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SCALE TRANSFORMATIONS FOR PRESENT POSITION-DEPENDENT CONDITIONAL EXPECTATIONS OVER CONTINUOUS PATHS

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Abstract. Let $C[0,t]$ denote a generalized Wiener space, the space of real-valued continuous functions on the interval $[0,t]$, and define a random vector $Z_n : C[0,t] \to \mathbb{R}^n$ by

$$Z_n(x) = \Big(\int_0^{t_1} h(s)\,dx(s), \dots, \int_0^{t_n} h(s)\,dx(s)\Big),$$

where $0 < t_1 < \cdots < t_n = t$ is a partition of $[0,t]$ and $h \in L_2[0,t]$ with $h \neq 0$ almost everywhere. Using a simple formula for a generalized conditional Wiener integral on $C[0,t]$ with the conditioning function Z_n, we evaluate the generalized analytic conditional Wiener and Feynman integrals of the cylinder function

$$G(x) = f\big((e,x)\big)\phi\big((e,x)\big)$$

for $x \in C[0,t]$, where $f \in L_p(\mathbb{R})(1 \le p \le \infty)$, e is a unit element in $L_2[0,t]$, and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}$. We then express the generalized analytic conditional Feynman integral of G as two kinds of limits of nonconditional generalized Wiener integrals with a polygonal function and cylinder functions using a change-of-scale transformation. The choice of a complete orthonormal subset of $L_2[0,t]$ used in the transformation is independent of e.

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1. Introduction and preliminaries

Let $C_0[0,t]$ denote the Wiener space, the space of continuous real-valued functions x on $[0,t]$ with $x(0)=0$. As mentioned in [1] and [2], the Wiener measure and Wiener measurability behave badly under change-of-scale transformation and under translation. Various kinds of change-of-scale formulas for Wiener integrals of bounded and unbounded functions were developed on the classical and abstract Wiener spaces (see [4], [9], [13], [12], [14]). Furthermore, the author and his coauthors [6], [11] introduced various kinds of change-of-scale formulas for the conditional Wiener integrals of the functions defined on $C_0[0,t]$, the infinite-dimensional Wiener space, and $C[0,t]$, an analogue of Wiener space that is the space of real-valued continuous paths on $[0,t]$ (see [8]).

Let $h \in L_2[0,t]$ with $h \neq 0$ almost everywhere on $[0,t]$. Define a stochastic process $Z : C[0,t] \times [0,t] \to \mathbb{R}$ by

$$Z(x,s) = \int_0^s h(u)\,dx(u)$$

for $x \in C[0,t]$ and $s \in [0,t]$, where the integral denotes the Paley–Wiener–Zygmund integral, and let

$$Z_n(x) = \big(Z(x,t_1),\dots,Z(x,t_n)\big)$$

for $x \in C[0,t]$, where $0=t_0<t_1<\cdots<t_n=t$ is a partition of $[0,t]$. On the space $C[0,t]$, the author [5] derived a simple formula for a generalized conditional Wiener integral given the vector-valued conditioning function Z_n. Using the simple formula, Yoo and the author [7] evaluated a generalized analytic conditional Wiener integral of the function G_r having the form

$$G_r(x) = F(x)\Psi\Big(\int_0^t v_1(s)\,dx(s),\dots,\int_0^t v_r(s)\,dx(s)\Big)$$

for F in a Banach algebra, which corresponds to Cameron–Storvick's Banach algebra $\mathcal{S}$ (see [3]), and for $\Psi = f+\phi$, which need not be bounded or continuous, where $f \in L_p(\mathbb{R}^r)(1\le p\le\infty)$, $\{v_1,\dots,v_r\}$ is an orthonormal subset of $L_2[0,t]$, and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^r$. They then established various kinds of change-of-scale formulas for the generalized analytic conditional Wiener integral of G_r with the conditioning function Z_n. Except for the results in [9], the choices of the orthonormal bases of $L_2[0,t]$ in the existing change-of-scale formulas depend on the orthonormal set $\{v_1,\dots,v_r\}$ used in the definition of a cylinder function.

In this paper, using the simple formula derived in [5], we evaluate the generalized analytic conditional Wiener and Feynman integrals of the cylinder function G having the form

$$G(x) = f\big((e,x)\big)\phi\big((e,x)\big)$$

for $x \in C[0,t]$, where $f \in L_p(\mathbb{R})(1\le p\le\infty)$ and e is a unit element in $L_2[0,t]$. We then express the generalized analytic conditional Feynman integral of G as two kinds of limits of nonconditional generalized Wiener integrals with a polygonal function and cylinder functions using a change-of-scale transformation. The choice of a complete orthonormal subset of $L_2[0,t]$ used in the transformation is independent of e. We note that the results of this paper are different from those in [6] and [11].

2. A generalized conditional Wiener integral

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively. Let $(C[0,t],\mathcal{B}(C[0,t]),w_\varphi)$ be the analogue of Wiener space associated with a probability measure φ on the Borel class of $\mathbb{R}$, where $\mathcal{B}(C[0,t])$ denotes the Borel class of $C[0,t]$ (see [8]). For $v\in L_2[0,t]$ and $x\in C[0,t]$, let $(v,x)=\int_0^t v(s)\,dx(s)$ denote the Paley–Wiener–Zygmund integral of v according to x (see [8]). The inner product on the real Hilbert space $L_2[0,t]$ is denoted by $\langle\cdot,\cdot\rangle$.

Let $F:C[0,t]\to\mathbb{C}$ be integrable, and let X be a random vector on $C[0,t]$. Then we have the conditional expectation $E[F\mid X]$ given X from a well-known probability theory (see [10, Definition 6.1.1.]). Furthermore, there exists a P_X-integrable function ψ on the value space of X such that $E[F\mid X](x)=(\psi\circ X)(x)$ for w_φ-a.e. $x\in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the *conditional Wiener w_φ-integral* of F given X, and it is also denoted by $E[F\mid X]$.

