Banach J. Math. Anal. 10 (2016), no. 1, 71–95
http://dx.doi.org/10.1215/17358787-3319378
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

MULTIPLIERS AND HADAMARD PRODUCTS IN THE VECTOR-VALUED SETTING

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Communicated by R. Villena

Abstract. Let E_i be Banach spaces, and let X_{E_i} be Banach spaces continuously contained in the spaces of E_i-valued sequences $(\hat{x}(j))_j \in E_i^{\mathbb{N}}$, for $i = 1, 2, 3$. Given a bounded bilinear map $B : E_1 \times E_2 \to E_3$, we define $(X_{E_2}, X_{E_3})_B$, the space of B-multipliers between X_{E_2} and X_{E_3}, to be the set of sequences $(\lambda_j)_j \in E_1^{\mathbb{N}}$ such that $(B(\lambda_j, \hat{x}(j)))_j \in X_{E_3}$ for all $(\hat{x}(j))_j \in X_{E_2}$, and we define the Hadamard projective tensor product $X_{E_1} \circledast_B X_{E_2}$ as consisting of those elements in $E_3^{\mathbb{N}}$ that can be represented as $\sum_n \sum_j B(\hat{x}_n(j), \hat{y}_n(j))$, where $(x_n)_n \in X_{E_1}$, $(y_n)_n \in X_{E_2}$, and $\sum_n \|x_n\|_{X_{E_1}} \|y_n\|_{X_{E_2}} < \infty$.

We will analyze some properties of these two spaces, relate them, and compute the Hadamard tensor products and the spaces of vector-valued multipliers in several cases, getting applications in the particular case where $E = \mathcal{L}(E_1, E_2)$ and $B(T, x) = T(x)$.

1. Introduction and preliminaries

One of the classic problems in Fourier analysis is the description of the space of coefficient multipliers between function spaces. Several papers have shown mathematicians' interest in determining this space in particular cases (see the recent monograph [18]; see also [20] for the historical situation for Hardy spaces and [16] and [17] for several techniques and results regarding mixed norm; we refer the

Received Dec. 11, 2014; Accepted Apr. 14, 2015.
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2010 *Mathematics Subject Classification.* Primary 46B28; Secondary 46E40.
Keywords. vector-valued multipliers, Hadamard product, bilinear map.

reader to [11], [24], and [15] for the notation and results concerning Hardy and Bergman spaces that are used in this paper.

The operator-valued multipliers $(X(E_1), Y(E_2))$ corresponding to sequences of operators $(T_j)_j \in \mathcal{L}(E_1, E_2)$, for which $(T_j(x_j))_j \in Y(E_2)$ for all $(x_j) \in X(E_1)$ where $X(E_1)$ and $Y(E_2)$ stand for different spaces of vector-valued sequences (see [2]) or different spaces of vector-valued analytic functions (see [6] and references therein), have been deeply investigated.

Recently, O. Blasco and M. Pavlovic (see [7]) have considered general properties on the spaces of analytic functions in an abstract context to be able to carry over the study of multipliers between these spaces relying on the construction of certain Hadamard tensor products. These techniques allow them to recover many old results on concrete examples. Motivated by their paper (see also the recent monograph [21]), we shall introduce the notion of $\mathcal{S}(E)$-admissibility and consider the vector-valued analogues of several of the results in [7]. In particular, we shall develop a very general theory of vector-valued multipliers adapted to bilinear maps, which will cover most of the known cases in the vector-valued setting and will generate new ones, as well as another point of view.

A Banach space E, $\mathcal{S}(E)$ stands for the space of sequences $f = (x_j)_j \subseteq E$ endowed with the locally convex topology given by the seminorms $p_j(f) = \|x_j\|_E$, $j \geq 0$. We shall say that X_E is $\mathcal{S}(E)$-*admissible* if X_E is a Banach space contained with continuity in $\mathcal{S}(E)$ and the maps $x \to xe_j$, where $e_j : \mathcal{D} \to \mathcal{K}$, $e_j(z) = z^j$, from $E \hookrightarrow X_E$ are also continuous for each j.

It is easy to check that most of the well-known vector-valued sequence spaces such as $\ell^p(E)$, $\ell_{\mathrm{weak}}(E)$, and $\ell^p \hat{\otimes}_\pi E$, and most vector-valued spaces of analytic functions, such as vector-valued Hardy, Bergman, Bloch, or bounded mean oscillation analytic (BMOA) spaces, turn out to be $\mathcal{S}(E)$-admissible.

Let us now introduce the basic notions in this paper. For a given bounded bilinear map $B : E \times E_1 \to E_2$, we define the space of multipliers between X_{E_1} and X_{E_2} to be

$$(X_{E_1}, X_{E_2})_B = \big\{(\lambda_j)_j \in E^{\mathbb{N}} \text{ s.t. } \big(B(\lambda_j, x_j)\big)_j \in X_{E_2} \ \forall (x_j)_j \in X_{E_1}\big\}.$$

Then, if B verifies that there exists $C > 0$ such that

$$\|e\|_E \leq C \sup_{\|x\|_{E_1}=1} \big\|B(e,x)\big\|_{E_2}, \quad e \in E, \tag{1.1}$$

$(X_{E_1}, X_{E_2})_B$ becomes an $\mathcal{S}(E)$-admissible Banach space with its natural norm (see Theorem 3.3).

The particular instances of bilinear maps such as $B_0 : \mathbb{K} \times E \longrightarrow E$ given by $(\alpha, x) \mapsto \alpha x$, $B_{\mathcal{D}} : E' \times E \longrightarrow \mathbb{K}$ given by $(x', x) \mapsto \langle x', x\rangle$, and $B_{\mathcal{L}} : \mathcal{L}(E, F) \times E \longrightarrow F$ given by $(T, x) \mapsto T(x)$ have been considered in the literature quite often, and the corresponding spaces of B-multipliers have been described in some cases.

Given now two admissible spaces X_{E_1} and X_{E_2} and a bilinear map $B : E_1 \times E_2 \to E$, we define $X_{E_1} \circledast_B X_{E_2}$ as the space of elements $h \in \mathcal{S}(E)$ such that $h = \sum_n \sum_j B(x_n(j), y_n(j))$, where the series converges in $\mathcal{S}(E)$, $(x_n)_n \in X_{E_1}$, $(y_n)_n \in X_{E_2}$, and $\sum_n \|x_n\|_{X_{E_1}} \|y_n\|_{X_{E_2}} < \infty$.

It is not difficult to see that this space, normed in a natural way, is also $\mathcal{S}(E)$-admissible for bilinear maps satisfying the following condition: $\exists C > 0$ such that for each $e \in E$ there exists $(x_n, y_n) \in E_1 \times E_2$ verifying

$$e = \sum_n B(x_n, y_n), \qquad \sum_n \|x_n\|_{E_1} \|y_n\|_{E_2} \leq C\|e\|_E \tag{1.2}$$

(see Theorem 4.3).

A particular example with such a condition, and one very important for our purposes, is the following bilinear map, defined using the projective tensor product

$$B_\pi : E_1 \times E_2 \longrightarrow E_1 \hat{\otimes}_\pi E_2, \qquad (x, y) \mapsto x \otimes y.$$

We refer the reader to [10] or [22] for the definitions and properties of the projective tensor product and norm.

Hadamard tensor products and multipliers are closely related. One first connection with multipliers comes using the topological dual and the vector-valued Köthe dual $X_E^K = (X_E, \ell^1)_{B_{\mathcal{D}}}$. It will be shown that

$$(X_{E_1} \circledast_B X_{E_2})^K = (X_{E_1}, X_{E_2}^K)_{B^*}$$

and

$$(X_{E_1} \circledast_B X_{E_2})' = (X_{E_1}, X'_{E_2})_{B^*},$$

where $B^* : E' \times E_1 \to E'_2$ is the bounded bilinear map defined by

$$\big\langle B^*(e', x), y\big\rangle = \big\langle e', B(x, y)\big\rangle, \quad x \in E_1, y \in E_2, e' \in E'$$

(see Proposition 4.6).

