigl\{\delta_k(r,s)\colon J_r\times K_s \to J_k(r,s)\bigr\}_{(r,s,k)\in\mathbb{J}_n}$$

and

$$\bigl\{\delta'(t_1,\ldots,t_m)\colon K_m\times J_{t_1}\times\cdots\times J_{t_m} \to J'(t_1,\ldots,t_m)\bigr\}_{(t_1,\ldots,t_m)\in\mathbb{J}'_n}$$

and degeneracy operators $\{\zeta_j\colon J_n\to J_{n-1}\}_{1\leq j\leq n}$. As in the case of associahedra, the facets $J_k(r,s)\cong J_r\times K_s$ and $J'(t_1,\ldots,t_m)\cong K_m\times J_{t_1}\times\cdots\times J_{t_m}$ via $\delta_k(r,s)$ and $\delta'(t_1,\ldots,t_m)$, respectively.

We next outline the higher homotopy associativity of H-spaces and H-maps.

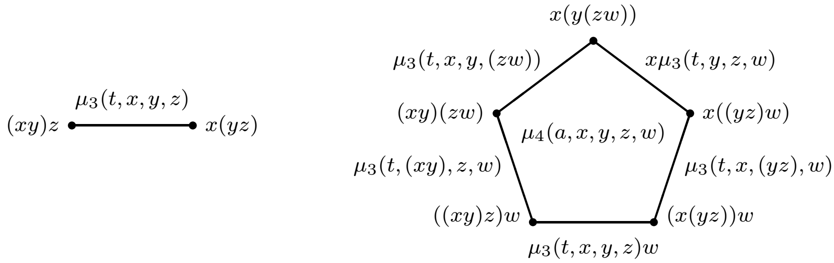

Figure 5. The A_n-forms on X for $n=3$ and 4.

According to Stasheff [27, I, Section 2], an A_n-form on a space X is a family $\{\mu_i\colon K_i \times X^i \to X\}_{1\le i\le n}$ of maps with the following relations:

$$\mu_1(*,x)=x, \tag{2.1}$$

$$\begin{aligned} &\mu_i\big(\partial_k(r,s)(a,b),x_1,\dots,x_i\big)\\ &\quad=\mu_r\big(a,x_1,\dots,x_{k-1},\mu_s(b,x_k,\dots,x_{k+s-1}),x_{k+s},\dots,x_i\big)\\ &\qquad\text{for } (r,s,k)\in\mathbb{K}_i, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\mu_i(a,x_1,\dots,x_{j-1},e,x_{j+1},\dots,x_i)\\ &\quad=\mu_{i-1}\big(\sigma_j(a),x_1,\dots,x_{j-1},x_{j+1},\dots,x_i\big)\quad\text{for } 1\le j\le i. \end{aligned} \tag{2.3}$$

A space with an A_n-form is called an A_n-space for $n\ge 1$ (see Figure 5). If there is a family $\{\mu_i\}_{i\ge 1}$ of maps such that $\{\mu_i\}_{1\le i\le n}$ is an A_n-form on X for any $n\ge 1$, then X is called an A_∞-space.

REMARK 2.1

(1) An A_1-space is just a space. Since $\mu_2(*,x,e)=\mu_2(*,e,x)=x$,

$$\mu_3\big(\partial_1(2,2)(*,*),x,y,z\big)=(xy)z,\qquad\text{and}\qquad \mu_3\big(\partial_2(2,2)(*,*),x,y,z\big)=x(yz),$$

an A_2-space and an A_3-space are an H-space and a homotopy associative H-space, respectively.

(2) From the result of Stasheff [27, I, Theorem 5], a space is an A_∞-space if and only if it is of the homotopy type of a topological monoid (see also [28, Theorem 11.4] and [14, Sections 5 and 6]).

The concept of higher homotopy associativity for maps was first introduced by Sugawara [30, Section 2] and Stasheff [27, II, Definition 4.4] in the case of maps between topological monoids. Later Stasheff [28, Definition 11.9] also considered A_n-maps from A_n-spaces to topological monoids by using the associahedra $\{K_i\}_{i\ge 1}$.

The full generalization was described by Iwase and Mimura [11, Section 3]. They defined A_n-maps between A_n-spaces by using the multiplihedra $\{J_i\}_{i\ge 1}$.

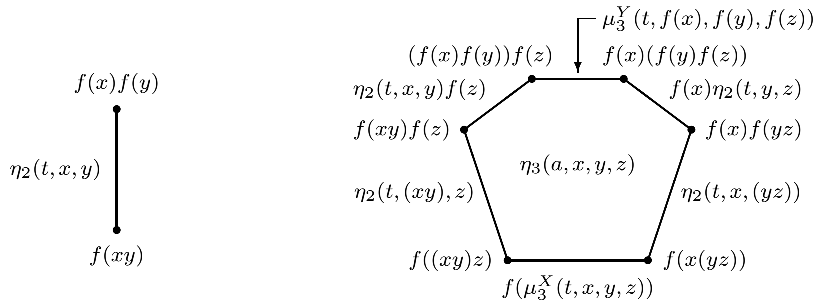

Figure 6. The A_n-forms on f for $n=2$ and 3.

Let X and Y be A_n-spaces with A_n-forms $\{\mu_i^X\}_{1\le i\le n}$ and $\{\mu_i^Y\}_{1\le i\le n}$, respectively. An A_n-form on a map $f\colon X\to Y$ is a family $\{\eta_i\colon J_i\times X^i\to Y\}_{1\le i\le n}$ of maps with the following relations:

$$\eta_1(*,x)=f(x), \tag{2.4}$$

$$\begin{aligned}&\eta_i\big(\delta_k(r,s)(a,b),x_1,\dots,x_i\big)\\ &\quad=\eta_r\big(a,x_1,\dots,x_{k-1},\mu_s^X(b,x_k,\dots,x_{k+s-1}),x_{k+s},\dots,x_i\big)\\ &\qquad\text{for } (r,s,k)\in\mathbb{J}_i,\end{aligned}\tag{2.5}$$

$$\begin{aligned}&\eta_i\big(\delta'(t_1,\dots,t_m)(a,b_1,\dots,b_m),x_1,\dots,x_i\big)\\ &\quad=\mu_m^Y\big(a,\eta_{t_1}(b_1,x_1,\dots,x_{t_1}),\dots,\eta_{t_m}(b_m,x_{t_1+\cdots+t_{m-1}+1},\dots,x_i)\big)\\ &\qquad\text{for } (t_1,\dots,t_m)\in\mathbb{J}_i',\end{aligned}\tag{2.6}$$

$$\begin{aligned}&\eta_i(a,x_1,\dots,x_{j-1},e,x_{j+1},\dots,x_i)\\ &\quad=\eta_{i-1}\big(\zeta_j(a),x_1,\dots,x_{j-1},x_{j+1},\dots,x_i\big)\quad\text{for } 1\le j\le i.\end{aligned}\tag{2.7}$$

A map between A_n-spaces admitting an A_n-form is called an A_n-map for $n\ge 1$ (see Figure 6). From the definition, an A_1-map is just a map. Since

$$\eta_2\big(\delta_1(1,2)(*,*),x,y\big)=f(xy)\qquad\text{and}\qquad \eta_2\big(\delta'(1,1)(*,*),x,y\big)=f(x)f(y),$$

an A_2-map is the same as an H-map. In general, an A_n-map is an H-map between A_n-spaces preserving homotopically their A_n-forms for $n\ge 2$.

If there is a family $\{\eta_i\}_{i\ge 1}$ of maps such that $\{\eta_i\}_{1\le i\le n}$ is an A_n-form on f for any $n\ge 1$, then f is called an A_∞-map. From the result of Iwase and Mimura [11, Theorem 3.1], $f\colon X\to Y$ is an A_∞-map if and only if we have the induced map $Bf\colon BX\to BY$ with $f\simeq\Omega(Bf)$, where BX and BY denote the classifying spaces of X and Y, respectively (see also [14, p. 55]).

Assume that X and Y are A_n-spaces with A_n-forms $\{\mu_i^X\}_{1\le i\le n}$ and $\{\mu_i^Y\}_{1\le i\le n}$, respectively. According to Stasheff [27, II, Definition 4.1], a map $f\colon X\to Y$ is called an A_n-homomorphism if $f\mu_i^X=\mu_i^Y(1_{K_i}\times f^i)$ for $1\le i\le n$. From the definition, an A_n-homomorphism is an A_n-map.