Let $0=t_0<t_1<\cdots<t_n=t$ be a partition of $[0,t]$, where n is a fixed positive integer. Let $h\in L_2[0,t]$ be of bounded variation with $h\neq 0$ almost everywhere on $[0,t]$. For $j=1,\dots,n$, let

$$\alpha_j=\frac{1}{\|\chi_{(t_{j-1},t_j]}h\|}\chi_{(t_{j-1},t_j]}h,$$

and let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1,\dots,\alpha_n\}$. Let $V^\perp$ be the orthogonal complement of V. Let $\mathcal{P}:L_2[0,t]\to V$ be the orthogonal projection given by

$$\mathcal{P}v=\sum_{j=1}^n\langle v,\alpha_j\rangle\alpha_j,$$

and let $\mathcal{P}^\perp:L_2[0,t]\to V^\perp$ be the orthogonal projection. For $x\in C[0,t]$ define the stochastic process $Z:C[0,t]\times[0,t]\to\mathbb{R}$ by

$$Z(x,s)=\int_0^s h(u)\,dx(u),\quad 0\le s\le t,$$

and let $Z_n:C[0,t]\to\mathbb{R}^n$ be given by

$$Z_n(x)=\big(Z(x,t_1),\dots,Z(x,t_n)\big). \tag{2.1}$$

Let $b(s)=\int_0^s(h(u))^2\,du$, and, for $x\in C[0,t]$, define the polygonal function $[Z(x,\cdot)]_b$ of $Z(x,\cdot)$ by

$$\begin{aligned}&\big[Z(x,\cdot)\big]_b(s)\\ &\quad=\sum_{j=1}^n\chi_{(t_{j-1},t_j]}(s)\Big[Z(x,t_{j-1})+\frac{b(s)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}\big(Z(x,t_j)-Z(x,t_{j-1})\big)\Big]\end{aligned} \tag{2.2}$$

for $s \in [0,t]$. Similarly, for $\vec{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, the polygonal function $[\vec{\xi}]_b$ of $\vec{\xi}$ is given by (2.2) replacing $Z(x,t_j)$ by $\xi_j (j = 1, \ldots, n)$ with $\xi_0 = 0$. For a function $F : C[0,t] \to \mathbb{C}$ such that $F_Z(x) \equiv F(Z(x,\cdot))$ is integrable over x, we have, by an application of Theorem 2.9 in [5],

$$E[F_Z \mid Z_n](\vec{\xi}) = E\big[F\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big] \tag{2.3}$$

for P_{Z_n}-a.e. $\vec{\xi} \in \mathbb{R}^n$ (for almost every $\vec{\xi} \in \mathbb{R}^n$), where P_{Z_n} is the probability distribution of Z_n on the Borel class $\mathcal{B}(\mathbb{R}^n)$ of $\mathbb{R}^n$. For $\lambda > 0$, let $F_Z^\lambda(x) = F_Z(\lambda^{-\frac{1}{2}} x)$, and let $Z_n^\lambda(x) = Z_n(\lambda^{-\frac{1}{2}} x)$ for $x \in C[0,t]$, where Z_n is given by (2.1). Suppose that $E[F_Z^\lambda]$ exists. By the definition of the conditional Wiener w_φ-integral and (2.3),

$$E[F_Z^\lambda \mid Z_n^\lambda](\vec{\xi}) = E\big[F\big(\lambda^{-\frac{1}{2}}\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b\big) + [\vec{\xi}]_b\big)\big] \tag{2.4}$$

for $P_{Z_n^\lambda}$-a.e. $\vec{\xi} \in \mathbb{R}^n$, where $P_{Z_n^\lambda}$ is the probability distribution of Z_n^λ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Let $I_{F_Z}^\lambda(\vec{\xi})$ be the right-hand side of (2.4). If $I_{F_Z}^\lambda(\vec{\xi})$ has an analytic extension $J_\lambda^*(F_Z)(\vec{\xi})$ on $\mathbb{C}_+$, then it is called the *conditional analytic Wiener w_φ-integral* of F_Z, given Z_n with the parameter λ, and is denoted by

$$E^{anw_\lambda}[F_Z \mid Z_n](\vec{\xi}) = J_\lambda^*(F_Z)(\vec{\xi})$$

for $\vec{\xi} \in \mathbb{R}^n$. Moreover, if, for nonzero real q, $E^{anw_\lambda}[F_Z \mid Z_n](\vec{\xi})$ has a limit as λ approaches $-iq$ through $\mathbb{C}_+$, then it is called the *conditional analytic Feynman w_φ-integral* of F_Z, given Z_n with the parameter q, and is denoted by

$$E^{anf_q}[F_Z \mid Z_n](\vec{\xi}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F_Z \mid Z_n](\vec{\xi}).$$

If $E[F(\lambda^{-\frac{1}{2}}\cdot)]$ exists for $\lambda > 0$ and it has an analytic extension $J_\lambda^*(F)$ on $\mathbb{C}_+$, then we call $J_\lambda^*(F)$ the *analytic Wiener w_φ-integral* of F over $C[0,t]$ with parameter λ, and it is denoted by

$$E^{anw_\lambda}[F] = J_\lambda^*(F).$$

The following lemmas are useful to prove the results in the next sections (see [9]).

Lemma 2.1. *Let a and b be positive real numbers. Then, for any real u,*

$$\int_{\mathbb{R}} \exp\big\{-av^2 - b(v-u)^2\big\}\, dv = \Big(\frac{\pi}{a+b}\Big)^{\frac{1}{2}} \exp\Big\{-\frac{ab}{a+b}u^2\Big\}.$$

Lemma 2.2. *Let $v \in L_2[0,t]$. Then, for w_φ-a.e. $x \in C[0,t]$,*

$$\big(v, \big[Z(x,\cdot)\big]_b\big) = \big(\mathcal{P}(vh), x\big).$$

Applying Theorem 3.5 in [8], we can easily prove the following theorem.