Given a continuous bilinear map $B : X \times Y \longrightarrow Z$, there then exist unique bounded linear operators $T_B : X \hat{\otimes}_\pi Y \longrightarrow Z$ and $\Phi_B : X \to \mathcal{L}(Y, Z)$ satisfying

$$T_B(x \otimes y) = B(x, y) = \Phi_B(x)(y), \quad x \in X, y \in Y. \tag{1.3}$$

Using these identifications, one gets that

$$\mathcal{B}(X \times Y, Z) = \mathcal{L}(X \hat{\otimes}_\pi Y, Z) = \mathcal{L}\big(X, \mathcal{L}(Y, Z)\big)$$

are isometric isomorphisms. These identifications will give us a basic formula (see Theorem 4.7),

$$(X_{E_1} \circledast_{B_\pi} X_{E_2}, X_{E_3})_{B_{\mathcal{L}}} = \big(X_{E_1}, (X_{E_2}, X_{E_3})_{B_{\mathcal{L}}}\big)_{B_{\mathcal{L}}}, \tag{1.4}$$

which shows that describing Hadamard tensor products helps to determine multipliers.

We shall get the description of Hadamard tensor products in some cases. A particularly interesting example is the description of $H^1(\mathbb{D}) \circledast_{B_0} H^1(\mathbb{D}, L^p)$ for the values $1 < p \leq 2$ in Theorem 5.5. We will use the above formula and the previously mentioned description to recover some known results on vector-valued multipliers (see [6]):

$$\big(H^1(\mathbb{T}), \mathrm{BMOA}(\mathbb{T}, L^p)\big)_{B_{\mathcal{L}}} = \mathcal{B}\mathrm{loch}\big(\mathbb{D}, \mathcal{L}(L^p, L^p)\big), \quad 2 \leq p < \infty,$$
$$\big(H^1(\mathbb{T}, L^p), \mathrm{BMOA}(\mathbb{T})\big)_{B_{\mathcal{L}}} = \mathcal{B}\mathrm{loch}\big(\mathbb{D}, \mathcal{L}(L^{p'}, L^{p'})\big), \quad 1 \leq p \leq 2,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ (see Corollary 5.6).

The paper is organized as follows: Section 2 is devoted to introducing $\mathcal{S}(E)$-admissibility and gives some examples. In Section 3, we introduce coefficient multipliers through a bilinear map, deal with solid spaces, and relate multipliers with the Köthe dual. The Hadamard tensor product is defined in Section 4, where we find its connection with multipliers via the Köthe dual and show the formula (1.4). In the last section, we first use multipliers to determine the Hadamard tensor product of some spaces and, in the other direction, we also use the Hadamard product to obtain some vector-valued multiplier spaces showing applications to vector-valued Hardy spaces.

2. Vector-valued $\mathcal{S}$-admissibility

Let E be a Banach space. We use the notation $\mathcal{S}(E)$ for the space of sequences $f = (x_j)_{j\geq 0}$, where $x_j \in E$, endowed with the locally convex topology given by the seminorms $p_j(f) = \|x_j\|_E$, $j \geq 0$. We shall think of f as a formal power series with coefficients in E, that is, $f(z) = \sum_{j\geq 0} x_j z^j$, and most of the time we will write $\hat{f}(j)$ instead of x_j. Hence a sequence $(f_n)_n \subset \mathcal{S}(E)$ converges to $f \in \mathcal{S}(E)$ if and only if $p_j(f - f_n) \to 0$ for all $j \geq 0$ if and only if $\|\hat{f}(j) - \hat{f}_n(j)\|_E \to 0$ as $n \to \infty$ for all $j \geq 0$.

We will write $e_j(z) = z^j$ for each $j \geq 0$ and $\mathcal{P}(E)$ for the vector space of the analytic polynomials with coefficients in E; that is, $\sum_j^N x_j e_j$, where $x_j \in E$.

We first introduce the basic notion that plays a fundamental role in what follows.

Definition 2.1. Let E be a Banach space, and let X_E be a subspace of $\mathcal{S}(E)$. We will say that X_E is $\mathcal{S}(E)$-*admissible* (or simply *admissible*) if

(i) $(X_E, \|\cdot\|_{X_E})$ is a Banach space,
(ii) the projection $\pi_j : X_E \longrightarrow E$, $f \mapsto \hat{f}(j)$ is continuous, and
(iii) the inclusion $i_j : E \longrightarrow X_E$, $x \mapsto x e_j$ is continuous.

We denote $\pi_j(X_E) = \|\pi_j\|$ and $i_j(X_E) = \|i_j\|$.

Hence, for each $j \geq 0$, we have

$$\big\|\hat{f}(j)\big\|_E \leq \pi_j(X_E)\|f\|_{X_E}, \qquad \|x e_j\|_{X_E} \leq i_j(X_E)\|x\|_E.$$

Remark 2.1. Let X_{E_2} be $\mathcal{S}(E_2)$-admissible, and let E_1 be isomorphic to a closed subspace of E_2, say, $I(E_1)$. Define

$$X_{E_1} = \big\{(x_j)_j : x_j \in E_1, \big(I(x_j)\big)_j \in X_{E_2}\big\}$$

and the norm

$$\big\|(x_j)_j\big\|_{X_{E_1}} = \big\|\big(I(x_j)\big)_j\big\|_{X_{E_2}}.$$

Then X_{E_1} is $\mathcal{S}(E_1)$-admissible.

Also, we have that if Z is a Banach space and $X_E \subset Z \subset Y_E$ where X_E and Y_E are $\mathcal{S}(E)$-admissible, then Z is $\mathcal{S}(E)$-admissible.

Let us give a method to generate $\mathcal{S}(E)$-admissible spaces from classical $\mathcal{S}$-admissible spaces (i.e., keeping our notation, $\mathcal{S}(\mathcal{K})$-admissible spaces).

Proposition 2.2. *Let E be a Banach space, and let X be $\mathcal{S}$-admissible. We denote*

$$X[E] = \big\{(x_j)_{j\geq 0} \in \mathcal{S}(E) : \big\|\big(\|x_j\|_E\big)_j\big\|_X < \infty\big\},$$
$$X_{\text{weak}}(E) = \big\{(x_j)_{j\geq 0} \in \mathcal{S}(E) : \big\|(x_j)_j\big\|_{X_{\text{weak}}(E)} = \sup_{\|x'\|_{E'}=1}\big\|\big(\langle x_j, x'\rangle\big)_j\big\|_X < \infty\big\}.$$

Then $X \hat{\otimes}_\pi E, X[E]$ and $X_{\text{weak}}(E)$ are $\mathcal{S}(E)$-admissible.

Proof. The fact that $X[E]$ is a Banach space is easy and left to the reader. Clearly, $X_{\text{weak}}(E) = \mathcal{L}(E', X)$ and $X \hat{\otimes}_\pi E$ have complete norms.

Due to the continuous embeddings

$$X \hat{\otimes}_\pi E \subset X[E] \subset X_{\text{weak}}(E),$$

we only need to see that $\mathcal{P}(E) \subset X \hat{\otimes}_\pi E$ with continuous injections i_j for $j \geq 0$ and that $X_{\text{weak}}(E) \subset \mathcal{S}(E)$ with continuity. Both assertions follow trivially from the facts

$$\|xe_j\|_{X \hat{\otimes}_\pi E} = \|x\|_E\|e_j\|_X \leq i_j(X)\|x\|_E$$

and

$$\|x_j\|_E = \sup_{\|x'\|_{E'}=1}\big|\langle x_j, x'\rangle\big| \leq \pi_j(X)\big\|(x_k)_k\big\|_{X_{\text{weak}}(E)}. \qquad \square$$

Definition 2.3. Let X_E be $\mathcal{S}(E)$-admissible, and denote $X_E^0 = \overline{\mathcal{P}(E)}^{X_E}$. We say that X_E is *minimal* whenever $\mathcal{P}(E)$ is dense in X_E; that is to say $X_E^0 = X_E$.

Of course, X_E^0 is $\mathcal{S}(E)$-admissible whenever X_E is.

Proposition 2.4. *Let X_E be $\mathcal{S}(E)$-admissible, and let F be a Banach space. Then $\mathcal{L}(X_E, F)$ is $\mathcal{S}(\mathcal{L}(E, F))$-admissible.*

In particular, $(X_E)'$ and $(X_E^0)'$ are $\mathcal{S}(E')$-admissible.