Let p be an odd prime, and let $t \geq 1$. The double suspension $\Sigma_2 \colon S^{2t-1}_{(p)} \to \Omega^2 S^{2t+1}_{(p)}$ is defined as the double adjoint of the identity $1_{S^{2t+1}_{(p)}}$ on $S^{2t+1}_{(p)} \simeq \Sigma^2 S^{2t-1}_{(p)}$. Then $S^{2t-1}_{(p)}$ is an A_{p-1}-space so that Σ_2 is an A_{p-1}-homomorphism by Stasheff [27, I, Theorem 17]. We denote $S^{2t-1}_{(p)}$ with this A_{p-1}-structure by Z_t.

EXAMPLE 2.2

Let $p > 3$ and $t \geq 1$. Then the power map Φ_λ on the A_{p-1}-space Z_t is an A_{p-1}-map for any $\lambda \in \mathbb{Z}$.

Proof

For simplicity, we write $\Omega_t = \Omega^2 S^{2t+1}_{(p)}$. Since Ω_t is a double loop space, the power map $\widehat{\Phi}_\lambda$ on Ω_t is an A_∞-map for any $\lambda \in \mathbb{Z}$. We denote the A_∞-form on $\widehat{\Phi}_\lambda$ by $\{\widehat{\eta}_i\}_{i \geq 1}$. Let $\omega_i \colon J_i \times (Z_t)^i \to \Omega_t$ be defined by $\omega_i = \widehat{\eta}_i(1_{J_i} \times (\Sigma_2)^i)$ for $i \geq 1$.

By induction on i, we construct an A_{p-1}-form $\{\eta_i\}_{1 \leq i \leq p-1}$ on Φ_λ with $\Sigma_2 \eta_i = \omega_i$ for $1 \leq i \leq p-1$. Put $\eta_1(*, x) = x$ for $x \in Z_t$. Assume inductively that $\{\eta_j\}_{1 \leq j < i}$ is constructed for some i with $2 \leq i \leq p-1$. Let $\Gamma_i(X) = \partial J_i \times X^i \cup J_i \times X^{[i]}$ for a space X, and let $i \geq 1$, where $X^{[i]}$ denotes the i-fold fat wedge of X defined as

$$X^{[i]} = \bigl\{(x_1, \ldots, x_i) \in X^i \mid x_j = e \text{ for some } j \text{ with } 1 \leq j \leq i\bigr\}.$$

Then $(J_i \times (Z_t)^i)/\Gamma_i(Z_t) \simeq S^{2ti-1}_{(p)}$.

Now we define $\nu_i \colon \Gamma_i(Z_t) \to Z_t$ by (2.5)–(2.7). By inductive hypothesis, $\Sigma_2 \nu_i = \omega_i|_{\Gamma_i(Z_t)}$. The obstructions to obtain $\eta_i \colon J_i \times (Z_t)^i \to Z_t$ with $\eta_i|_{\Gamma_i(Z_t)} = \nu_i$ and $\Sigma_2 \eta_i = \omega_i$ appear in the following cohomology groups (cf. [1, Proposition 9.2.3]):

$$H^{k+1}\bigl(J_i \times (Z_t)^i, \Gamma_i(Z_t); \pi_k(F_t)\bigr) \cong \widetilde{H}^k\bigl(S^{2ti-2}_{(p)}; \pi_k(F_t)\bigr) \quad \text{for } k \geq 1, \tag{2.8}$$

where F_t denotes the homotopy fiber of Σ_2. Then (2.8) is nontrivial only if $k = 2ti - 2 \leq 2tp - 2t - 2 \leq 2tp - 4$.

On the other hand, $\pi_k(F_t) = 0$ for $k \leq 2tp - 4$ by Toda [34, Corollary 13.2]. Hence, (2.8) is trivial for any k, and we have a map η_i. This completes the induction, and we have an A_{p-1}-form $\{\eta_i\}_{1 \leq i \leq p-1}$ on Φ_λ. □

REMARK 2.3

When $p = 3$ and $t > 1$, Z_t is not a grouplike space from the following facts.

(1) If $S^{2t-1}_{(3)}$ is an A_3-space, then $t = 1$ or 2 by [4, Theorem 1.2].

(2) Z_2 for $p = 3$ is a homotopy commutative H-space (cf. [15, Example 4.8 and Remark 4.5(1)]). Then it is not an A_3-space by [2, Proposition 3.1 and Theorem 3.3].

The following propositions are used to prove Theorems A and B in Section 3.

PROPOSITION 2.4

Assume that p, t, and m are as in Theorem A. Let $\lambda \in \mathbb{Z}$ and $1 \leq j \leq p$. If the power map Φ_λ on Y_t is an $A_{(j-1)m+1}$-map, then it is also an A_{jm}-map.

PROPOSITION 2.5

Let p, t, and m be as in Theorem A. If the power map Φ_λ on Y_t is an A_{m+1}-map, then $\lambda(\lambda^m - 1) \equiv 0 \bmod p$.

In a similar way to the proof of Example 2.2, we can show Proposition 2.4 as follows.

Proof of Proposition 2.4

By induction on i, we construct an A_{jm}-form $\{\eta_i\}_{1\leq i\leq jm}$ on Φ_λ. From the assumption, we have an $A_{(j-1)m+1}$-form $\{\eta_i\}_{1\leq i\leq (j-1)m+1}$ on Φ_λ. Assume inductively that $\{\eta_j\}_{1\leq j<i}$ is constructed for some i with

$$(j-1)m + 2 \leq i \leq jm. \tag{2.9}$$

Define $\widetilde{\eta}_i \colon \Gamma_i(Y_t) \to Y_t$ by (2.5)–(2.7). Then the obstructions to obtain $\eta_i \colon J_i \times (Y_t)^i \to Y_t$ with $\eta_i|_{\Gamma_i(Y_t)} = \widetilde{\eta}_i$ appear in the cohomology groups (cf. [1, Proposition 9.3.3])

$$H^{k+1}\big(J_i \times (Y_t)^i, \Gamma_i(Y_t); \pi_k(Y_t)\big) \cong \widetilde{H}^k\big(S^{2ti-2}_{(p)}; \pi_k(Y_t)\big) \quad \text{for } k \geq 1. \tag{2.10}$$

The above is nontrivial only if k is an even integer with

$$2t + 2(j-1)(p-1) - 2 < k < 2t + 2j(p-1) - 2 \tag{2.11}$$

since $(j-1)(p-1) + 2t \leq ti \leq j(p-1)$ by (2.9).

On the other hand, $\pi_k(Y_t) = 0$ for any even integer k with (2.11) by [34, Theorem 13.4]. Hence, (2.10) is trivial for any k, and we have a map η_i. This completes the induction, and we have an A_{jm}-form $\{\eta_i\}_{1\leq i\leq jm}$ on Φ_λ. $\square$

Let X be an A_n-space. Stasheff [27, I, Theorem 5] constructed the projective spaces $\{P_i(X)\}_{0\leq i\leq n}$ associated to the A_n-form on X. From the construction, $P_0(X) = \{*\}$, $P_1(X) = \Sigma X$, and we have a fibration

$$X \longrightarrow \Sigma^{i-1}X^{\wedge i} \xrightarrow{\gamma_{i-1}} P_{i-1}(X) \tag{2.12}$$

and a long cofibration sequence

$$\Sigma^{i-1}X^{\wedge i} \xrightarrow{\gamma_{i-1}} P_{i-1}(X) \xrightarrow{\iota_{i-1}} P_i(X) \xrightarrow{\rho_i} \Sigma^i X^{\wedge i} \xrightarrow{\Sigma\gamma_{i-1}} \cdots \quad \text{for } 1 \leq i \leq n,$$

where $X^{\wedge i}$ denotes the i-fold smash product of X. When X is an A_∞-space, we have $P_\infty(X) = BX$.