Theorem 2.3. *Let $\{h_1, h_2, \ldots, h_n\}$ be an orthonormal system of $L_2[0,t]$. For $i = 1, 2, \ldots, n$, let $X_i(x) = (h_i, x)$ on $C[0,t]$. Then $X_1, \ldots, X_n$ are independent* *and each X_i has the standard normal distribution. Moreover, if $f : \mathbb{R}^n \to \mathbb{R}$ is Borel-measurable, then*

$$\int_{C[0,t]} f\big(X_1(x), \dots, X_n(x)\big)\, dw_\varphi(x) \stackrel{*}{=} \Big(\frac{1}{2\pi}\Big)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, u_2, \dots, u_n) \exp\Big\{-\frac{1}{2}\sum_{j=1}^{n} u_j^2\Big\}\, d(u_1, u_2, \dots, u_n),$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

3. Generalized analytic conditional Feynman integrals

In this section we establish the analytic conditional Wiener and Feynman integrals of cylinder functions.

Let e be in $L_2[0,t]$ with $\|e\| = 1$. For $1 \le p \le \infty$, let $\mathcal{A}^{(p)}$ be the space of the cylinder functions F having the following form:

$$F(x) = f\big((e, x)\big) \tag{3.1}$$

for w_φ-a.e. $x \in C[0,t]$, where $f \in L_p(\mathbb{R})$. Without loss of generality, we can take f to be Borel-measurable. Let $\hat{\mathrm{M}}(\mathbb{R})$ be the space of all functions ϕ on $\mathbb{R}$ defined by

$$\phi(u) = \int_{\mathbb{R}} \exp\{iuz\}\, d\rho(z), \tag{3.2}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}$.

Theorem 3.1. *Let $1 \le p \le \infty$. Let Z_n and $F \in \mathcal{A}^{(p)}$ be given by (2.1) and (3.1), respectively. Then, for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F_Z \mid Z_n](\vec{\xi})$ exists for almost every $\vec{\xi} \in \mathbb{R}^n$ and it is given by*

$$E^{anw_\lambda}[F_Z \mid Z_n](\vec{\xi}) = \Big[\frac{\lambda}{2\pi\|\mathcal{P}^\perp(eh)\|^2}\Big]^{\frac{1}{2}} \int_{\mathbb{R}} f(u) \exp\Big\{-\frac{\lambda}{2\|\mathcal{P}^\perp(eh)\|^2}\big(u - (e, [\vec{\xi}]_b)\big)^2\Big\}\, du$$

if $\mathcal{P}^\perp(eh) \neq 0$ or, equivalently, $eh \notin V$. Furthermore, if $p = 1$ and $\mathcal{P}^\perp(eh) \neq 0$, then for a nonzero real q $E^{anf_q}, [F_Z \mid Z_n](\vec{\xi})$ is given by the right-hand side of the above equality, replacing λ by $-iq$. If $\mathcal{P}^\perp(eh) = 0$ or, equivalently, $eh \in V$, then

$$E^{anw_\lambda}[F_Z \mid Z_n](\vec{\xi}) = E^{anf_q}[F_Z \mid Z_n](\vec{\xi}) = F\big([\vec{\xi}]_b\big) = f\big((e, [\vec{\xi}]_b)\big)$$

for almost every $\vec{\xi} \in \mathbb{R}^n$.

Proof. For $\lambda > 0$ and almost every $\vec{\xi} \in \mathbb{R}^n$, we have, by Lemma 2.2 and Theorem 2.3,

$$\begin{aligned} I^\lambda_{F_Z}(\vec{\xi}) &= \int_{C[0,t]} f\big(\lambda^{-\frac{1}{2}}\big(e, Z(x,\cdot) - [Z(x,\cdot)]_b\big) + \big(e, [\vec{\xi}]_b\big)\big)\, dw_\varphi(x) \\ &= \int_{C[0,t]} f\big(\lambda^{-\frac{1}{2}}\big(eh - \mathcal{P}(eh), x\big) + \big(e, [\vec{\xi}]_b\big)\big)\, dw_\varphi(x) \end{aligned}$$

$$= \int_{C[0,t]} f\big(\lambda^{-\frac{1}{2}}\big(\mathcal{P}^{\perp}(eh), x\big) + \big(e, [\vec{\xi}]_b\big)\big)\, dw_{\varphi}(x)$$

$$= \Big[\frac{\lambda}{2\pi\|\mathcal{P}^{\perp}(eh)\|^2}\Big]^{\frac{1}{2}} \int_{\mathbb{R}} f\big(u + \big(e, [\vec{\xi}]_b\big)\big) \exp\Big\{-\frac{\lambda}{2\|\mathcal{P}^{\perp}(eh)\|^2} u^2\Big\}\, du$$

if $\mathcal{P}^{\perp}(eh) \neq 0$. If $\mathcal{P}^{\perp}(eh) = 0$, then it is not difficult to show that $I^{\lambda}_{F_Z}(\vec{\xi}) = f((e, [\vec{\xi}]_b))$. By Morera's theorem we have the existence of $E^{anw_\lambda}[F_Z \mid Z_n](\vec{\xi})$. If $p = 1$, then the existence of $E^{anf_q}[F_Z \mid Z_n](\vec{\xi})$ follows from the dominated convergence theorem. □

By the boundedness of ϕ and Theorem 3.1, we have the following theorem.