Proof. Identifying each $T \in \mathcal{L}(X_E, F)$ with the sequence $(\hat{T}(j))_j \in \mathcal{S}(\mathcal{L}(E, F))$ given by $\hat{T}(j)(x) = T(xe_j)$, we have that $\mathcal{L}(X_E, F) \hookrightarrow \mathcal{S}(\mathcal{L}(E, F))$. Moreover, $\pi_j(\mathcal{L}(X_E, F)) \leq i_j(X_E)$ due to the estimate $\|\hat{T}(j)\|_{\mathcal{L}(E,F)} \leq i_j(X_E)\|T\|_{\mathcal{L}(X_E,F)}$.

To show $\mathcal{P}(\mathcal{L}(E, F)) \subset \mathcal{L}(X_E, F)$, we use that, for each $j \geq 0$ and $S \in \mathcal{L}(E, F)$, Se_j defines an operator in $\mathcal{L}(X_E, F)$ by means of

$$Se_j(f) = S(x_j), \quad f = (x_j) \in X_E.$$

Moreover, $i_j(\mathcal{L}(E, F)) \leq \pi_j(X_E)$ because $\|Se_j\|_{\mathcal{L}(X_E,F)} \leq \pi_j(X_E)\|S\|_{\mathcal{L}(E,F)}$. $\square$

Example 2.1. Some examples of $\mathcal{S}(E)$-admissible spaces are $\ell^p(E), \ell^p_{\text{weak}}(E)$, and $\ell^p \hat{\otimes}_\pi E$ for $1 \leq p \leq \infty$, where

$$\ell^p(E) = \ell^p[E] = \Big\{(x_n)_{n\geq 0} : \big\|(x_n)\big\|_{\ell^p(E)} = \Big(\sum_{n=0}^{\infty}\|x_n\|_E^p\Big)^{1/p} < \infty\Big\},$$

$$\ell^p_{\text{weak}}(E) = \Big\{(x_n)_{n\geq 0} : \big\|(x_n)\big\|_{\ell^p_{\text{weak}}(E)} = \sup_{\|x'\|_{E'}=1}\Big(\sum_{n=0}^{\infty}\big|\langle x_n, x'\rangle\big|^p\Big)^{1/p} < \infty\Big\},$$

with the obvious modifications for $p = \infty$.

In particular, $c_0(E) = (\ell^\infty(E))^0$ and

$$UC(E) = (\ell^1_{\text{weak}})^0(E) = \Big\{(x_n)_{n\geq 0} \in \ell^1_{\text{weak}}(E); \sum_n x_n \text{ converges unconditionally}\Big\}$$

are $S(E)$-admissible spaces.

Another interesting space, not coming from the above constructions, is

$$\text{Rad}(E) = \Big\{(x_j)_{j\geq 0} : \sup_N \Big[\int_0^1 \Big\|\sum_{j=0}^N x_j r_j(t)\Big\|_E^2 \, dt\Big]^{1/2} < \infty\Big\},$$

where r_j stands for the Rademacher functions (see [9]).

It is well known (see [9]) that

$$\ell^1_{\text{weak}}(E) \subset \text{Rad}(E) \subset \ell^2_{\text{weak}}(E)$$

with continuous embeddings, and therefore $\text{Rad}(E)$ is $\mathcal{S}(E)$-admissible.

Let us mention the interplay with the geometry of Banach spaces when comparing the space $\text{Rad}(E)$ and

$$\text{Rad}[E] = \big\{(x_j)_j \in \mathcal{S}(E) : \big\|\big(\|x_j\|_E\big)_j\big\|_{\text{Rad}}\big\}$$

and

$$\begin{aligned}\text{Rad}_{\text{weak}}(E) = \big\{(x_j)_{j\geq 0} \in \mathcal{S}(E) :& \\ \big\|(x_j)_j\big\|_{\text{Rad}_{\text{weak}}(E)} &= \sup_{\|x'\|_{E'}=1} \big\|\big(\langle x_j, x'\rangle\big)_j\big\|_{\text{Rad}} < \infty\big\},\end{aligned}$$

where $\text{Rad} = \text{Rad}(\mathbb{K})$. Recall that the notions of type 2 and cotype 2 correspond to $\ell^2(E) \subset \text{Rad}(E)$ and $\text{Rad}(E) \subset \ell^2(E)$, respectively (see [9]).

Proposition 2.5. *Let E be a Banach space.*

(i) $\text{Rad}(E) = \text{Rad}[E]$ *if and only if E is isomorphic to a Hilbert space.*
(ii) $\text{Rad}_{\text{weak}}(E) = \text{Rad}[E]$ *if and only if E is finite-dimensional.*

Proof. Note that, using the orthonormality of r_n, Plancherel's theorem gives $\text{Rad}[E] = \ell^2(E)$ and $\text{Rad}_{\text{weak}}(E) = \ell^2_{\text{weak}}(E)$. Of course, if E is a Hilbert space, then $\text{Rad}(E) = \ell^2(E)$ and, for finite-dimensional spaces, $\text{Rad}_{\text{weak}}(E) = \ell^2_{\text{weak}}(E) = \ell^2(E)$.

On the other hand, clearly $\text{Rad}[E] \subset \text{Rad}(E)$ if and only if E has type 2, and $\text{Rad}(E) \subset \text{Rad}[E]$ if and only if E has cotype 2. Now use Kwapien's theorem (see [9, Corollary 12.20, p. 246]) to conclude (i).

To see the direct implication in (ii), simply use that if $\dim(E) = \infty$, then $\ell^2(E) \subsetneq \ell^2_{\text{weak}}(E)$ (see [9, Theorem 2.18, p. 50]). □

Example 2.2. Let E be a complex Banach space, and denote by $\mathcal{H}(\mathbb{D}, E)$ the space of holomorphic functions from the unit disk $\mathbb{D}$ into E; that is,

$$f(z) = \sum_{j=0}^\infty x_j z^j, \quad x_j \in E, |z| < 1.$$

Then, with the notation in the Introduction, f would be written $\sum_{j\geq 0} \hat{f}(j) e_j$ and $\mathcal{P}(E)$ would actually be the E-valued polynomials.

In particular, for $E = \mathbb{C}$, most of the classical examples—such as Hardy spaces, Bergman spaces, Besov spaces, Bloch functions, and so on—become $\mathcal{S}$-admissible.

Let us introduce the vector-valued versions of those used in this paper. The vector-valued disk algebra and the bounded analytic functions will be denoted by

$$A(\mathbb{D}, E) = \big\{ f \in \mathcal{H}(\mathbb{D}, E), f \in C(\overline{\mathbb{D}}, E) \big\}$$

and

$$H^\infty(\mathbb{D}, E) = \big\{ f \in \mathcal{H}(\mathbb{D}, E), \sup_{|z|<1} \big\| f(z) \big\|_E < \infty \big\},$$

respectively, where we define

$$\|f\|_{A(\mathbb{D},E)} = \sup_{|z|=1} \big\| f(z) \big\|_E, \qquad \|f\|_{H^\infty(\mathbb{D},E)} = \sup_{|z|<1} \big\| f(z) \big\|_E.$$

It is easy to see that $(H^\infty(\mathbb{D}, E))^0 = A(\mathbb{D}, E)$.

Given $1 \le p < \infty$, the E-valued Bergman space $A^p(\mathbb{D}, E)$ is defined as the space of E-valued analytic functions on the unit disk such that

$$\|f\|_{A^p(\mathbb{D},E)} = \Big[\int_{\mathbb{D}} \big\| f(z) \big\|_E^p \, dA(z) \Big]^{1/p} = \Big[\int_0^1 M_p(f, r)^p r \, dr \Big]^{1/p} < \infty,$$

where

$$M_p(f, r) = \Big[\frac{1}{2\pi} \int_0^{2\pi} \big\| f(re^{it}) \big\|_E^p \, dt \Big]^{1/p}.$$

It is known that $A^p(\mathbb{D}, E)$ are minimal for $1 \le p < \infty$ (see, e.g., [1]).