Proof of Proposition 2.5

It is known that (cf. [14, Sections 7 and 24])

$$H^*\big(P_{m+1}(Y_t);\mathbb{F}_p\big) \cong \mathbb{F}_p[x]/(x^{m+2}) \quad \text{with } \deg x = 2t$$

and

$$\mathscr{P}^1(x) = \xi x^{m+1} \quad \text{with } \xi \not\equiv 0 \bmod p. \tag{2.13}$$

Since Φ_λ is an A_{m+1}-map, we have the induced map

$$P_{m+1}(\Phi_\lambda)\colon P_{m+1}(Y_t) \to P_{m+1}(Y_t) \quad \text{with } P_{m+1}(\Phi_\lambda)\varepsilon_m \simeq \varepsilon_m(\Sigma\Phi_\lambda)$$

by [28, Theorem 8.4], where $\varepsilon_i = \iota_i \cdots \iota_1\colon \Sigma Y_t = P_1(Y_t) \to P_{i+1}(Y_t)$ for $i \geq 1$. Then $P_{m+1}(\Phi_\lambda)^*(x) = \lambda x$, and so we have that

$$\mathscr{P}^1 P_{m+1}(\Phi_\lambda)^*(x) = \xi\lambda x^{m+1} \qquad \text{and} \qquad P_{m+1}(\Phi_\lambda)^*\mathscr{P}^1(x) = \xi\lambda^{m+1}x^{m+1}.$$

Hence, $\lambda(\lambda^m - 1) \equiv 0 \bmod p$. □

3. Brown–Peterson cohomology

Let X be a connected space with the homotopy type of a CW-complex of finite type. The Brown–Peterson cohomology $BP^*(X)$ of X is a module over

$$BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{with } \deg v_i = -2(p^i - 1) \text{ for } i \geq 1,$$

where $\mathbb{Z}_{(p)}$ denotes the p-localized integers. When $H^*(X;\mathbb{Z}_{(p)})$ is torsion-free, $BP^*(X)$ is a free BP^*-module and the Thom maps

$$\widetilde{\mathscr{T}}\colon BP^*(X) \to H^*(X;\mathbb{Z}_{(p)}) \qquad \text{and} \qquad \mathscr{T}\colon BP^*(X) \to H^*(X;\mathbb{F}_p)$$

are epimorphisms with $\ker\widetilde{\mathscr{T}} = (v_1, v_2, \dots)$ and $\ker\mathscr{T} = (p, v_1, v_2, \dots)$, respectively.

As in the case of the reduced power operations $\{\mathscr{P}^i\}_{i\geq 1}$ on $H^*(X;\mathbb{F}_p)$, there are operations $\{r_i\}_{i\geq 1}$ on $BP^*(X)$ with the following commutative diagram:

$$\begin{array}{ccc} BP^*(X) & \xrightarrow{r_i} & BP^{*+2i(p-1)}(X) \\ {\scriptstyle\mathscr{T}}\big\downarrow & & \big\downarrow{\scriptstyle\mathscr{T}} \\ H^*(X;\mathbb{F}_p) & \xrightarrow[\chi(\mathscr{P}^i)]{} & H^{*+2i(p-1)}(X;\mathbb{F}_p) \end{array}$$

where χ denotes the canonical antiautomorphism on $\mathscr{A}_p$. In particular, we have

$$\mathscr{T} r_1 = -\mathscr{P}^1\mathscr{T} \tag{3.1}$$

since $\chi(\mathscr{P}^1) = -\mathscr{P}^1$ by [22, p. 167].

According to Kane [13, Sections 1 and 2], the Brown–Peterson operations $\{r_i\}_{i\geq 1}$ have many useful properties similar to those of $\{\mathscr{P}^i\}_{i\geq 1}$ (see also [14, Appendix C]).

In order to prove Theorems A and B, we first show the following propositions.

PROPOSITION 3.1

Assume that p, t, and m are as in Theorem A. If $0 \leq j \leq p-1$ and $\lambda \equiv 0 \bmod p^j$, then the power map Φ_λ on Y_t is an A_{jm+1}-map.

PROPOSITION 3.2

Let p, t, and m be as in Theorem A. Assume that $1 \le j \le t$ and $\lambda \equiv 0 \bmod p$. If the power map Φ_λ on Y_t is an A_{jm+1}-map, then $\lambda \equiv 0 \bmod p^j$.

From the result of Toda [34, Theorem 13.4], we have

$$\pi_{2t+2j(p-1)-2}(Y_t) \cong \mathbb{Z}/p\{\alpha_j\} \quad \text{for } 1 \le j \le p-1. \tag{3.2}$$

Put $\varphi_j = \Sigma\alpha_j \colon S^{2t+2j(p-1)-1}_{(p)} \to \Sigma Y_t$ for $1 \le j \le p-1$. Let $C(\varphi_j)$ be the cofiber of φ_j. Then

$$H^*\big(C(\varphi_j);\mathbb{Z}_{(p)}\big) = \mathbb{Z}_{(p)}\{z, w\} \quad \text{as a } \mathbb{Z}_{(p)}\text{-algebra}$$
$$\text{with } \deg z = 2t \text{ and } \deg w = 2t + 2j(p-1).$$

Take $\boldsymbol{z} \in BP^{2t}(C(\varphi_j))$ and $\boldsymbol{w} \in BP^{2t+2j(p-1)}(C(\varphi_j))$ with $\widetilde{\mathscr{T}}(\boldsymbol{z}) = z$ and $\widetilde{\mathscr{T}}(\boldsymbol{w}) = w$, respectively. For dimensional reasons, we can write that

$$r_1(\boldsymbol{z}) = \zeta v_1^{j-1}\boldsymbol{w} \quad \text{for some } \zeta \in \mathbb{Z}_{(p)}. \tag{3.3}$$

In the proof of Proposition 3.1, we need the following lemma.

LEMMA 3.3

We have $\zeta \not\equiv 0 \bmod p$ in (3.3).

Proof

Put $\varphi'_j = \Sigma^k\varphi_j \colon S^{2t+2j(p-1)+k-1}_{(p)} \to \Sigma^{k+1}Y_t$, where k is an integer with $2t + k > 2j(p-1)$. Then $\varphi'_j \in \pi^S_{2j(p-1)-1}$. Since $C(\varphi'_j) = \Sigma^k C(\varphi_j)$, we have that $\sigma^k \colon BP^*(C(\varphi_j)) \to BP^{*+k}(C(\varphi'_j))$ is an isomorphism, where σ denotes the suspension isomorphism.

Put $\boldsymbol{z}' = \sigma^k(\boldsymbol{z}) \in BP^{2t+k}(C(\varphi'_j))$ and $\boldsymbol{w}' = \sigma^k(\boldsymbol{w}) \in BP^{2t+2j(p-1)+k}(C(\varphi'_j))$, respectively. Then by (3.3),

$$r_1(\boldsymbol{z}') = \zeta v_1^{j-1}\boldsymbol{w}'. \tag{3.4}$$

Applying r_{j-1} to (3.4), we have $r_{j-1}r_1(\boldsymbol{z}') = \zeta p^{j-1}\boldsymbol{w}'$ by [13, p. 458, (2.2)]. On the other hand, $r_{j-1}r_1(\boldsymbol{z}') \equiv jr_j(\boldsymbol{z}') \bmod \ker\widetilde{\mathscr{T}}$ by [13, p. 455, (1.2)] and [22, p. 164]. Now $r_j(\boldsymbol{z}') = \gamma\boldsymbol{w}'$ with $\gamma \not\equiv 0 \bmod p^j$ by [26, Proposition 1.1 and Theorem 2.1]. Hence, $\zeta \not\equiv 0 \bmod p$. $\square$

Since $\varphi_j = \Sigma\alpha_j$ is a suspension map, we have a self-map $\Lambda_j \colon C(\varphi_j) \to C(\varphi_j)$ with the following commutative diagram:

$$\begin{tikzcd} S^{2t+2j(p-1)-1}_{(p)} \arrow[r, "\varphi_j"] \arrow[d, "{[\lambda]}"'] & \Sigma Y_t \arrow[r] \arrow[d, "\Sigma\Phi_\lambda"] & C(\varphi_j) \arrow[d, "\Lambda_j"] \\ S^{2t+2j(p-1)-1}_{(p)} \arrow[r, "\varphi_j"] & \Sigma Y_t \arrow[r] & C(\varphi_j) \end{tikzcd} \tag{3.5}$$

where $[\lambda]$ denotes the self-map of degree λ.

Proof of Proposition 3.1

We work by induction on j. The result is clear for $j=0$. Assume inductively that the result is proved for $j-1$ with $1 \le j \le p-1$. Now $\lambda \equiv 0 \bmod p^j$. By inductive hypothesis, Φ_λ is an $A_{(j-1)m+1}$-map, and so Proposition 2.4 implies that it is also an A_{jm}-map. Then we have the induced map

$$P_{jm}(\Phi_\lambda)\colon P_{jm}(Y_t) \to P_{jm}(Y_t) \quad \text{with } P_{jm}(\Phi_\lambda)\varepsilon_{jm-1} \simeq \varepsilon_{jm-1}(\Sigma\Phi_\lambda)$$

by [28, Theorem 8.4].