Theorem 3.2. *Let $G(x) = F(x)\phi((e, x))$ for w_φ-a.e. $x \in C[0, t]$, where $F \in \mathcal{A}^{(p)} (1 \le p \le \infty)$ and $\phi \in \hat{\mathrm{M}}(\mathbb{R})$ are given by (3.1) and (3.2), respectively. Then, for $\lambda \in \mathbb{C}_+$ and almost every $\vec{\xi} \in \mathbb{R}^n$,*

$$E^{anw_\lambda}[G_Z \mid Z_n](\vec{\xi})$$
$$= \Big[\frac{\lambda}{2\pi\|\mathcal{P}^{\perp}(eh)\|^2}\Big]^{\frac{1}{2}} \int_{\mathbb{R}} f(u)\phi(u) \exp\Big\{-\frac{\lambda}{2\|\mathcal{P}^{\perp}(eh)\|^2}\big(u - \big(e, [\vec{\xi}]_b\big)\big)^2\Big\}\, du$$

if $\mathcal{P}^{\perp}(eh) \neq 0$. Furthermore, if $p = 1$ and $\mathcal{P}^{\perp}(eh) \neq 0$, then for a nonzero real q, $E^{anf_q}[G_Z \mid Z_n](\vec{\xi})$ is given by the right-hand side of the above equality, replacing λ by $-iq$. If $\mathcal{P}^{\perp}(eh) = 0$, then

$$E^{anw_\lambda}[G_Z \mid Z_n](\vec{\xi}) = E^{anf_q}[G_Z \mid Z_n](\vec{\xi}) = G\big([\vec{\xi}]_b\big) = \phi\big(\big(e, [\vec{\xi}]_b\big)\big) f\big(\big(e, [\vec{\xi}]_b\big)\big)$$

for almost every $\vec{\xi} \in \mathbb{R}^n$.

4. Change-of-scale formulas using the polygonal function

In this section we derive a change-of-scale formula for the generalized conditional Wiener integrals of cylinder functions on the analogue of Wiener space using the polygonal function as given in the previous section.

Throughout this paper, let $\{e_1, e_2, \ldots\}$ be a complete orthonormal basis of $L_2[0, t]$. For $v \in L_2[0, t]$, let

$$c_j(v) = \langle v, e_j\rangle \quad \text{for } j = 1, 2, \ldots. \tag{4.1}$$

For $m \in \mathbb{N}$, $\lambda \in \mathbb{C}_+$, and $x \in C[0, t]$, let

$$K_m(\lambda, x) = \exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{m}(e_j, x)^2\Big\}. \tag{4.2}$$

Lemma 4.1. *Let m be a fixed positive integer, and let K_m be given by (4.2). Let $1 \le p \le \infty$ and $F \in \mathcal{A}^{(p)}$ be given by (3.1). Suppose that $\{e_1, \ldots, e_m, \mathcal{P}^{\perp}(eh)\}$ is a linearly independent set. Then, for $\lambda \in \mathbb{C}_+$ and $\vec{\xi} \in \mathbb{R}^n$,*

$$E\big[K_m(\lambda, x)F\big(Z(x, \cdot) - \big[Z(x, \cdot)\big]_b + [\vec{\xi}]_b\big)\big]$$
$$= \lambda^{-\frac{m}{2}}\Big[\frac{\lambda}{2\pi A(m, \lambda)}\Big]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2A(m, \lambda)}\big(u - \big(e, [\vec{\xi}]_b\big)\big)^2\Big\} f(u)\, du, \tag{4.3}$$

where

$$A(m,\lambda)=\sum_{j=1}^{m}\big[c_j\big(\mathcal{P}^{\perp}(eh)\big)\big]^2+\lambda\Big[\big\|\mathcal{P}^{\perp}(eh)\big\|^2-\sum_{j=1}^{m}\big[c_j\big(\mathcal{P}^{\perp}(eh)\big)\big]^2\Big] \tag{4.4}$$

and the c_j's are given by (4.1).

Proof. For $\lambda>0$ and $\vec{\xi}\in\mathbb{R}^n$, let $\Gamma(\lambda,m,\vec{\xi})$ be the left-hand side of (4.3). Then

$$\begin{aligned}\Gamma(\lambda,m,\vec{\xi})&=\int_{C[0,t]}K_m(\lambda,x)F\big(Z(x,\cdot)-\big[Z(x,\cdot)\big]_b+[\vec{\xi}]_b\big)\,dw_\varphi(x)\\&=\int_{C[0,t]}\exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{m}(e_j,x)^2\Big\}f\big((\mathcal{P}^{\perp}(eh),x)+\big(e,[\vec{\xi}]_b\big)\big)\,dw_\varphi(x)\end{aligned}$$

by Lemma 2.2. Let g_{m+1} be the unit element in $L_2[0,t]$ obtained from $\{e_1,\dots,e_m,\mathcal{P}^{\perp}(eh)\}$ using the Gram–Schmidt orthonormalization process. Then

$$\mathcal{P}^{\perp}(eh)=\sum_{j=1}^{m}c_j\big(\mathcal{P}^{\perp}(eh)\big)e_j+c_{m+1}\big(\mathcal{P}^{\perp}(eh)\big)g_{m+1},$$

where $c_j(\mathcal{P}^{\perp}(eh))$ is given by (4.1) for $j=1,\dots,m$ and

$$\big[c_{m+1}\big(\mathcal{P}^{\perp}(eh)\big)\big]^2=\big\|\mathcal{P}^{\perp}(eh)\big\|^2-\sum_{j=1}^{m}\big[c_j\big(\mathcal{P}^{\perp}(eh)\big)\big]^2.$$