The E-valued Hardy space $H^p(\mathbb{D}, E)$ is defined as the space of E-valued analytic functions on the unit disk such that

$$\|f\|_{H^p(\mathbb{D},E)} = \sup_{0<r<1} M_p(f, r) < \infty.$$

We also have the space defined at the boundary

$$H^p(\mathbb{T}, E) = \Big\{ f \in L^p(\mathbb{T}, E) : \hat{f}(n) = \int_0^{2\pi} f(e^{it}) e^{-int} \frac{dt}{2\pi} = 0, n \le 0 \Big\},$$

where $L^p(\mathbb{T}, E)$ stands for the functions which are p-integrable Bochner with values in E. It is not difficult to see that $H^p(\mathbb{T}, E) = (H^p(\mathbb{D}, E))^0$.

It is also well known that, for $1 \le p < \infty$,

$$A(\mathbb{D}, E) \subset H^\infty(\mathbb{D}, E) \subset H^p(\mathbb{D}, E) \subset A^p(\mathbb{D}, E) \subseteq A^1(\mathbb{D}, E).$$

Observe that $A(\mathbb{D}) \hat{\otimes}_\pi E \subset A(\mathbb{D}, E)$ and $A^1(\mathbb{D}, E) \subset A^1_{\text{weak}}(\mathbb{D}, E)$. Using that $A(\mathbb{D}) = A(\mathbb{D}, \mathbb{K})$ and $A^1(\mathbb{D}) = A^1(\mathbb{D}, \mathbb{K})$ are $\mathcal{S}$-admissible, we have that all the previous spaces of analytic functions are $\mathcal{S}(E)$-admissible.

Finally, we define the E-valued Bloch space, $\mathcal{B}\text{loch}(\mathbb{D}, E)$, to be the set of E-valued holomorphic functions on the disk that verify

$$\sup_{z \in \mathbb{D}} \big(1 - |z|\big) \big\| f'(z) \big\|_E < \infty.$$

It is a Banach space under the norm

$$\|f\|_{\mathcal{B}\text{loch}(\mathbb{D},E)} = \big\| f(0) \big\|_E + \sup_{z \in \mathbb{D}} \big(1 - |z|\big) \big\| f'(z) \big\|_E.$$

We will denote by $\mathrm{BMOA}(\mathbb{T}, E)$ the space of functions in $L^1(\mathbb{T}, E)$ with Fourier coefficients $\hat{f}(n) = 0$ for $n < 0$ and such that

$$\sup \frac{1}{|I|} \int_I \big\| f(e^{it}) - f_I \big\|_E \frac{dt}{2\pi} < \infty,$$

where the supremum is taken over all intervals $I \subseteq [0, 2\pi)$, $|I|$ is normalized I's Lebesgue measure, and $f_I = \frac{1}{|I|} \int_I f(e^{it}) \frac{dt}{2\pi}$. This becomes a Banach space under the norm

$$\|f\|_{\mathrm{BMOA}(\mathbb{T},E)} = \big\| f(0) \big\|_E + \sup \frac{1}{|I|} \int_I \big\| f(e^{it}) - f_I \big\|_E \frac{dt}{2\pi}.$$

Again we can use that

$$A(\mathbb{D}, E) \subset \mathrm{BMOA}(\mathbb{T}, E) \subset \mathcal{B}\mathrm{loch}(\mathbb{D}, E)$$

and $\mathcal{B}\mathrm{loch}(\mathbb{D}, E) = \mathcal{B}\mathrm{loch}_{\mathrm{weak}}(\mathbb{D}, E)$ to obtain that both spaces are $\mathcal{S}(E)$-admissible.

Remark 2.2. The spaces $X(E)$ and $X[E]$ are quite different whenever $X \subset \mathcal{H}(\mathbb{D})$ for infinite-dimensional Banach spaces E.

For instance, let $E = c_0$, and denote by $(e_n)_n$ the canonical basis. Consider the functions $f_N(z) = \sum_{n=0}^N e_n z^n$.

Let us analyze its norm in $H^p(\mathbb{D}, E)$ and $H^p(\mathbb{D})[E]$. We have

$$\|f_N\|_{H^p(\mathbb{D},c_0)} \leq \|f_N\|_{H^\infty(\mathbb{D},c_0)} = 1, \quad p \geq 1.$$

However,

$$\begin{aligned} \|f_N\|_{H^\infty(\mathbb{D})[c_0]} &= N + 1, \\ \|f_N\|_{H^p(\mathbb{D})[c_0]} &\geq \|f_N\|_{H^2(\mathbb{D})[c_0]} = (N+1)^{1/2}, \quad 2 \leq p < \infty, \end{aligned}$$

and, using Hardy's inequality for functions in H^1 (see [11]),

$$\|f_N\|_{H^p(\mathbb{D})[c_0]} \geq \|f_N\|_{H^1(\mathbb{D})[c_0]} \geq C \sum_{n=0}^N \frac{1}{n+1} \geq C \log(N+1), \quad 1 \leq p < 2.$$

Similarly,

$$A^2(\mathbb{D})[E] = \Big\{ (x_j)_j \in E^{\mathbb{N}} : \sum_{j=0}^\infty \frac{\|x_j\|^2}{j+1} < \infty \Big\}$$

and then, for $p \geq 2$,

$$\|f_N\|_{A^p(\mathbb{D},c_0)} \leq 1, \qquad \|f_N\|_{A^p(\mathbb{D})[c_0]} \geq C \big(\log(N+1) \big)^{1/2},$$

which exhibits the difference between the spaces above and the vector-valued interpretation $X[E]$.

3. Multipliers associated to bilinear maps

Now that we have introduced new classes of sequence spaces, we define a general convolution using bilinear maps, which will be the main notion in this paper.

Definition 3.1. Let E_1, E_2, and E_3 be Banach spaces, and let $B : E_1 \times E_2 \longrightarrow E_3$ be a bounded bilinear map.

We define the *B-convolution product* as the continuous bilinear map $\mathcal{S}(E_1) \times \mathcal{S}(E_2) \to \mathcal{S}(E_3)$ given by $(\lambda, f) \to \lambda *_B f$, where

$$\widehat{\lambda *_B f}(j) = B\big(\hat{\lambda}(j), \hat{f}(j)\big), \quad j \geq 0.$$

In particular, our results in the sequel could be applied to the following bilinear maps:

- For $B_0 : E \times \mathbb{K} \longrightarrow E$, $(x, \alpha) \mapsto \alpha x$, we get
$$\lambda *_{B_0} f = (\alpha_j x_j)_j.$$
- For $B_{\mathcal{D}} : E' \times E \longrightarrow \mathbb{K}$, $(x', x) \mapsto \langle x', x \rangle$, we get
$$\lambda *_{\mathcal{D}} f = \big(\langle x'_j, x_j \rangle\big)_j.$$
- For $B_{\mathcal{L}} : \mathcal{L}(E_1, E_2) \times E_1 \longrightarrow E_2$, $(T, x) \mapsto T(x)$, we get
$$\lambda *_{\mathcal{L}} f = \big(T_j(x_j)\big)_j.$$
- For $B_\pi : E_1 \times E_2 \longrightarrow E_1 \hat{\otimes}_\pi E_2$, $(x, y) \mapsto x \otimes y$, we get
$$f *_\pi g = (x_j \otimes y_j)_j.$$
- For a Banach algebra $(A, \cdot)$ and $P : A \times A \longrightarrow A$, $(a, b) \mapsto ab$, we get
$$\lambda *_P f = (a_j b_j)_j.$$

Associated to a bilinear convolution we have the spaces of multipliers.

Definition 3.2. Let E_1, E_2, and E be Banach spaces, and let $B : E \times E_1 \longrightarrow E_2$ be a bounded bilinear map. Let X_{E_1} and X_{E_2} be $S(E_1)$- and $S(E_2)$-admissible Banach spaces, respectively. We define the multipliers space between X_{E_1} and X_{E_2} through the bilinear map B as

$$(X_{E_1}, X_{E_2})_B = \big\{\lambda \in \mathcal{S}(E) : \lambda *_B f \in X_{E_2} \ \forall f \in X_{E_1}\big\}$$

with the norm

$$\|\lambda\|_{(X_{E_1}, X_{E_2})_B} = \sup_{\|f\|_{X_{E_1}} \leq 1} \|\lambda *_B f\|_{X_{E_2}}.$$

In the particular case where $E = \mathcal{L}(E_1, E_2)$ and $B = B_{\mathcal{L}}$, we simply write (X_{E_1}, X_{E_2}).