Let $\widetilde{\varphi}_j = \varepsilon_{jm-1}\varphi_j\colon S^{2t+2j(p-1)-1}_{(p)} \to P_{jm}(Y_t)$. Since there is a fibration

$$Y_t \longrightarrow S^{2t+2j(p-1)-1}_{(p)} \xrightarrow{\gamma_{jm}} P_{jm}(Y_t)$$

by (2.12), we have

$$\pi_{2t+2j(p-1)-1}\big(P_{jm}(Y_t)\big) \cong \mathbb{Z}_{(p)}\{\gamma_{jm}\} \oplus \mathbb{Z}/p\{\widetilde{\varphi}_j\}.$$

Let $\widehat{\varphi}_j = \iota_{jm}\widetilde{\varphi}_j = \varepsilon_{jm}\varphi_j\colon S^{2t+2j(p-1)-1}_{(p)} \to P_{jm+1}(Y_t)$. Put $X_j = C(\widehat{\varphi}_j)$. Then $C(\varphi_j) \subset X_j$ and we see that $\pi_{2t+2j(p-1)-1}(X_j) = 0$ by using the Blakers–Massey theorem (cf. [1, Theorem 5.6.4]). Since $P_{jm+1}(Y_t) = C(\gamma_{jm})$, there is a map $\widetilde{\Psi}_j\colon P_{jm+1}(Y_t) \to X_j$ with the following commutative diagram:

$$\begin{tikzcd}
S^{2t}_{(p)} \arrow[r, equal] \arrow[d, "{[\lambda]}"'] & \Sigma Y_t \arrow[r, "\varepsilon_{jm-1}"] \arrow[d, "\Sigma\Phi_\lambda"'] & P_{jm}(Y_t) \arrow[r, "\iota_{jm}"] \arrow[d, "P_{jm}(\Phi_\lambda)"] & P_{jm+1}(Y_t) \arrow[d, "\widetilde{\Psi}_j"] \\
S^{2t}_{(p)} \arrow[r, equal] & \Sigma Y_t \arrow[r, "\varepsilon_{jm-1}"'] & P_{jm}(Y_t) \arrow[r, "\widetilde{\iota}_{jm}"'] & X_j
\end{tikzcd}$$

where $\widetilde{\iota}_{jm}$ denotes the composition of ι_{jm} and the inclusion $P_{jm+1}(Y_t) \subset X_j$.

Consider the self-map $\Psi_j\colon X_j \to X_j$ defined by $\Psi_j|_{P_{jm+1}(Y_t)} = \widetilde{\Psi}_j$ and $\Psi_j|_{C(\varphi_j)} = \Lambda_j$ in (3.5). From the definition of X_j, we have that

$$H^*(X_j;\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[x]/(x^{jm+2}) \oplus \mathbb{Z}_{(p)}\{y\} \quad \text{as a } \mathbb{Z}_{(p)}\text{-algebra}$$
$$\text{with } \deg x = 2t \text{ and } \deg y = 2t + 2j(p-1).$$

Since $\Psi_j|_{C(\varphi_j)} = \Lambda_j$, the induced homomorphism

$$\Psi_j^*\colon H^*(X_j;\mathbb{Z}_{(p)}) \to H^*(X_j;\mathbb{Z}_{(p)})$$

is given by $\Psi_j^*(x) = \lambda x$ and $\Psi_j^*(y) = \lambda y + \eta x^{jm+1}$ for some $\eta \in \mathbb{Z}_{(p)}$.

In order to complete the proof, we need to show that

$$\eta \equiv 0 \bmod p. \tag{3.6}$$

Take $\boldsymbol{x} \in BP^{2t}(X_j)$ and $\boldsymbol{y} \in BP^{2t+2j(p-1)}(X_j)$ with $\widetilde{\mathscr{T}}(\boldsymbol{x}) = x$ and $\widetilde{\mathscr{T}}(\boldsymbol{y}) = y$, respectively. Then we can assume that $\boldsymbol{z} = \tau_j^*(\boldsymbol{x})$ and $\boldsymbol{w} = \tau_j^*(\boldsymbol{y})$ are as in (3.3), where $\tau_j\colon C(\varphi_j) \to X_j$ denotes the inclusion. For dimensional reasons, we can write that

$$\Psi_j^*(\boldsymbol{x}) = \lambda\boldsymbol{x} + \sum_{k=1}^{j} \theta_k v_1^k \boldsymbol{x}^{km+1} + \delta v_1^j \boldsymbol{y} \quad \text{with } \theta_k, \delta \in \mathbb{Z}_{(p)} \text{ for } 1 \le k \le j,$$

$$r_1(\boldsymbol{x}) = \sum_{\ell=1}^{j} \xi_\ell v_1^{\ell-1} \boldsymbol{x}^{\ell m+1} + \zeta v_1^{j-1} \boldsymbol{y} \quad \text{with } \xi_\ell \in \mathbb{Z}_{(p)} \text{ for } 1 \le \ell \le j,$$

$$\Psi_j^*(\boldsymbol{y}) = \lambda \boldsymbol{y} + \eta \boldsymbol{x}^{jm+1}, \qquad \text{and} \qquad r_1(\boldsymbol{y}) = 0.$$

Then

$$\begin{aligned} r_1\big(\Psi_j^*(\boldsymbol{x})\big) &= \sum_{k=1}^{j} (pk\theta_k + \lambda \xi_k) v_1^{k-1} \boldsymbol{x}^{km+1} \\ &\quad + \sum_{\substack{k,\ell \ge 1 \\ k+\ell \le j}} (km+1)\theta_k \xi_\ell v_1^{k+\ell-1} \boldsymbol{x}^{(k+\ell)m+1} + (pj\delta + \lambda\zeta) v_1^{j-1} \boldsymbol{y}. \end{aligned} \tag{3.7}$$

On the other hand,

$$\begin{aligned} \Psi_j^*\big(r_1(\boldsymbol{x})\big) &= \sum_{\ell=1}^{j} \xi_\ell v_1^{\ell-1} \Big(\lambda \boldsymbol{x} + \sum_{k=1}^{j} \theta_k v_1^k \boldsymbol{x}^{km+1}\Big)^{\ell m+1} \\ &\quad + \zeta\eta v_1^{j-1} \boldsymbol{x}^{jm+1} + \lambda\zeta v_1^{j-1} \boldsymbol{y}. \end{aligned} \tag{3.8}$$

To show (3.6), we first prove that if $\lambda \equiv 0 \bmod p^j$, then

$$\theta_k \equiv 0 \bmod p^{j-k} \quad \text{for } 1 \le k \le j. \tag{3.9}$$

We work by induction on k. When $k = 1$, we compare the coefficients mod p^j of $\boldsymbol{x}^{m+1}$ in (3.7) and (3.8). From the assumption, we have $p\theta_1 \equiv 0 \bmod p^j$. Hence, $\theta_1 \equiv 0 \bmod p^{j-1}$.

Assume inductively that $\theta_i \equiv 0 \bmod p^{j-i}$ for $1 \le i \le k-1$ with $2 \le k \le j$. Compare the coefficients mod p^{j-k+1} of $\boldsymbol{x}^{km+1}$ in (3.7) and (3.8). By inductive hypothesis, we have $pk\theta_k \equiv 0 \bmod p^{j-k+1}$. Then $\theta_k \equiv 0 \bmod p^{j-k}$ since $k \le j \le p-1$. This completes the induction, and we have (3.9).

We next compare the coefficients mod p of $\boldsymbol{x}^{jm+1}$ in (3.7) and (3.8). Then $\zeta\eta \equiv 0 \bmod p$ by (3.9). Now $\zeta \not\equiv 0 \bmod p$ by Lemma 3.3, and so we have (3.6).