By the independence of $\{e_1,\dots,e_m,\mathcal{P}^{\perp}(eh)\}$, $c_{m+1}(\mathcal{P}^{\perp}(eh))\neq 0$. By Theorem 2.3,

$$\begin{aligned}&\Gamma(\lambda,m,\vec{\xi})\\&=\Big(\frac{1}{2\pi}\Big)^{\frac{m+1}{2}}\int_{\mathbb{R}^{m+1}}\exp\Big\{\frac{1-\lambda}{2}\sum_{j=1}^{m}u_j^2-\frac12\sum_{j=1}^{m+1}u_j^2\Big\}f\Big(\sum_{j=1}^{m+1}c_j\big(\mathcal{P}^{\perp}(eh)\big)u_j\\&\quad+\big(e,[\vec{\xi}]_b\big)\Big)\,d(u_1,\dots,u_m,u_{m+1})\\&=\Big(\frac{1}{2\pi}\Big)^{\frac{m+1}{2}}\int_{\mathbb{R}^{m+1}}\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{m}u_j^2-\frac12u_{m+1}^2\Big\}f\Big(\sum_{j=1}^{m+1}c_j\big(\mathcal{P}^{\perp}(eh)\big)u_j\\&\quad+\big(e,[\vec{\xi}]_b\big)\Big)\,d(u_1,\dots,u_m,u_{m+1}).\end{aligned}$$

Suppose that $c_j(\mathcal{P}^{\perp}(eh))\neq 0$ for $j=1,\dots,m$. For $j=1,\dots,m+1$, let $z_j=\sum_{k=1}^{j}c_k(\mathcal{P}^{\perp}(eh))u_k$ and $z_0=0$. Then $u_j=\frac{1}{c_j(\mathcal{P}^{\perp}(eh))}(z_j-z_{j-1})$ for $j=1,\dots,m+1$ so that, by Lemma 2.1 and the change-of-variable theorem,

$$\begin{aligned}&\Gamma(\lambda,m,\vec{\xi})\\&=\Big(\frac{1}{2\pi}\Big)^{\frac{m+1}{2}}\frac{1}{\prod_{j=1}^{m+1}c_j(\mathcal{P}^{\perp}(eh))}\int_{\mathbb{R}^{m+1}}\exp\Big\{-\frac{\lambda}{2}\sum_{j=1}^{m}\frac{(z_j-z_{j-1})^2}{[c_j(\mathcal{P}^{\perp}(eh))]^2}\\&\quad-\frac{(z_{m+1}-z_m)^2}{2[c_{m+1}(\mathcal{P}^{\perp}(eh))]^2}\Big\}f\big(z_{m+1}+\big(e,[\vec{\xi}]_b\big)\big)\,d(z_1,\dots,z_m,z_{m+1})\end{aligned}$$

$$= \lambda^{-\frac{1}{2}}\Big(\frac{1}{2\pi}\Big)^{\frac{m}{2}}\Big[\frac{1}{[\sum_{j=1}^{2}[c_j(\mathcal{P}^{\perp}(eh))]^2]\prod_{j=3}^{m+1}[c_j(\mathcal{P}^{\perp}(eh))]^2}\Big]^{\frac{1}{2}}$$
$$\times \int_{\mathbb{R}^m}\exp\Big\{-\frac{\lambda}{2\sum_{j=1}^{2}[c_j(\mathcal{P}^{\perp}(eh))]^2}z_2^2 - \frac{\lambda}{2}\sum_{j=3}^{m}\frac{(z_j - z_{j-1})^2}{[c_j(\mathcal{P}^{\perp}(eh))]^2}$$
$$- \frac{(z_{m+1} - z_m)^2}{2[c_{m+1}(\mathcal{P}^{\perp}(eh))]^2}\Big\} f\big(z_{m+1} + \big(e, [\vec{\xi}]_b\big)\big)\, d(z_2, \dots, z_m, z_{m+1}).$$

Applying this process repeatedly,

$$\Gamma(\lambda, m, \vec{\xi})$$
$$= \lambda^{-\frac{m-1}{2}}\frac{1}{2\pi}\Big[\frac{1}{[\sum_{j=1}^{m}[c_j(\mathcal{P}^{\perp}(eh))]^2][c_{m+1}(\mathcal{P}^{\perp}(eh))]^2}\Big]^{\frac{1}{2}}$$
$$\times \int_{\mathbb{R}^2}\exp\Big\{-\frac{\lambda}{2\sum_{j=1}^{m}[c_j(\mathcal{P}^{\perp}(eh))]^2}z_m^2 - \frac{(z_{m+1} - z_m)^2}{2[c_{m+1}(\mathcal{P}^{\perp}(eh))]^2}\Big\}$$
$$\times f\big(z_{m+1} + \big(e, [\vec{\xi}]_b\big)\big)\, d(z_m, z_{m+1})$$
$$= \lambda^{-\frac{m}{2}}\Big(\frac{\lambda}{2\pi}\Big)^{\frac{1}{2}}\Big[\frac{1}{\sum_{j=1}^{m}[c_j(\mathcal{P}^{\perp}(eh))]^2 + \lambda[c_{m+1}(\mathcal{P}^{\perp}(eh))]^2}\Big]^{\frac{1}{2}}$$
$$\times \int_{\mathbb{R}}\exp\Big\{-\frac{\lambda}{2(\sum_{j=1}^{m}[c_j(\mathcal{P}^{\perp}(eh))]^2 + \lambda[c_{m+1}(\mathcal{P}^{\perp}(eh))]^2)}z_{m+1}^2\Big\}$$
$$\times f\big(z_{m+1} + \big(e, [\vec{\xi}]_b\big)\big)\, dz_{m+1}$$
$$= \lambda^{-\frac{m}{2}}\Big[\frac{\lambda}{2\pi A(m, \lambda)}\Big]^{\frac{1}{2}}\int_{\mathbb{R}}\exp\Big\{-\frac{\lambda}{2A(m, \lambda)}\big(z - \big(e, [\vec{\xi}]_b\big)\big)^2\Big\} f(z)\, dz.$$

Since

$$\Big(\frac{1}{2\pi}\Big)^{\frac{1}{2}}\int_{\mathbb{R}}\exp\Big\{-\frac{\lambda}{2}u^2\Big\}\, du = \lambda^{-\frac{1}{2}}, \tag{4.5}$$

we have (4.3) for $\lambda > 0$, even if $c_j(\mathcal{P}^{\perp}(eh)) = 0$ for some $j \in \{1, \dots, m\}$. Each side of (4.3) is an analytic function of λ in $\mathbb{C}_+$ so that, by the uniqueness of the analytic extension, we have (4.3) for any $\lambda \in \mathbb{C}_+$. □

Using (4.5) and the same process as used in the proof of Lemma 4.1, we have the following corollary.