It is easy to prove that $\|\cdot\|_{(X_{E_1}, X_{E_2})_B}$ is a norm on $(X_{E_1}, X_{E_2})_B$ whenever B satisfies the condition

$$B(e, x) = 0, \quad \forall x \in E_1 \quad \Longrightarrow \quad e = 0.$$

In other words, the mapping $E \to \mathcal{L}(E_1, E_2)$, given by $e \to T_e$ where $T_e(x) = B(e, x)$, is injective.

Theorem 3.3. *Let E_1, E_2, and E be Banach spaces, and let $B : E \times E_1 \longrightarrow E_2$ be a bounded bilinear map for which there exists $C > 0$ such that*

$$\|e\|_E \le C \sup_{\|x\|_{E_1}=1} \big\|B(e,x)\big\|_{E_2}, \quad e \in E. \tag{3.1}$$

If X_{E_1} and X_{E_2} are $\mathcal{S}(E_1)$, $\mathcal{S}(E_2)$-admissible Banach spaces, respectively, then $(X_{E_1}, X_{E_2})_B$ is $\mathcal{S}(E)$-admissible.

Proof. We shall consider first the case where $E = \mathcal{L}(E_1, E_2)$ and $B = B_{\mathcal{L}}$.

Let $\lambda = (T_j)_j \in (X_{E_1}, X_{E_2})$ and $j \ge 0$. For each $x \in E_1$, using the admissibility of X_{E_1} and X_{E_2}, we have

$$\begin{aligned}\big\|T_j(x)\big\|_{E_2} &\le \pi_j(X_{E_2})\big\|T_j(x)e_j\big\|_{X_{E_2}}\\ &= \pi_j(X_{E_2})\|\lambda *_{\mathcal{L}} xe_j\|_{X_{E_2}}\\ &\le \pi_j(X_{E_2})\|\lambda\|_{(X_{E_1},X_{E_2})}\|xe_j\|_{X_{E_1}}\\ &\le \pi_j(X_{E_2})i_j(X_{E_1})\|\lambda\|_{(X_{E_1},X_{E_2})}\|x\|_{E_1}.\end{aligned}$$

This gives $(X_{E_1}, X_{E_2}) \hookrightarrow \mathcal{S}(\mathcal{L}(E_1, E_2))$ with continuity.

On the other hand, if $p \in \mathcal{P}(\mathcal{L}(E_1, E_2))$ and $f \in X_{E_1}$, we have $p *_{\mathcal{L}} f \in \mathcal{P}(E_2) \subset X_{E_2}$. Hence $p \in (X_{E_1}, X_{E_2})$. For each $j \ge 0$ and $T \in \mathcal{L}(E_1, E_2)$, we have to show that $\|Te_j\|_{(X_{E_1},X_{E_2})} \le C_j\|T\|$. Now given $f \in X_{E_1}$, again by the admissibility of X_{E_1} and X_{E_2},

$$\begin{aligned}\|Te_j *_{\mathcal{L}} f\|_{X_{E_2}} &= \big\|T\big(\hat f(j)\big)e_j\big\|_{X_{E_2}}\\ &\le i_j(X_{E_2})\big\|T\big(\hat f(j)\big)\big\|_{E_2}\\ &\le i_j(X_{E_2})\|T\|\big\|\hat f(j)\big\|_{E_1}\\ &\le i_j(X_{E_2})\pi_j(X_{E_1})\|T\|\|f\|_{X_{E_1}}.\end{aligned}$$

Therefore $C_j = i_j(X_{E_2})\pi_j(X_{E_1})$.

Let us now show the completeness of (X_{E_1}, X_{E_2}). Let $(\lambda_n)_n \subset (X_{E_1}, X_{E_2})$ be a Cauchy sequence of multipliers. Since the sequence of operators $\Lambda_n(f) = \lambda_n *_{\mathcal{L}} f$ is a Cauchy sequence in $\mathcal{L}(X_{E_1}, X_{E_2})$, we define $\Lambda \in \mathcal{L}(X_{E_1}, X_{E_2})$ to be its limit in the norm. Therefore

$$\|\Lambda - \Lambda_n\| \to 0 \quad \Rightarrow \quad \big\|\Lambda(f) - \Lambda_n(f)\big\|_{X_{E_2}} \to 0 \quad \Rightarrow \quad \lambda_n *_{\mathcal{L}} f \to \Lambda(f) \in \mathcal{S}(E_2).$$

On the other hand, we know that $(X_{E_1}, X_{E_2}) \hookrightarrow \mathcal{S}(\mathcal{L}(E_1, E_2))$, and then there exists $\lambda \in \mathcal{S}(\mathcal{L}(E_1, E_2))$ such that

$$\lambda_n *_{\mathcal{L}} f \to \lambda *_{\mathcal{L}} f$$

in $\mathcal{S}(\mathcal{L}(E_1, E_2))$. Hence, necessarily, $\Lambda(f) = \lambda *_{\mathcal{L}} f$.

For the general case, assumption (3.1) allows us to use Remark 2.1 where the isomorphism is given by $e \in E \to T_e \in \mathcal{L}(E_1, E_2)$ where $T_e(x) = B(e, x)$ for each $e \in E$ and $x \in E_1$. Just note that

$$(X_{E_1}, X_{E_2})_B = \Big\{\big(\hat\lambda(j)\big)_j \in E^{\mathbb{N}} : (T_{\hat\lambda(j)})_j \in (X_{E_1}, X_{E_2})\Big\}. \qquad \square$$

Let us consider the particular cases $B = B_0$ and $B = B_{\mathcal{D}}$.

Definition 3.4. Let X_E be $\mathcal{S}(E)$-admissible. We define

$$X_E^S = \{f = (x_j)_j \in \mathcal{S}(E) : (\alpha_j x_j)_j \in X_E, \forall (\alpha_j)_j \in \ell^\infty\}$$

with the norm

$$\|(x_j)_j\|_{X_E^S} = \sup_{\|(\alpha_j)_j\|_\infty = 1} \|(\alpha_j x_j)_j\|_{X_E}$$

and

$$X_E^K = \Big\{f = (x'_j)_j \in \mathcal{S}(E') : \sum_j |\langle x'_j, x_j\rangle| < \infty, \forall (x_j)_j \in X_E\Big\},$$

where

$$\|(x_j)_j\|_{X_E^K} = \sup_{\|(x_j)_j\|_{X_E} = 1} \sum_j |\langle x'_j, x_j\rangle|.$$

We also denote

$$X_E^{KK} = \Big\{f = (x_j)_j \in \mathcal{S}(E) : \sum_j |\langle x'_j, x_j\rangle| < \infty, \forall (x'_j)_j \in X_E^K\Big\}.$$

In general we have

$$X_E^S \subseteq X_E \subseteq X_E^{KK}.$$

One basic concept in the theory of multipliers is the notion of solid space (see [3]). We have the analogue notion in our setting.

Definition 3.5. We say that $X_E \subset \mathcal{S}(E)$ is $\mathcal{S}(E)$-*solid* (or simply *solid*) whenever X_E is an $\mathcal{S}(E)$-admissible space verifying $(\alpha_j \hat{f}(j))_j \in X_E$ for $f \in X_E$ and $(\alpha_j)_j \in \ell^\infty$; that is to say $X_E = X_E^S$.

Using that $(\ell^\infty, X_E)_{B_0} = X_E^S$ and $X_E^K = (X_E, \ell^1)_{B_\mathcal{D}}$ together with Theorem 3.3, we obtain the following corollary.

Corollary 3.6. *Let X_E be $\mathcal{S}(E)$-admissible. Then X_E^S and X_E^K are $\mathcal{S}(E)$-solid and $\mathcal{S}(E')$-solid, respectively.*

Remark 3.1. Let us collect here some observations of solid spaces.

(a) $X[E], X_{\text{weak}}(E)$, and $X \hat{\otimes}_\pi E$ are $\mathcal{S}(E)$-solid if and only if X is a solid space.
(b) $\mathrm{Rad}(E)$ is a $\mathcal{S}(E)$-solid space. (This follows from Kahane's contraction principle [9, Contraction Principle 12.2, p. 231].)
(c) Neither $H^p(\mathbb{D}, E)$ nor $A^p(\mathbb{D}, E)$ are $\mathcal{S}(E)$-solid unless $p = 2$.