Let $\boldsymbol{a}, \boldsymbol{b} \in H_{2t+2j(p-1)}(X_j; \mathbb{Z}_{(p)})$ denote the Kronecker duals of

$$x^{jm+1}, y \in H^{2t+2j(p-1)}(X_j; \mathbb{Z}_{(p)}),$$

respectively. Using the duality, we can show that

$$(\Psi_j)_*(\boldsymbol{a}) = \lambda^{jm+1} \boldsymbol{a} + \eta \boldsymbol{b} \qquad \text{and} \qquad (\Psi_j)_*(\boldsymbol{b}) = \lambda \boldsymbol{b}.$$

Consider the homomorphism

$$\mathscr{E}_j \colon H_{2t+2j(p-1)}(X_j; \mathbb{Z}_{(p)}) \to \pi_{2t+2j(p-1)-1}\big(P_{jm}(Y_t)\big)$$

defined by the following composition:

$$\begin{aligned} H_{2t+2j(p-1)}(X_j; \mathbb{Z}_{(p)}) &\longrightarrow H_{2t+2j(p-1)}\big(X_j, P_{jm}(Y_t); \mathbb{Z}_{(p)}\big) \\ &\xrightarrow[\cong]{\mathscr{H}^{-1}} \pi_{2t+2j(p-1)}\big(X_j, P_{jm}(Y_t)\big) \xrightarrow{\partial} \pi_{2t+2j(p-1)-1}\big(P_{jm}(Y_t)\big), \end{aligned}$$

where $\mathscr{H}$ denotes the Hurewicz isomorphism. Then $P_{jm}(\Phi_\lambda)_\# \mathscr{E}_j = \mathscr{E}_j (\Psi_j)_*$. Since $\mathscr{E}_j(\boldsymbol{a}) = \gamma_{jm}$ and $\mathscr{E}_j(\boldsymbol{b}) = \widetilde{\varphi}_j$, we have that

$$P_{jm}(\Phi_\lambda)_\#(\gamma_{jm}) = \lambda^{jm+1}\gamma_{jm} + \eta\widetilde{\varphi}_j = \lambda^{jm+1}\gamma_{jm}$$

by (3.6). Hence, $\iota_{jm}P_{jm}(\Phi_\lambda)\gamma_{jm}$ is null-homotopic, and so there is a self-map

$$\psi_j\colon P_{jm+1}(Y_t) \to P_{jm+1}(Y_t) \quad \text{with } \psi_j\iota_{jm} \simeq \iota_{jm}P_{jm}(\Phi_\lambda).$$

Then Φ_λ is an A_{jm+1}-map by [28, Theorem 8.4]. This completes the proof of Proposition 3.1. □

Proof of Proposition 3.2

We work by induction on j. From the assumption, the result is clear for $j=1$. Assume inductively that the result is proved for $j-1$ with $2 \le j \le t$. Now Φ_λ is an A_{jm+1}-map. Then we have the induced map

$$P_{jm+1}(\Phi_\lambda)\colon P_{jm+1}(Y_t) \to P_{jm+1}(Y_t) \quad \text{with } P_{jm+1}(\Phi_\lambda)\varepsilon_{jm} \simeq \varepsilon_{jm}(\Sigma\Phi_\lambda)$$

by [28, Theorem 8.4]. By inductive hypothesis, we have

$$\lambda \equiv 0 \bmod p^{j-1} \tag{3.10}$$

since Φ_λ is also an $A_{(j-1)m+1}$-map.

It is known that

$$H^*\big(P_{jm+1}(Y_t);\mathbb{Z}_{(p)}\big) = \mathbb{Z}_{(p)}[x]/(x^{jm+2}) \quad \text{as a } \mathbb{Z}_{(p)}\text{-algebra with } \deg x = 2t.$$

Take $\boldsymbol{x} \in BP^{2t}(P_{jm+1}(Y_t))$ with $\widetilde{\mathscr{T}}(\boldsymbol{x}) = x$. For dimensional reasons, we can write that

$$P_{jm+1}(\Phi_\lambda)^*(\boldsymbol{x}) = \lambda\boldsymbol{x} + \sum_{k=1}^{j}\theta_k v_1^k \boldsymbol{x}^{km+1} \quad \text{with } \theta_k \in \mathbb{Z}_{(p)} \text{ for } 1 \le k \le j$$

and

$$r_1(\boldsymbol{x}) = \sum_{\ell=1}^{j}\xi_\ell v_1^{\ell-1}\boldsymbol{x}^{\ell m+1} \quad \text{with } \xi_\ell \in \mathbb{Z}_{(p)} \text{ for } 1 \le \ell \le j.$$

Then

$$\begin{aligned} r_1\big(P_{jm+1}(\Phi_\lambda)^*(\boldsymbol{x})\big) &= \sum_{k=1}^{j}(pk\theta_k + \lambda\xi_k)v_1^{k-1}\boldsymbol{x}^{km+1} \\ &\quad + \sum_{\substack{k,\ell\ge 1\\ k+\ell\le j}}(km+1)\theta_k\xi_\ell v_1^{k+\ell-1}\boldsymbol{x}^{(k+\ell)m+1}. \end{aligned} \tag{3.11}$$

On the other hand,

$$P_{jm+1}(\Phi_\lambda)^*\big(r_1(\boldsymbol{x})\big) = \sum_{\ell=1}^{j}\xi_\ell v_1^{\ell-1}\Big(\lambda\boldsymbol{x} + \sum_{k=1}^{j}\theta_k v_1^k\boldsymbol{x}^{km+1}\Big)^{\ell m+1}. \tag{3.12}$$

To complete the proof, we first show that

$$\theta_k \equiv 0 \bmod p^{j-k} \quad \text{for } 1 \le k \le j. \tag{3.13}$$

We work by downward induction on k. The result is clear for $k=j$. Assume inductively that the result is proved for $k+1$ with $1 \le k \le j-1$. Then

$$\theta_{k+1} \equiv 0 \bmod p^{j-k-1}. \tag{3.14}$$

Using the same way as in the proof of (3.9), we have that $\theta_i \equiv 0 \bmod p^{j-i-1}$ for $1 \le i \le j-1$ by (3.10). Hence,

$$\theta_i \equiv 0 \bmod p^{j-k} \quad \text{for } 1 \le i \le k-1. \tag{3.15}$$

Compare the coefficients mod p^{j-k} of $\boldsymbol{x}^{(k+1)m+1}$ in (3.11) and (3.12). Then $(km+1)\theta_k\xi_1 \equiv 0 \bmod p^{j-k}$ by (3.10), (3.14), and (3.15). Now we note that $\xi_1 \not\equiv 0 \bmod p$ by (2.13) and (3.1). Then $\theta_k \equiv 0 \bmod p^{j-k}$ since $k \le j-1 \le t-1$. This completes the induction, and so we have (3.13).

We next compare the coefficients mod p^j of $\boldsymbol{x}^{m+1}$ in (3.11) and (3.12). Then $p\theta_1 + \lambda\xi_1 \equiv 0 \bmod p^j$. Since $\xi_1 \not\equiv 0 \bmod p$ and $\theta_1 \equiv 0 \bmod p^{j-1}$ by (3.13), we have $\lambda \equiv 0 \bmod p^j$. This completes the proof of Proposition 3.2. □

We are now in position to prove Theorems A and B.

Proof of Theorem A
We see that (1) follows from Proposition 2.4 in the case of $j=1$. We have (2) by Propositions 2.5 and 3.1 for $j=1$ and Corollary 1.4. □

Proof of Theorem B
Proposition 2.4 implies (1). We have (2) by Propositions 3.1 and 3.2. □

4. Modified projective spaces

Let p be an odd prime. Assume that X is a simply connected A_p-space whose mod p cohomology $H^*(X;\mathbb{F}_p)$ is an exterior algebra given as

$$H^*(X;\mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(x_1,\dots,x_\ell) \quad \text{with } \deg x_i = 2t_i - 1 \text{ for } 1 \le i \le \ell. \tag{4.1}$$

Iwase [9] gave a structure theorem for the K-cohomology of the projective spaces $\{P_n(X)\}_{1\le n\le p}$. Later Hemmi [5, Section 3] used his method to determine the mod p cohomology of them. Consider the homomorphisms

$$\mathscr{F}_n\colon \widetilde{H}^*(X;\mathbb{F}_p)^{\otimes n} \to \widetilde{H}^*\big(P_n(X);\mathbb{F}_p\big) \quad \text{for } 1 \le n \le p$$

and

$$\mathscr{G}_n\colon \widetilde{H}^*\big(P_n(X);\mathbb{F}_p\big) \to \widetilde{H}^*(X;\mathbb{F}_p)^{\otimes n+1} \quad \text{for } 0 \le n \le p-1$$

defined by the following compositions:

$$\widetilde{H}^*(X;\mathbb{F}_p)^{\otimes n} \cong \widetilde{H}^*(X^{\wedge n};\mathbb{F}_p) \xrightarrow[\cong]{\sigma^n} \widetilde{H}^*(\Sigma^n X^{\wedge n};\mathbb{F}_p) \xrightarrow{\rho_n^*} \widetilde{H}^*\big(P_n(X);\mathbb{F}_p\big)$$

and

$$\widetilde{H}^*\big(P_n(X);\mathbb{F}_p\big) \xrightarrow{\gamma_n^*} \widetilde{H}^*(\Sigma^n X^{\wedge n+1};\mathbb{F}_p)$$
$$\xrightarrow[\cong]{(\sigma^{-1})^n} \widetilde{H}^*(X^{\wedge n+1};\mathbb{F}_p) \cong \widetilde{H}^*(X;\mathbb{F}_p)^{\otimes n+1},$$

respectively. Here $M^{\otimes j}$ is the j-fold tensor product of an $\mathbb{F}_p$-module M, and σ denotes the suspension isomorphism. From the definition, $\deg \mathscr{F}_n = -\deg \mathscr{G}_n = n$ and $\mathscr{F}_1 = \sigma$.