Corollary 4.2. *Let m be a fixed positive integer, and let K_m be given by (4.2). Let $1 \le p \le \infty$ and $F \in \mathcal{A}^{(p)}$ be given by (3.1). Suppose that $\{e_1, \dots, e_m, \mathcal{P}^{\perp}(eh)\}$ is a linearly dependent set. If $\mathcal{P}^{\perp}(eh) \neq 0$ or, equivalently, $eh \notin V$, then, for $\lambda \in \mathbb{C}_+$ and $\vec{\xi} \in \mathbb{R}^n$,*

$$E\big[K_m(\lambda, x)F\big(Z(x, \cdot) - \big[Z(x, \cdot)\big]_b + [\vec{\xi}]_b\big)\big]$$
$$= \lambda^{-\frac{m}{2}}\Big[\frac{\lambda}{2\pi A(m, 0)}\Big]^{\frac{1}{2}}\int_{\mathbb{R}}\exp\Big\{-\frac{\lambda}{2A(m, 0)}\big(u - \big(e, [\vec{\xi}]_b\big)\big)^2\Big\} f(u)\, du,$$

where A is given by (4.4). Furthermore, if $\mathcal{P}^\perp(eh) = 0$ or, equivalently, $eh \in V$, then, for $\lambda \in \mathbb{C}_+$ and $\vec{\xi} \in \mathbb{R}^n$,

$$E\big[K_m(\lambda, x)F\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big] = \lambda^{-\frac{m}{2}} F\big([\vec{\xi}]_b\big).$$

We now have the following theorem by the boundedness of ϕ.

Theorem 4.3. *Let $G(x) = F(x)\phi((e,x))$ for w_φ-a.e. $x \in C[0,t]$, where $F \in \mathcal{A}^{(p)}(1 \le p \le \infty)$ and ϕ are given by (3.1) and (3.2), respectively. Let m be a fixed positive integer, and let K_m be given by (4.2). If $\{e_1, \dots, e_m, \mathcal{P}^\perp(eh)\}$ is a linearly independent set, then, for $\lambda \in \mathbb{C}_+$ and $\vec{\xi} \in \mathbb{R}^n$,*

$$\begin{aligned} &E\big[K_m(\lambda, x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big] \\ &\quad = \lambda^{-\frac{m}{2}} \Big[\frac{\lambda}{2\pi A(m,\lambda)}\Big]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2A(m,\lambda)}\big(u - \big(e, [\vec{\xi}]_b\big)\big)^2\Big\} f(u)\phi(u)\, du, \end{aligned}$$

where A is given by (4.4). If $\{e_1, \dots, e_m, \mathcal{P}^\perp(eh)\}$ is a linearly dependent set and $\mathcal{P}^\perp(eh) \neq 0$ or, equivalently, $eh \notin V$, then, for $\lambda \in \mathbb{C}_+$ and $\vec{\xi} \in \mathbb{R}^n$,

$$\begin{aligned} &E\big[K_m(\lambda, x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big] \\ &\quad = \lambda^{-\frac{m}{2}} \Big[\frac{\lambda}{2\pi A(m,0)}\Big]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2A(m,0)}\big(u - \big(e, [\vec{\xi}]_b\big)\big)^2\Big\} f(u)\phi(u)\, du. \end{aligned}$$

If $\mathcal{P}^\perp(eh) = 0$ or, equivalently, $eh \in V$, then, for $\lambda \in \mathbb{C}_+$ and $\vec{\xi} \in \mathbb{R}^n$,

$$\begin{aligned} E\big[K_m(\lambda, x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big] &= \lambda^{-\frac{m}{2}} G\big([\vec{\xi}]_b\big) \\ &= \lambda^{-\frac{m}{2}} \phi\big(\big(e, [\vec{\xi}]_b\big)\big) f\big(\big(e, [\vec{\xi}]_b\big)\big). \end{aligned}$$

Theorem 4.4. *Let Z_n be given by (2.1), and let G be as given in Theorem 4.3. Then, for $\lambda \in \mathbb{C}_+$ and almost every $\vec{\xi} \in \mathbb{R}^n$,*

$$\begin{aligned} &E^{anw_\lambda}[G_Z \mid Z_n](\vec{\xi}) \\ &\quad = \lim_{m \to \infty} \lambda^{\frac{m}{2}} E\big[K_m(\lambda, x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big], \end{aligned} \tag{4.6}$$

where K_m is given by (4.2). Moreover, if $p = 1$, q is a nonzero real number, and $\{\lambda_m\}_{m=1}^\infty$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞, then $E^{anf_q}[G_Z \mid Z_n](\vec{\xi})$ is given by the right-hand side of (4.6) replacing λ by λ_m.