Assuming that they are $\mathcal{S}(E)$-solid, and restricting to $\phi(z)x$ for $\phi \in \mathcal{H}(\mathbb{D})$ and $x \in E$, we will have that also H^p or A^p must be solid for $p \neq 2$, which is not the case.

Proposition 3.7. *Let X be $\mathcal{S}$-solid, and let E be a Banach space. Then*

(i) $(X \hat{\otimes}_\pi E)^K = (X^K)_{\text{weak}}(E')$;
(ii) $(X[E])^K = X^K[E']$.

Proof. (i) We first claim that $(x'_j)_j \in (X^K)_{\mathrm{weak}}(E')$ if and only if $(\langle x'_j, x\rangle)_j \in X^K$ for all $x \in E$. We only need to see that if

$$\sup_{\|x\|_E=1} \big\|\big(\langle x'_j, x\rangle\big)_j\big\|_{X^K} < \infty,$$

then $(\langle x'', x'_j\rangle)_j \in X^K$ for $x'' \in E''$.

For each $(\alpha_j)_j \in X$, $\|(\alpha_j)_j\|_X \leq 1$, and $N \in \mathbb{N}$, there are ϵ_j with $|\epsilon_j| = 1$,

$$\begin{aligned}
\sum_{j=0}^{N} \big|\langle x'', x'_j\rangle \alpha_j\big| &= \Big|\sum_{j=0}^{N} \langle x'', x'_j\rangle \alpha_j \epsilon_j\Big| \\
&= \Big|\Big\langle x'', \sum_{j=0}^{N} x'_j \alpha_j \epsilon_j\Big\rangle\Big| \\
&\leq \|x''\|_{E''} \Big\|\sum_{j=0}^{N} x'_j \alpha_j \epsilon_j\Big\|_{E'} \\
&\leq \|x''\|_{E''} \sup_{\|x\|_E=1} \sum_{j=0}^{N} \big|\langle x'_j, x\rangle \alpha_j\big| \\
&\leq \|x''\|_{E''} \sup_{\|x\|_E=1} \big\|\big(\langle x'_j, x\rangle\big)_j\big\|_{X^K}.
\end{aligned}$$

This concludes the claim.

We show first that $(X \hat{\otimes}_\pi E)^K \subseteq (X^K)_{\mathrm{weak}}(E')$. Take $\lambda = (x'_j)_j \in (X \hat{\otimes}_\pi E)^K$, $x \in E$, and $(\alpha_j)_j \in X$. Note that

$$\lambda *_{\mathcal{D}} \big((\alpha_j) \otimes x\big) = \big(\langle x'_j, x\rangle \alpha_j\big)_j \in \ell^1 \tag{3.2}$$

and then we obtain $(x'_j)_j \in (X^K)_{\mathrm{weak}}(E')$ with $\|(x'_j)_j\|_{(X^K)_{\mathrm{weak}}(E')} \leq \|\lambda\|$ from the previous result.

Assume now that $\lambda = (x'_j)_j \in (X^K)_{\mathrm{weak}}(E')$, and let us show that $\lambda \in (X \hat{\otimes}_\pi E)^K$. If $\epsilon > 0$ and $f = \sum_n f_n \otimes x_n \in X \hat{\otimes}_\pi E$ with $\hat{f}_n(j) = \alpha_j^n$ and $\sum_n \|f_n\|_X \|x_n\|_E < \|f\|_{X \hat{\otimes}_\pi E} + \epsilon$, then we have

$$\begin{aligned}
\sum_j \big|\widehat{\lambda *_{\mathcal{D}} f}(j)\big| &\leq \sum_j \sum_n \big|\langle x'_j, x_n\rangle \alpha_j^n\big| \\
&= \sum_n \sum_j \big|\langle x'_j, x_n\rangle \alpha_j^n\big| \\
&\leq \sum_n \|x_n\|_E \Big\|\Big(\Big\langle x'_j, \frac{x_n}{\|x_n\|}\Big\rangle\Big)_j\Big\|_{X^K} \|f_n\|_X \\
&\leq \big\|(x'_j)_j\big\|_{(X^K)_{\mathrm{weak}}(E')} \Big(\sum_n \|x_n\|_E \|f_n\|_X\Big) \\
&\leq \big\|(x'_j)_j\big\|_{(X^K)_{\mathrm{weak}}(E')} \big(\|f\|_{X \hat{\otimes}_\pi E} + \epsilon\big).
\end{aligned}$$

(ii) We first notice that

$$\sum_j \big|\langle x'_j, x_j\rangle\big| \le \sum_j \|x'_j\|_{E'}\|x_j\|_E \le \big\|\big(\|x'_j\|_{E'}\big)_j\big\|_{X^K}\big\|\big(\|x_j\|_E\big)_j\big\|_X.$$

This shows that $X^K[E'] \subseteq (X[E])^K$.

To see the other inclusion, let $\lambda = (x'_j)_j \in (X[E])^K$ and show that $(\|x'_j\|_{E'})_{j\ge 0} \in X^K$. Fix $(\alpha_j)_j \in X$, $\epsilon > 0$, and $j \ge 0$. Select $x_j \in E$ with $\|x_j\|_E = 1$ and $\|x'_j\|_{E'} = |\langle x'_j, x_j\rangle| + \epsilon 2^{-(j+1)}|\alpha_j|^{-1}$ for $\alpha_j \ne 0$. Consider now $f = (\alpha_j x_j)_j \in X[E]$ and observe that, using that X is solid, we get

$$\begin{aligned}\sum_j \|x'_j\|_{E'}|\alpha_j| &= \sum_j \big|\langle x'_j, x_j\rangle\big||\alpha_j| + \epsilon\\ &= \|\lambda *_{\mathcal{D}} f\|_{\ell^1} + \epsilon\\ &\le \|\lambda\|_{(X[E])^K}\|f\|_{X[E]} + \epsilon\\ &\le \|\lambda\|_{(X[E])^K}\big\|(\alpha_j)_j\big\|_X + \epsilon.\end{aligned}$$

This finishes the proof. □

Remark 3.2. In general, $X^K \hat{\otimes}_\pi E' \subseteq (X_{\text{weak}}(E))^K$.

Indeed, for each $g = (\beta_j)_j \in X^K$, $x' \in E'$, and $f = (x_j)_j \in X_{\text{weak}}(E)$, we have that

$$(g \otimes x') *_{\mathcal{D}} f = \big(\langle x', x_j\rangle\beta_j\big)_j, \tag{3.3}$$

which satisfies

$$\sum_j \big|\langle x', x_j\rangle\beta_j\big| \le \|g\|_{X^K}\|x'\|_{E'}\|f\|_{X_{\text{weak}}(E)},$$

and then

$$\|g \otimes x'\|_{(X_{\text{weak}}(E))^K} \le \|g\|_{X^K}\|x'\|_{E'}.$$

Now we extend using linearity and density to obtain $X^K \hat{\otimes}_\pi E' \subseteq (X_{\text{weak}}(E))^K$.

For the case $X = \ell^p$, $1 < p < \infty$, it has been shown (see [8], [13], [2]) that

$$\big(\ell^p_{\text{weak}}(E)\big)^K = \ell^{p'} \hat{\otimes}_\pi E'.$$

Theorem 3.8. *Let E_1, E_2, and E be Banach spaces, and let $B : E \times E_1 \longrightarrow E_2$ be a bounded bilinear map satisfying (3.1).*

Define $B_* : E \times E'_2 \to E'_1$ given by

$$\big\langle B_*(e, y'), x\big\rangle = \big\langle y', B(e, x)\big\rangle, \quad e \in E, x \in E_1, y' \in E'_2.$$

If X_{E_1} and X_{E_2} are admissible spaces and $X_{E_2} = X_{E_2}^{KK}$, then

$$(X_{E_1}, X_{E_2})_B = (X_{E_2}^K, X_{E_1}^K)_{B_*}.$$

Proof. From the definition we can write, for $\lambda \in \mathcal{S}(E)$, $f \in \mathcal{S}(E_1)$, $g \in \mathcal{S}(E'_2)$, and $j \ge 0$,

$$\big\langle \hat{g}(j), \widehat{\lambda *_B f}(j)\big\rangle = \big\langle \widehat{\lambda *_{B_*} g}(j), \hat{f}(j)\big\rangle.$$