Consider the reduced coproduct $\widetilde{\Delta}\colon \widetilde{H}^*(X;\mathbb{F}_p) \to \widetilde{H}^*(X;\mathbb{F}_p)^{\otimes 2}$ on $\widetilde{H}^*(X;\mathbb{F}_p)$. Then by [5, p. 100],

$$\mathscr{G}_n \mathscr{F}_n = \sum_{j=1}^{n} (-1)^j 1^{\otimes(j-1)} \otimes \widetilde{\Delta} \otimes 1^{\otimes(n-j)} \quad \text{for } 1 \le n \le p-1. \tag{4.2}$$

Put $S(n) = \mathscr{F}_n(D(n)) \subset \widetilde{H}^*(P_n(X);\mathbb{F}_p)$, where

$$D(n) = \sum_{j=1}^{n} \widetilde{H}^*(X;\mathbb{F}_p)^{\otimes(j-1)} \otimes DH^*(X;\mathbb{F}_p) \otimes \widetilde{H}^*(X;\mathbb{F}_p)^{\otimes(n-j)}$$

and DA denotes the decomposable module of an $\mathbb{F}_p$-algebra A. Then $S(n)$ is an ideal of $H^*(P_n(X);\mathbb{F}_p)$ closed under the action of $\mathscr{A}_p$ with (see [5, Theorem 3.5(1)])

$$\iota_{n-1}^*\big(S(n)\big) = 0 \qquad \text{and} \qquad S(n) \cdot \widetilde{H}^*\big(P_n(X);\mathbb{F}_p\big) = 0.$$

Let $\mathbb{F}_p[z_1, \dots, z_\ell]$ be a polynomial algebra over $\mathbb{F}_p$ with generators $\{z_i\}_{1 \le i \le \ell}$. Then the truncated polynomial algebra $T_{\mathbb{F}_p}^{[k]}[z_1, \dots, z_\ell]$ at height k is defined by

$$T_{\mathbb{F}_p}^{[k]}[z_1, \dots, z_\ell] = \mathbb{F}_p[z_1, \dots, z_\ell]/D^k\mathbb{F}_p[z_1, \dots, z_\ell],$$

where D^kA denotes the k-fold decomposable module of an $\mathbb{F}_p$-algebra A for $k \ge 2$ with $D^2A = DA$.

Iwase [9] and Hemmi [5] proved the following result.

THEOREM 4.1 ([9, THEOREM A] AND [5, THEOREM 3.5])
Let p be an odd prime, and let $1 \le n \le p-1$. Assume that X is a simply connected A_p-space whose mod p *cohomology $H^*(X;\mathbb{F}_p)$ is as in (4.1). Then there are classes*

$$y_i \in \widetilde{H}^{2t_i}\big(P_n(X);\mathbb{F}_p\big) \quad \textit{with } \iota_1^* \cdots \iota_{n-1}^*(y_i) = \sigma(x_i) \textit{ for } 1 \le i \le \ell$$

such that

$$H^*\big(P_n(X);\mathbb{F}_p\big) \cong T(n) \oplus S(n) \quad \textit{as an } \mathbb{F}_p\textit{-algebra,}$$

where $T(n) = T_{\mathbb{F}_p}^{[n+1]}[y_1, \dots, y_\ell]$.

We remark that they also proved Theorem 4.1 in the case of $n = p$ under an additional assumption that the generators $\{x_i\}_{1 \le i \le \ell}$ are A_p-primitive, where a class $x \in \widetilde{H}^*(X;\mathbb{F}_p)$ is called A_n-primitive if there is a class

$$y \in \widetilde{H}^{*+1}\big(P_n(X);\mathbb{F}_p\big) \quad \text{with } \iota_1^* \cdots \iota_{n-1}^*(y) = \sigma(x).$$

Since $\gamma_1^* = \sigma\widetilde{\Delta}\sigma^{-1}$, we see that a class is A_2-primitive if and only if it is primitive. From Theorem 4.1, if X is a simply connected A_p-space whose mod p cohomology $H^*(X;\mathbb{F}_p)$ is as in (4.1), then $\{x_i\}_{1 \le i \le \ell}$ are A_{p-1}-primitive.

Hemmi [7, Section 2] modified the construction of $P_p(X)$ to obtain the truncated polynomial algebra $T(p)$ without the assumption that $\{x_i\}_{1\leq i\leq \ell}$ are A_p-primitive. He proved the following result.

THEOREM 4.2 ([7, THEOREM 1.1])
Let p and X be as in Theorem 4.1. Then we have a space $R_p(X)$ and a map $\varepsilon\colon \Sigma X \to R_p(X)$ with the following properties.

(1) *There is a subalgebra $A^* \subset H^*(R_p(X);\mathbb{F}_p)$ with*

$$A^* \cong T^{[p+1]}_{\mathbb{F}_p}[y_1,\dots,y_\ell] \oplus M \quad \text{as an } \mathbb{F}_p\text{-algebra},$$

where

$$y_i \in \widetilde{H}^{2t_i}\bigl(R_p(X);\mathbb{F}_p\bigr) \quad \text{with } \varepsilon^*(y_i) = \sigma(x_i) \text{ for } 1 \leq i \leq \ell$$

and M is an ideal of $H^*(R_p(X);\mathbb{F}_p)$ with

$$\varepsilon^*(M) = 0 \qquad \text{and} \qquad M \cdot \widetilde{H}^*\bigl(R_p(X);\mathbb{F}_p\bigr) = 0.$$

(2) *A^* and M are closed under the action of $\mathscr{A}_p$. Hence,*

$$T(p) = T^{[p+1]}_{\mathbb{F}_p}[y_1,\dots,y_\ell] \cong A^*/M \tag{4.3}$$

is an unstable $\mathscr{A}_p$-algebra.

(3) *We have that $\sigma^{-1}\varepsilon^*|_{A^*}\colon A^* \to H^{*-1}(X;\mathbb{F}_p)$ induces an isomorphism*

$$\mathscr{Q}\colon QT(p) \to QH^{*-1}(X;\mathbb{F}_p) \quad \text{of } \mathscr{A}_p\text{-modules.}$$

Let p be a prime, and let $n \geq 1$. According to Hemmi and Kawamoto [8, Definition 2.4], an unstable $\mathscr{A}_p$-algebra A is called a $\mathscr{D}_n$-algebra if the following condition is satisfied: for any $z_j \in A$ and $\mathscr{O}_j \in \mathscr{A}_p$ for $1 \leq j \leq m$ with

$$\sum_{j=1}^{m} \mathscr{O}_j(z_j) \in DA, \tag{4.4}$$

there are decomposable classes $d_j \in DA$ for $1 \leq j \leq m$ with

$$\sum_{j=1}^{m} \mathscr{O}_j(z_j - d_j) \in D^{n+1}A. \tag{4.5}$$

From the definition, any unstable $\mathscr{A}_p$-algebra is a $\mathscr{D}_1$-algebra. On the other hand, if X is a simply connected A_p-space whose mod p cohomology $H^*(X;\mathbb{F}_p)$ is as in (4.1) with $\ell \geq 1$, then $T(p)$ in (4.3) cannot be a $\mathscr{D}_p$-algebra by [8, Remark 2.5].

In order to prove Theorem C, we need the following result.

THEOREM 4.3
Let p and λ be as in Theorem C, and let $1 \leq n \leq p-1$. If X is a simply connected A_p-space whose mod p *cohomology $H^*(X;\mathbb{F}_p)$ is as in (4.1) and the power map Φ_λ on X is an A_n-map, then $T(p)$ in (4.3) is a $\mathscr{D}_n$-algebra.*

The proof of Theorem 4.3 is similar to that of [8, Theorem 2.6]. In the proof, we use the following lemma instead of [8, Lemma 2.7].

LEMMA 4.4

Let p, λ, n, and X be as in Theorem 4.3. If $z_j \in H^*(P_n(X);\mathbb{F}_p)$ and $\mathscr{O}_j \in \mathscr{A}_p$ for $1 \le j \le m$ satisfy

$$\sum_{j=1}^{m} \mathscr{O}_j(z_j) = w + u \quad \text{with } w \in DT(n) \text{ and } u \in S(n), \tag{4.6}$$

then there are decomposable classes $d_j \in DT(n)$ for $1 \le j \le m$ with

$$\sum_{j=1}^{m} \mathscr{O}_j(z_j - d_j) = u.$$

Proof

We first prove the case of $z_j \in T(n) \setminus DT(n)$ for $1 \le j \le m$. We work by induction on n. Since $DT(1) = 0$, the result is clear for $n = 1$. Assume that the result is proved for $n-1$ with $2 \le n \le p-1$.