Proof. Suppose that $\{e_1, \dots, e_m, \mathcal{P}^\perp(eh)\}$ is a linearly independent set for any positive integer m. Then, for $\lambda \in \mathbb{C}_+$ and $\vec{\xi} \in \mathbb{R}^n$,

$$\begin{aligned} &\lambda^{\frac{m}{2}} E\big[K_m(\lambda, x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big] \\ &\quad = \Big[\frac{\lambda}{2\pi A(m,\lambda)}\Big]^{\frac{1}{2}} \int_{\mathbb{R}} \exp\Big\{-\frac{\lambda}{2A(m,\lambda)}\big(u - \big(e, [\vec{\xi}]_b\big)\big)^2\Big\} f(u)\phi(u)\, du \end{aligned}$$

by Theorem 4.3. By (4.4),

$$\lim_{m\to\infty} A(m,\lambda) = \lim_{m\to\infty}\Big[\sum_{j=1}^{m}\big[c_j\big(\mathcal{P}^{\perp}(eh)\big)\big]^2 + \lambda\Big[\big\|\mathcal{P}^{\perp}(eh)\big\|^2 - \sum_{j=1}^{m}\big[c_j\big(\mathcal{P}^{\perp}(eh)\big)\big]^2\Big]\Big]$$
$$= \big\|\mathcal{P}^{\perp}(eh)\big\|^2 + \lambda\big[\big\|\mathcal{P}^{\perp}(eh)\big\|^2 - \big\|\mathcal{P}^{\perp}(eh)\big\|^2\big] = \big\|\mathcal{P}^{\perp}(eh)\big\|^2$$

so that we have (4.6) by Theorem 3.2. If $\{e_1,\dots,e_l,\mathcal{P}^{\perp}(eh)\}$ is a linearly dependent set for some positive integer l and $\mathcal{P}^{\perp}(eh)\neq 0$, then, for $m\geq l$,

$$A(m,\lambda) = A(m,0) = A(l,0) = \sum_{j=1}^{l}\big[c_j\big(\mathcal{P}^{\perp}(eh)\big)\big]^2 = \big\|\mathcal{P}^{\perp}(eh)\big\|^2,$$

and hence

$$\lambda^{\frac{m}{2}} E\big[K_m(\lambda,x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big] = E^{anw_\lambda}[G_Z \mid Z_n](\vec{\xi})$$

by Theorem 3.2 and the second equality of Theorem 4.3. Finally, if $\mathcal{P}^{\perp}(eh)=0$, then we have (4.6) by Theorem 3.2 and the third equality of Theorem 4.3. □

The following corollary follows immediately from the proof of Theorem 4.4.

Corollary 4.5. *Let $K_0(\lambda,x)=1$ for $\lambda\in\mathbb{C}_+$ and $x\in C[0,t]$, let G be as given in Theorem 4.3, and let l be the smallest positive integer such that $\{e_1,\dots,e_l,\mathcal{P}^{\perp}(eh)\}$ is a linearly dependent set if $\mathcal{P}^{\perp}(eh)\neq 0$. Moreover, let $l=0$ if $\mathcal{P}^{\perp}(eh)=0$. Then, for any nonnegative integer r with $r\geq l$, for $\lambda\in\mathbb{C}_+$ and for almost every $\vec{\xi}\in\mathbb{R}^n$,*

$$E^{anw_\lambda}[G_Z \mid Z_n](\vec{\xi}) = \lambda^{\frac{r}{2}} E\big[K_r(\lambda,x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big]. \tag{4.7}$$

Letting $\lambda=\gamma^{-2}$ in (4.6) and (4.7), we have the following change-of-scale formulas for the generalized conditional Wiener integral on the analogue of Wiener space using the polygonal function.

Corollary 4.6.

(1) *Under the assumptions as given in Theorem 4.4, we have, for $\gamma>0$ and almost every $\vec{\xi}\in\mathbb{R}^n$,*

$$E\big[G\big(\gamma Z(x,\cdot)\big) \mid \gamma Z_n(x)\big](\vec{\xi})$$
$$= \lim_{m\to\infty}\gamma^{-m} E\big[K_m(\gamma^{-2},x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big].$$

(2) *Under the assumptions as given in Corollary 4.5, we have for any nonnegative integer r with $r\geq l$, for $\gamma>0$, and for almost every $\vec{\xi}\in\mathbb{R}^n$,*

$$E\big[G\big(\gamma Z(x,\cdot)\big) \mid \gamma Z_n(x)\big](\vec{\xi})$$
$$= \gamma^{-r} E\big[K_r(\gamma^{-2},x)G\big(Z(x,\cdot) - \big[Z(x,\cdot)\big]_b + [\vec{\xi}]_b\big)\big].$$

5. Change-of-scale formulas without the polygonal function

In this section we derive change-of-scale formulas for the generalized conditional Wiener integral of the cylinder function on the analogue of Wiener space without the polygonal functions used in Section 4.

Theorem 5.1. *Let Z_n be given by (2.1), and let G be as given in Theorem 4.3. Then, for $\lambda \in \mathbb{C}_+$ and almost every $\vec{\xi} \in \mathbb{R}^n$,*

$$E^{anw_\lambda}[G_Z \mid Z_n](\vec{\xi}) = \lim_{m\to\infty} \lambda^{\frac{m}{2}} E\big[K_m(\lambda,\cdot) f\big((v,\cdot)\big\|\mathcal{P}^\perp(eh)\big\| + \big(e,[\vec{\xi}]_b\big)\big)\phi\big((v,\cdot)\big\|\mathcal{P}^\perp(eh)\big\| + \big(e,[\vec{\xi}]_b\big)\big)\big] \tag{5.1}$$

for any unit element $v \in L_2[0,t]$, where K_m is given by (4.2). Moreover, if $p = 1$, q is a nonzero real number, and $\{\lambda_m\}_{m=1}^\infty$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞, then $E^{anf_q}[G_Z \mid Z_n](\vec{\xi})$ is given by the right-hand side of (5.1) replacing λ by λ_m.