Assume now that $\lambda \in (X_{E_1}, X_{E_2})_B$ and $g \in X^K_{E_2}$. We have

$$\begin{aligned}
\|\lambda *_{B_*} g\|_{X^K_{E_1}} &= \sup\Big\{\sum_j \big|\big\langle \widehat{\lambda *_{B_*} g}(j), \hat{f}(j)\big\rangle\big| : \|f\|_{X_{E_1}} \le 1\Big\} \\
&= \sup\Big\{\sum_j \big|\big\langle \hat{g}(j), \widehat{\lambda *_B f}(j)\big\rangle\big| : \|f\|_{X_{E_1}} \le 1\Big\} \\
&\le \|g\|_{X^K_{E_2}} \sup\big\{\big\|(\lambda *_B f)\big\|_{X_{E_2}} : \|f\|_{X_{E_1}} \le 1\big\} \\
&\le \|\lambda\|_{(X_{E_1}, X_{E_2})_B} \|g\|_{X^K_{E_2}}.
\end{aligned}$$

Using the assumption $X_{E_2} = X^{KK}_{E_2}$, one can argue as above for $\lambda \in (X^K_{E_2}, X^K_{E_1})_{B_*}$ and $f \in X_{E_1}$ to obtain

$$\begin{aligned}
\|\lambda *_B f\|_{X_{E_2}} &= \sup\Big\{\sum_j \big|\big\langle \hat{g}(j), \widehat{\lambda *_B f}(j)\big\rangle\big| : \|g\|_{X^K_{E_2}} \le 1\Big\} \\
&= \sup\Big\{\sum_j \big|\langle \widehat{\lambda *_{B_*} g}(j), \hat{f}(j)\big| : \|g\|_{X^K_{E_2}} \le 1\Big\} \\
&\le \|f\|_{X_{E_1}} \sup\big\{\big\|(\lambda *_{B_*} g)\big\|_{X^K_{E_1}} : \|g\|_{X^K_{E_2}} \le 1\big\} \\
&\le \|\lambda\|_{(X^K_{E_2}, X^K_{E_1})_{B_*}} \|f\|_{X_{E_1}}.
\end{aligned}$$

□

4. The B-Hadamard tensor product

Let us now generate a new $\mathcal{S}(E)$-admissible space using bilinear maps and tensor products.

Definition 4.1. Let E_1, E_2, and E_3 be Banach spaces, and let $B : E_1 \times E_2 \longrightarrow E_3$ be a bounded bilinear map. Let X_{E_1}, X_{E_2} be $\mathcal{S}(E_1), \mathcal{S}(E_2)$-admissible, respectively. We define the Hadamard projective tensor product $X_{E_1} \circledast_B X_{E_2}$ as the space of elements $h \in \mathcal{S}(E_3)$ that can be represented as

$$h = \sum_n f_n *_B g_n,$$

where the convergence of $\sum_n f_n *_B g_n$ is considered in $\mathcal{S}(E_3)$, being $f_n \in X_{E_1}, g_n \in X_{E_2}$, and

$$\sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} < \infty.$$

The particular case $E_3 = E_1 \hat{\otimes}_\pi E_2$ and $B_\pi : E_1 \times E_2 \to E_3$ will be simply denoted as $X_{E_1} \circledast X_{E_2}$

Proposition 4.2. *Let E_1, E_2, and E_3 be Banach spaces, and let $B : E_1 \times E_2 \longrightarrow E_3$ be a bounded bilinear map. Let $h \in X_{E_1} \circledast_B X_{E_2}$, and define*

$$\|h\|_B = \inf \sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}},$$

where the infimum is taken over all possible representations of $h = \sum_n f_n *_B g_n$.

Then $(X_{E_1} \circledast_B X_{E_2}, \|\cdot\|_B)$ is a Banach space.

Proof. Let $\|h\|_B = 0$ and $\epsilon > 0$. Thus there exists a representation $h = \sum_n f_n *_B g_n$ such that $\sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} < \epsilon$. Since the series converges in $\mathcal{S}(E_3)$ we conclude that $\hat{h}(j) = \sum_n B(\hat{f}_n(j), \hat{g}_n(j))$. Using the admissibility of X_{E_1} and X_{E_2},

$$\begin{aligned}
\big\|\hat{h}(j)\big\|_{E_3} &\le \sum_n \big\|B\big(\hat{f}_n(j), \hat{g}_n(j)\big)\big\|_{E_3} \\
&\le \|B\| \sum_n \big\|\hat{f}_n(j)\big\|_{E_1} \big\|\hat{g}_n(j)\big\|_{E_2} \\
&\le \|B\| \pi_j(X_{E_1}) \pi_j(X_{E_2}) \sum_n \|\hat{f}_n\|_{X_{E_1}} \|\hat{g}_n\|_{X_{E_2}} < \epsilon.
\end{aligned}$$

Consequently, $\hat{h}(j) = 0$ for all $j \ge 0$.

Of course, $\|\alpha h\|_B = |\alpha| \|h\|_B$ for any $\alpha \in \mathbb{K}$ and $h \in X_{E_1} \circledast_B X_{E_2}$.

The triangle inequality follows using that if $h_1 = (f_n^1 *_B g_n^1)_n$ and $h_2 = (f_n^2 *_B g_n^2)_n$ such that

$$\sum_n \|f_n^i\|_{X_{E_1}} \|g_n^i\|_{X_{E_2}} < \|h_i\|_B + \frac{\epsilon}{2}, \quad i = 1, 2,$$

then $h_1 + h_2 = \sum_n f_n^1 *_B g_n^1 + \sum_m f_m^2 *_B g_m^2$ and then

$$\|h_1 + h_2\|_B \le \sum_n \|f_n^1\|_{X_{E_1}} \|g_n^1\|_{X_{E_2}} + \sum_m \|f_m^2\|_{X_{E_1}} \|g_m^2\|_{X_{E_2}} < \|h_1\|_B + \|h_2\|_B + \epsilon.$$

Finally, let us see that $X_{E_1} \circledast_B X_{E_2}$ is complete. Let $\sum_n h_n$ be an absolute convergent series in $X_{E_1} \circledast_B X_{E_2}$ with $h_n \in X_{E_1} \circledast_B X_{E_2}$. For each $n \in \mathbb{N}$ select a decomposition $h_n(z) = \sum_k f_k^n *_B g_k^n$ such that

$$\sum_k \|f_k^n\|_{X_{E_1}} \|g_k^n\|_{X_{E_2}} < 2\|h_n\|_B.$$

Let us now show that $\sum_n h_n = \sum_n \sum_k f_k^n *_B g_k^n$ in $\mathcal{S}(E_3)$. Indeed, for each $j \ge 0$, we have

$$\begin{aligned}
\sum_n \sum_k \big\|B\big(\widehat{f_k^n}(j), \widehat{g_k^n}(j)\big)\big\|_{E_3} &\le \|B\| \pi_j(X_{E_1}) \pi_j(X_{E_2}) \sum_n \sum_k \|f_k^n\|_{X_{E_1}} \|g_k^n\|_{X_{E_2}} \\
&< 2\|B\| \pi_j(X_{E_1}) \pi_j(X_{E_2}) \sum_n \|h_n\|_B,
\end{aligned}$$

and since E_3 is complete we get the result.

Moreover, $h = \sum_n h_n \in X_{E_1} \circledast_B X_{E_2}$ because $\sum_n \sum_k \|f_k^n\|_{X_{E_1}} \|g_k^n\|_{X_{E_2}} < \infty$. Now use that

$$\Big\|\sum_{n=N}^{\infty} h_n\Big\|_B \le \sum_{n=N}^{\infty} \sum_k^{\infty} \|f_k^n\|_{X_{E_1}} \|g_k^n\|_{X_{E_2}} < 2 \sum_{n=N}^{\infty} \|h_n\|_B$$

to conclude that the series $\sum_n h_n$ converges to h in $X_{E_1} \circledast_B X_{E_2}$. $\square$

Remark 4.1. If $h = \sum_n f_n *_\pi g_n \in X_{E_1} \circledast_B X_{E_2}$, then $\sum_n \|f_n *_B g_n\|_B < \infty$ and $h = \sum_n f_n *_B g_n$ converges in $X_{E_1} \circledast_B X_{E_2}$.