Applying ι_{n-1}^* to (4.6), we have that $\iota_{n-1}^*(z_j) \in T(n-1) \setminus DT(n-1)$ and

$$\sum_{j=1}^{m} \mathscr{O}_j\big(\iota_{n-1}^*(z_j)\big) = \iota_{n-1}^*\Big(\sum_{j=1}^{m} \mathscr{O}_j(z_j)\Big) = \iota_{n-1}^*(w) \in DT(n-1).$$

By inductive hypothesis, we have $\widehat{d}_j \in DT(n-1)$ for $1 \le j \le m$ with

$$\sum_{j=1}^{m} \mathscr{O}_j\big(\iota_{n-1}^*(z_j) - \widehat{d}_j\big) = 0.$$

Take $\widetilde{d}_j \in DT(n)$ with $\iota_{n-1}^*(\widetilde{d}_j) = \widehat{d}_j$, and put $\widetilde{z}_j = z_j - \widetilde{d}_j \in T(n) \setminus DT(n)$ for $1 \le j \le m$. Then

$$\iota_{n-1}^*\Big(\sum_{j=1}^{m} \mathscr{O}_j(\widetilde{z}_j)\Big) = \sum_{j=1}^{m} \mathscr{O}_j\big(\iota_{n-1}^*(z_j) - \widehat{d}_j\big) = 0,$$

and so

$$\sum_{j=1}^{m} \mathscr{O}_j(\widetilde{z}_j) = \widetilde{w} + u \quad \text{for some } \widetilde{w} \in D^n T(n).$$

From the definition of $S(n)$, we have that $P_n(\Phi_\lambda)^*(S(n)) \subset S(n)$ and $\mathscr{O}_j(S(n)) \subset S(n)$ for $1 \le j \le m$. Then

$$P_n(\Phi_\lambda)^*\Big(\sum_{j=1}^{m} \mathscr{O}_j(\widetilde{z}_j)\Big) \equiv P_n(\Phi_\lambda)^*(\widetilde{w}) = \lambda^n \widetilde{w} \mod S(n).$$

On the other hand,

$$\begin{aligned}P_n(\Phi_\lambda)^*\Big(\sum_{j=1}^m \mathscr{O}_j(\widetilde{z}_j)\Big) &= \sum_{j=1}^m \mathscr{O}_j\big(P_n(\Phi_\lambda)^*(\widetilde{z}_j)\big) \equiv \sum_{j=1}^m \mathscr{O}_j(\lambda\widetilde{z}_j + g_j)\\ &= \lambda\sum_{j=1}^m \mathscr{O}_j(\widetilde{z}_j) + \sum_{j=1}^m \mathscr{O}_j(g_j) \equiv \lambda\widetilde{w} + \sum_{j=1}^m \mathscr{O}_j(g_j) \mod S(n)\end{aligned}$$

since $P_n(\Phi_\lambda)^*(\widetilde{z}_j) \equiv \lambda\widetilde{z}_j + g_j \mod S(n)$ with $g_j \in DT(n)$ for $1 \le j \le m$. Then

$$\widetilde{w} \equiv \sum_{j=1}^m \mathscr{O}_j\Big(\frac{g_j}{\lambda^n - \lambda}\Big) \mod S(n). \tag{4.7}$$

Now we note that both sides of (4.7) are classes of $DT(n)$. Hence,

$$\widetilde{w} = \sum_{j=1}^m \mathscr{O}_j\Big(\frac{g_j}{\lambda^n - \lambda}\Big).$$

Let $d_j \in DT(n)$ be defined by

$$d_j = \widetilde{d}_j + \frac{g_j}{\lambda^n - \lambda} \quad \text{for } 1 \le j \le m.$$

Then

$$\sum_{j=1}^m \mathscr{O}_j(z_j - d_j) = u,$$

and so we have the required conclusion.

We next consider the general case. Let $z_j \in H^*(P_n(X); \mathbb{F}_p) \cong T(n) \oplus S(n)$ and $\mathscr{O}_j \in \mathscr{A}_p$ for $1 \le j \le m$ with (4.6). Write $z_j = z_j' + z_j''$ with $z_j' \in T(n)$ and $z_j'' \in S(n)$ for $1 \le j \le m$.

Now by permuting j suitably, we have an integer m' with $0 \le m' \le m$ such that $z_j' \in T(n) \setminus DT(n)$ for $1 \le j \le m'$ and $z_j' \in DT(n)$ for $m'+1 \le j \le m$. Define $w' \in DT(n)$ and $u' \in S(n)$ by

$$w' = w - \sum_{j=m'+1}^m \mathscr{O}_j(z_j') \qquad \text{and} \qquad u' = u - \sum_{j=1}^m \mathscr{O}_j(z_j''),$$

respectively. Then

$$\sum_{j=1}^{m'} \mathscr{O}_j(z_j') = w' + u' \quad \text{with } w' \in DT(n) \text{ and } u' \in S(n).$$

From the above proof, we have $d_j' \in DT(n)$ for $1 \le j \le m'$ with

$$\sum_{j=1}^{m'} \mathscr{O}_j(z_j' - d_j') = u'.$$

Put

$$d_j = \begin{cases} d_j' & \text{if } 1 \le j \le m', \\ z_j' & \text{if } m'+1 \le j \le m. \end{cases}$$

Then $d_j \in DT(n)$ for $1 \le j \le m$ with

$$\sum_{j=1}^{m} \mathscr{O}_j(z_j - d_j) = u,$$

which implies the required conclusion. This completes the proof of Lemma 4.4. $\square$

Proof of Theorem 4.3
From the construction of $R_p(X)$ in [7, Section 2], we have a space $R_{p-1}(X)$ with the following commutative diagram:

$$\begin{tikzcd} & & P_{p-2}(X) \arrow[r, "e_{p-1}"] \arrow[d, equal] & R_{p-1}(X) \arrow[r, "e_p"] \arrow[d, "f_{p-1}"'] & R_p(X) \\ P_n(X) \arrow[r, "\iota_n"'] & \cdots \arrow[r, "\iota_{p-3}"'] & P_{p-2}(X) \arrow[r, "\iota_{p-2}"'] & P_{p-1}(X) & \end{tikzcd} \tag{4.8}$$

Since $e_p^*(M) = 0$ by [7, p. 593], we have that $e_p^*|_{A^*}\colon A^* \to H^*(R_{p-1}(X);\mathbb{F}_p)$ induces a homomorphism $\mathscr{E}\colon T(p) = A^*/M \to H^*(R_{p-1}(X);\mathbb{F}_p)$ of $\mathscr{A}_p$-algebras by Theorem 4.2(2).

We first prove the case of $1 \le n \le p-2$. Let $\mathscr{K}_n\colon T(p) \to H^*(P_n(X);\mathbb{F}_p)$ be defined by $\mathscr{K}_n = \iota_n^* \cdots \iota_{p-3}^* e_{p-1}^* \mathscr{E}$. Put $\mathscr{K}_n(z_j) = \widetilde{z}_j$ for $1 \le j \le m$. Applying $\mathscr{K}_n$ to (4.4), we have

$$\sum_{j=1}^{m} \mathscr{O}_j(\widetilde{z}_j) \in DT(n).$$

Now we have $\widetilde{d}_j \in DT(n)$ for $1 \le j \le m$ with

$$\sum_{j=1}^{m} \mathscr{O}_j(\widetilde{z}_j - \widetilde{d}_j) = 0 \tag{4.9}$$

by Lemma 4.4. Take $d_j \in DT(p)$ with $\mathscr{K}_n(d_j) = \widetilde{d}_j$ for $1 \le j \le m$. Then by (4.9),

$$\sum_{j=1}^{m} \mathscr{O}_j(z_j - d_j) \in D^{n+1}T(p),$$

and so we have the required conclusion.