Proof. Suppose that $\mathcal{P}^\perp(eh) \neq 0$. For $\lambda \in \mathbb{C}_+$ and almost every $\vec{\xi} \in \mathbb{R}^n$, we have, by Theorem 3.2 and the change-of-variable theorem,

$$\begin{aligned}
&E^{anw_\lambda}[G_Z \mid Z_n](\vec{\xi}) \\
&= \Big[\frac{\lambda}{2\pi\|\mathcal{P}^\perp(eh)\|^2}\Big]^{\frac12} \int_{\mathbb{R}} f(u)\phi(u)\exp\Big\{-\frac{\lambda}{2\|\mathcal{P}^\perp(eh)\|^2}\big(u - \big(e,[\vec{\xi}]_b\big)\big)^2\Big\}\,du \\
&= \Big(\frac{\lambda}{2\pi}\Big)^{\frac12} \int_{\mathbb{R}} f\big(u\big\|\mathcal{P}^\perp(eh)\big\| + \big(e,[\vec{\xi}]_b\big)\big)\phi\big(u\big\|\mathcal{P}^\perp(eh)\big\| + \big(e,[\vec{\xi}]_b\big)\big) \\
&\quad \times \exp\Big\{-\frac{\lambda}{2}u^2\Big\}\,du \\
&= E^{anw_\lambda}\big[f\big((v,\cdot)\big\|\mathcal{P}^\perp(eh)\big\| + \big(e,[\vec{\xi}]_b\big)\big)\phi\big((v,\cdot)\big\|\mathcal{P}^\perp(eh)\big\| + \big(e,[\vec{\xi}]_b\big)\big)\big],
\end{aligned}$$

where the last equality follows from Theorem 3.1 in [6]. Applying the same method as used in the proofs of Lemma 2.2, Theorem 2.6, and Corollary 2.7 in [9], we have (5.1). If $\mathcal{P}^\perp(eh) = 0$, then we have (5.1) by (4.5) and Theorem 3.2. The second part of the theorem immediately follows from the dominated convergence theorem. □

Now we have the following corollaries by Corollary 4.5 and Theorem 5.1.

Corollary 5.2. *Under the assumptions as given in Corollary 4.5, we have, for any nonnegative integer r with $r \geq l$, for $\lambda \in \mathbb{C}_+$, and for almost every $\vec{\xi} \in \mathbb{R}^n$,*

$$\begin{aligned}
E^{anw_\lambda}[G_Z \mid Z_n](\vec{\xi}) = \lambda^{\frac{r}{2}} E\big[K_r(\lambda,\cdot) f\big((v,\cdot)\big\|\mathcal{P}^\perp(eh)\big\| + \big(e,[\vec{\xi}]_b\big)\big) \\
\times \phi\big((v,\cdot)\big\|\mathcal{P}^\perp(eh)\big\| + \big(e,[\vec{\xi}]_b\big)\big)\big]
\end{aligned}$$

for any unit element $v \in L_2[0,t]$.

Corollary 5.3.

(1) *Under the assumptions as given in Theorem 4.4, we have, for $\gamma > 0$ and almost every $\vec{\xi} \in \mathbb{R}^n$,*

$$\begin{aligned} &E\big[G\big(\gamma Z(x,\cdot)\big) \mid \gamma Z_n(x)\big](\vec{\xi}) \\ &\quad = \lim_{m\to\infty} \gamma^{-m} E\big[K_m(\gamma^{-2},\cdot) f\big((v,\cdot)\|\mathcal{P}^{\perp}(eh)\| \\ &\qquad + \big(e,[\vec{\xi}]_b\big)\big)\phi\big((v,\cdot)\|\mathcal{P}^{\perp}(eh)\| + \big(e,[\vec{\xi}]_b\big)\big)\big] \end{aligned}$$

for any unit element $v \in L_2[0,t]$.

(2) *Under the assumptions as given in Corollary 4.5, we have, for any non-negative integer r with $r \geq l$, for $\gamma > 0$, and for almost every $\vec{\xi} \in \mathbb{R}^n$,*

$$\begin{aligned} &E\big[G\big(\gamma Z(x,\cdot)\big) \mid \gamma Z_n(x)\big](\vec{\xi}) \\ &\quad = \gamma^{-r} E\big[K_r(\gamma^{-2},\cdot) f\big((v,\cdot)\|\mathcal{P}^{\perp}(eh)\| \\ &\qquad + \big(e,[\vec{\xi}]_b\big)\big)\phi\big((v,\cdot)\|\mathcal{P}^{\perp}(eh)\| + \big(e,[\vec{\xi}]_b\big)\big)\big] \end{aligned}$$

for any unit element $v \in L_2[0,t]$.

Remark 5.4.

(1) While the complete orthonormal set in [6] and [11] contain e used in the definition of the cylinder function, the complete orthonormal set $\{e_1, e_2, \ldots\}$ in this paper does not contain e. Furthermore, the v's in Theorem 5.1 and Corollaries 5.2 and 5.3 are independent of both $\{e_1, e_2, \ldots\}$ and e.

(2) Letting $\phi = 1$ or, equivalently, $\rho = \delta_0$, which is the Dirac measure concentrated at 0, Corollaries 4.5, 4.6, 5.2, and 5.3 and Theorems 4.4 and 5.1 still hold replacing G by F.

(3) The change-of-scale formulas in this paper still hold, even if $\mathcal{P}^{\perp}(eh) = 0$ or, equivalently, $eh \in V$. Since, for $\gamma > 0$ and almost every $\vec{\xi} \in \mathbb{R}^n$,

$$E\big[G\big(\gamma Z(x,\cdot)\big) \mid \gamma Z_n(x)\big](\vec{\xi}) = G\big([\vec{\xi}]_b\big) = E\big[G\big(Z(x,\cdot)\big) \mid Z_n(x)\big](\vec{\xi}),$$

they are surplus in this case.

(4) While the conditioning function Z_n does not contain the initial position $Z(x,0)$ of the path $Z(x,\cdot)$ because of $Z(x,0) = 0$, it does contain the position $Z(x,t)$ at the present time t. Furthermore, if $h = 1$ almost everywhere, then $Z_n(x) = (x(t_1) - x(0), \ldots, x(t_n) - x(0))$ so that the formulas in this paper do not extend the existing change-of-scale formulas on the (generalized) Wiener spaces (see [6], [11]).

(5) The results of this paper are independent of a particular choice of the probability measure φ.

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