We next consider the case of $n = p-1$. Since

$$\mathscr{E}\big(T(p)\big) = f_{p-1}^*\big(T(p-1)\big) \subset H^*\big(R_{p-1}(X);\mathbb{F}_p\big) \tag{4.10}$$

by [7, Proposition 5.2], there are classes $\widetilde{z}_j \in T(p-1)$ with $f_{p-1}^*(\widetilde{z}_j) = \mathscr{E}(z_j)$ for $1 \le j \le m$. Moreover, we take $w \in DT(p-1)$ with

$$f_{p-1}^*(w) = \mathscr{E}\Big(\sum_{j=1}^{m} \mathscr{O}_j(z_j)\Big)$$

by (4.4) and (4.10). Hence,

$$\sum_{j=1}^{m} \mathscr{O}_j(\widetilde{z}_j) = w + u \quad \text{for some } u \in H^*\big(P_{p-1}(X); \mathbb{F}_p\big) \text{ with } f_{p-1}^*(u) = 0.$$

Now $u \in S(p-1)$ by [7, Lemma 5.1], and so we have $\widetilde{d}_j \in DT(p-1)$ for $1 \le j \le m$ with

$$\sum_{j=1}^{m} \mathscr{O}_j(\widetilde{z}_j - \widetilde{d}_j) = u$$

by Lemma 4.4. Taking $d_j \in DT(p)$ with $\mathscr{E}(d_j) = f_{p-1}^*(\widetilde{d}_j)$ for $1 \le j \le m$, we have

$$\sum_{j=1}^{m} \mathscr{O}_j(z_j - d_j) \in D^p T(p).$$

This completes the proof of Theorem 4.3. □

From Theorems 4.2(3) and 4.3 and the result of Hemmi and Kawamoto [8, Proposition 3.2], we have the following proposition.

PROPOSITION 4.5

Let p and λ be as in Theorem C. If X is a simply connected A_p-space whose mod p *cohomology $H^*(X;\mathbb{F}_p)$ is as in (4.1) and the power map Φ_λ on X is an A_n-map with $n > (p-1)/2$, then we have the following.*

(1) *If $a \ge 0$, $b > 0$, and $0 < c < p$, then*

$$QH^{2p^a(pb+c)-1}(X;\mathbb{F}_p) = \mathscr{P}^{p^a k} QH^{2p^a(p(b-k)+c+k)-1}(X;\mathbb{F}_p)$$

for $1 \le k \le \min\{b, p-c\}$

and

$$\mathscr{P}^{p^a k} QH^{2p^a(pb+c)-1}(X;\mathbb{F}_p) = 0 \quad \text{for } c \le k < p.$$

(2) *If $a \ge 0$ and $0 < c < p$, then*

$$\mathscr{P}^{p^a k} \colon QH^{2p^a c-1}(X;\mathbb{F}_p) \to QH^{2p^a(kp+c-k)-1}(X;\mathbb{F}_p)$$

is an isomorphism for $1 \le k < c$.

REMARK 4.6

When $p = 3$ and X is a homotopy associative and homotopy commutative H-space, Proposition 4.5(1) was first proved by Hemmi [6, Theorem 1.1]. Later Lin [17, Theorem B] also proved (1) of the above result for any odd prime p under the additional assumptions that Φ_λ is an A_{p-1}-map and $H^*(X;\mathbb{F}_p)$ is generated by A_p-primitive classes.

LEMMA 4.7

Assume that p, λ, n, and X are as in Theorem C. Then the mod p *cohomology $H^*(X;\mathbb{F}_p)$ is as in (4.1) such that $t_i = p^{a_i}$ with $a_i > 0$ for $1 \le i \le \ell$.*

Proof

We first prove that there is no even-dimensional generator in $H^*(X;\mathbb{F}_p)$. Assume contrarily that $x \in QH^*(X;\mathbb{F}_p)$ is an even-dimensional generator. According to Lin [16, Theorem 4.3.1] (see also [14, Section 35]),

$$x = \beta\mathscr{P}^n(y) \quad \text{for some } y \in QH^{2n+1}(X;\mathbb{F}_p) \text{ with } n \ge 1.$$

Then $\mathscr{P}^n QH^{2n+1}(X;\mathbb{F}_p) \ne 0$. From the assumption, $\{\mathscr{P}^i\}_{i\ge 1}$ act trivially on $QH^*(X;\mathbb{F}_p)$, and so we have a contradiction. Hence, $H^*(X;\mathbb{F}_p)$ is as in (4.1).

Let $x \in QH^{2t-1}(X;\mathbb{F}_p)$ be one of the generators $\{x_i\}_{1\le i\le \ell}$ in (4.1). Write

$$t = p^a(pb + c) \quad \text{with } a, b \ge 0 \text{ and } 0 < c < p.$$

When $b > 0$, we have that

$$x \in \mathscr{P}^{p^a} QH^{2(t-p^a(p-1))-1}(X;\mathbb{F}_p)$$

by Proposition 4.5(1). If $b = 0$ and $1 < c < p$, then

$$\mathscr{P}^{p^a}(x) \ne 0 \quad \text{in } QH^{2(t+p^a(p-1))-1}(X;\mathbb{F}_p)$$

by Proposition 4.5(2). Now we note that $\{\mathscr{P}^i\}_{i\ge 1}$ act trivially on $QH^*(X;\mathbb{F}_p)$, and so $b = 0$ and $c = 1$. Since $t > 1$, we have $t = p^a$ with $a > 0$. $\square$

We are now in position to prove Theorem C.

Proof of Theorem C

We use a similar way to the proof of [5, Theorem 1.1]. From Theorem 4.1 and Lemma 4.7, there are classes

$$y_i \in \widetilde{H}^{2p^{a_i}}\big(P_{p-1}(X);\mathbb{F}_p\big) \quad \text{with } \iota_1^* \cdots \iota_{p-2}^*(y_i) = \sigma(x_i) \text{ for } 1 \le i \le \ell$$

such that

$$H^*\big(P_{p-1}(X);\mathbb{F}_p\big) \cong T(p-1) \oplus S(p-1) \quad \text{as an } \mathbb{F}_p\text{-algebra,}$$

where $T(p-1) = T^{[p]}_{\mathbb{F}_p}[y_1, \dots, y_\ell]$.

Assume contrarily that X is not $\mathbb{F}_p$-acyclic. Put $a = \min\{a_i\}_{1\le i\le \ell}$. Take $x \in QH^{2p^a-1}(X;\mathbb{F}_p)$ and $y \in T(p-1)$ with $\iota_1^* \cdots \iota_{p-2}^*(y) = \sigma(x) \ne 0$. Then the composition

$$H^t\big(P_p(X);\mathbb{F}_p\big) \xrightarrow{\iota_{p-1}^*} H^t\big(P_{p-1}(X);\mathbb{F}_p\big) \longrightarrow T(p-1) \tag{4.11}$$

is an isomorphism for $t < 2p^{a+1}$ and an epimorphism for $t < 2(p^{a+1} + p^a - 1)$ (see [5, p. 106, (4.10)]). From Lemma 4.7 and (4.11), we have

$$H^t\big(P_p(X);\mathbb{F}_p\big) \cap (\operatorname{Im}\beta \cup \operatorname{Im}\mathscr{P}^1) = 0 \quad \text{for } t \le 2p^{a+1}.$$

Then

$$H^t\big(P_p(X);\mathbb{F}_p\big)\cap \operatorname{Im}\mathscr{P}^{p^a}=0 \quad \text{for } t\le 2p^{a+1} \tag{4.12}$$

by Shimada and Yamanoshita [25, Theorem 5.3] or Liulevicius [19, Theorem 1.2.1].

Taking $z\in H^{2p^a}(P_p(X);\mathbb{F}_p)$ with $\iota_{p-1}^*(z)=y$ by (4.11), we have

$$\mathscr{F}_p(x^{\otimes p})=z^p=\mathscr{P}^{p^a}(z)=0$$

by (4.12) and [33, Theorem 2.4] (see also [9, Theorem 4.1]). Hence,

$$x^{\otimes p}=\mathscr{G}_{p-1}(u)\quad \text{for some } u\in H^{2p^{a+1}-1}\big(P_{p-1}(X);\mathbb{F}_p\big).$$

For dimensional reasons, we have $u\in S(p-1)$, and so

$$u=\mathscr{F}_{p-1}(v)\quad \text{for some } v\in D(p-1).$$

Let $\boldsymbol{c}\in PH_{2p^a-1}(X;\mathbb{F}_p)$ be a primitive class with $\langle x,\boldsymbol{c}\rangle\neq 0$. Then $\langle x^{\otimes p}, \boldsymbol{c}^{\otimes p}\rangle\neq 0$ by [22, p. 152, (3)]. On the other hand,

$$\langle x^{\otimes p},\boldsymbol{c}^{\otimes p}\rangle=\big\langle(\mathscr{G}_{p-1}\mathscr{F}_{p-1})(v),\boldsymbol{c}^{\otimes p}\big\rangle=0$$

by (4.2) and [12, Lemma 2.5] (see also [14, p. 98, Corollary C(i)]). This is a contradiction, and so X is $\mathbb{F}_p$-acyclic. □

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