Indeed, simply use that

$$\|f *_B g\|_B \le \|f\|_{X_{E_1}} \|g\|_{X_{E_2}}$$

for $f \in X_{E_1}$ and $g \in X_{E_2}$ and that, for $M > N$,

$$\Big\|\sum_{n=N}^{M} f_n *_B g_n\Big\|_B \leq \sum_{n=N}^{M} \|f_n *_B g_n\|_B \leq \sum_{n=N}^{M} \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}}.$$

Theorem 4.3. *Let E_1, E_2, and E be Banach spaces, and let $B : E_1 \times E_2 \longrightarrow E$ be a bounded bilinear map satisfying that there exists $C > 0$ such that for each $e \in E$ there exists $(x_n, y_n) \in E_1 \times E_2$ such that*

$$e = \sum_n B(x_n, y_n), \qquad \sum_n \|x_n\|_{E_1} \|y_n\|_{E_2} \leq C\|e\|_E. \tag{4.1}$$

If X_{E_1} and X_{E_2} are admissible spaces, then $X_{E_1} \circledast_B X_{E_2}$ is $\mathcal{S}(E)$-admissible.

In particular, $X_{E_1} \circledast X_{E_2}$ is admissible.

Proof. We show first that $X_{E_1} \circledast_B X_{E_2} \subset \mathcal{S}(E)$ with continuity. For $\epsilon > 0$ we can find a representation $h = \sum_n f_n *_B g_n$ such that $\sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} < \|h\|_B + \epsilon$. Therefore, for each $j \geq 0$,

$$\begin{aligned}
\big\|\hat{h}(j)\big\|_E &\leq \sum_n \big\|B\big(\hat{f}_n(j), \hat{g}_n(j)\big)\big\|_E \\
&\leq \|B\| \sum_n \big\|\hat{f}_n(j)\big\|_{E_1} \big\|\hat{g}_n(j)\big\|_{E_2} \\
&\leq \|B\| \pi_j(X_{E_1}) \pi_j(X_{E_2}) \sum_n \|f_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} \leq C_j \|h\|_B + \epsilon.
\end{aligned}$$

To show that $\mathcal{P}(E) \subset X_{E_1} \circledast_B X_{E_2}$, it suffices to see that $ee_j \in X_{E_1} \circledast_B X_{E_2}$ for each $j \geq 0$ and $e \in E$. Now use condition (4.1) to write $e = \sum_n B(x_n, y_n) \in E$, and therefore

$$ee_j = \sum_n (x_n e_j) *_B (y_n e_j)$$

and

$$\sum_n \|x_n e_j\|_{X_{E_1}} \|y_n e_j\|_{X_{E_2}} \leq i_j(X_{E_1}) i_j(X_{E_2}) \sum_n \|x_n\|_{E_1} \|y_n\|_{E_2} \leq C_j \|e\|_E.$$

Hence $ee_j \in X_{E_1} \circledast_B X_{E_2}$ and $\|ee_j\|_B \leq C i_j(X_{E_1}) i_j(X_{E_2}) \|e\|_E$. □

Remark 4.2. If E_1, E_2, and E are Banach spaces and $B : E_1 \times E_2 \longrightarrow E$ is a surjective bounded bilinear map such that there exists $C > 0$ such that for every $e \in E$ there exists $(x, y) \in E_1 \times E_2$ verifying

$$e = B(x, y), \qquad \|x\|_{E_1} \|y\|_{E_2} \leq C\|e\|_E, \tag{4.2}$$

then we can apply Theorem 4.3.

A simple application of (4.2) gives the following cases.

Corollary 4.4.

(i) *If X and X_E are admissible spaces and $B_0 : \mathbb{K} \times E \to E$ is given by $(\alpha, x) \to \alpha x$, then $X \circledast_{B_0} X_E$ is $\mathcal{S}(E)$-admissible.*

(ii) *Let (Σ, μ) be a measure space, $1 \leq p_j \leq \infty$ for $i = 1, 2, 3$ and $1/p_3 = 1/p_1 + 1/p_2$. Let $B : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{p_3}(\mu)$ be given by $(f, g) \to fg$. Then if*

$X_{L^{p_1}}$ and $X_{L^{p_2}}$ are admissible spaces, then $X_{L^{p_1}} \circledast_B X_{L^{p_2}}$ is admissible.

(iii) *Let A be a Banach algebra with identity and $P : A \times A \to A$ given by $(a, b) \to ab$. If X_A and Y_A are admissible spaces, then $X_A \circledast_P Y_A$ is admissible.*

Remark 4.3. It is straightforward to see that, under the assumptions of Theorem 4.3, if either X_{E_1} or X_{E_2} are solid spaces, then $X_{E_1} \circledast_B X_{E_2}$ is an $\mathcal{S}(E)$-solid space.

Proposition 4.5. *Let E_1, E_2, and E be Banach spaces, and let $B : E_1 \times E_2 \longrightarrow E$ be a bounded bilinear map satisfying (4.1). Let X_{E_1}, X_{E_2} be admissible Banach spaces such that either X_{E_1} or X_{E_2} are minimal spaces; then $X_{E_1} \circledast_B X_{E_2}$ is a minimal $\mathcal{S}(E)$-admissible space.*

Proof. We shall prove the case $X^0_{E_1} = X_{E_1}$. Let $h \in X_{E_1} \circledast_B X_{E_2}$. From Remark 4.1, there exist $f_n \in X_{E_1}$, $g_n \in X_{E_2}$, and $N \in \mathbb{N}$ such that

$$\Big\| h - \sum_{n=0}^{N} f_n *_B g_n \Big\|_B < \frac{\epsilon}{2}.$$

By density, choose polynomials p_n with coefficients in E_1 such that

$$\|f_n - p_n\|_{X_{E_1}} \leq \frac{\epsilon}{2(N+1)\|g_n\|_{X_{E_2}}}.$$

Then $\sum_{n=0}^{N} p_n *_B g_n \in \mathcal{P}(E)$ and

$$\begin{aligned}\Big\| h - \sum_{n=0}^{N} p_n *_B g_n \Big\|_B &\leq \Big\| h - \sum_{n=0}^{N} f_n *_B g_n \Big\|_B + \Big\| \sum_{n=0}^{N} (f_n - p_n) *_B g_n \Big\|_B \\ &\leq \frac{\epsilon}{2} + \sum_{n=0}^{N} \|f_n - p_n\|_{X_{E_1}} \|g_n\|_{X_{E_2}} \leq \frac{\epsilon}{2} + \sum_{n=0}^{N} \frac{\epsilon}{2(N+1)} \\ &= \epsilon.\end{aligned}$$

□

Proposition 4.6. *Let $B : E_1 \times E_2 \to E$ be a bounded bilinear map satisfying (4.1). Denote by $B^* : E' \times E_1 \to E'_2$ the bounded bilinear map defined by*

$$\big\langle B^*(e', x), y \big\rangle = \big\langle e', B(x, y) \big\rangle, \quad x \in E_1, y \in E_2, e' \in E'.$$

If X_{E_1} and X_{E_2} are admissible, then

$$(X_{E_1} \circledast_B X_{E_2})^K = (X_{E_1}, X^K_{E_2})_{B^*};$$
$$(X_{E_1} \circledast_B X_{E_2})' = (X_{E_1}, X'_{E_2})_{B^*}.$$

In particular, $(X_{E_1} \circledast X_{E_2})' = (X_{E_1}, X'_{E_2})$ and $(X_{E_1} \circledast X_{E_2})^K = (X_{E_1}, X^K_{E_2})$.

Proof. Let $\lambda \in (X_{E_1}, X^K_{E_2})_{B^*}$, and define, for $f \in X_{E_1}$ and $g \in X_{E_2}$,

$$\tilde{\lambda}(f *_B g)\widehat{\ }(j) = \big\langle (\lambda *_{B^*} f)\widehat{\ }(j), \hat{g}(j) \big\rangle, \quad j \geq 0.$$

Let us see that, $\tilde{\lambda} \in (X_{E_1} \circledast_B X_{E_2})^K$,

$$\begin{aligned}\sum_j \big|\tilde{\lambda}(f *_B g)\hat{}(j)\big| &= \sum_j \big|\big\langle(\lambda *_{B^*} f)\hat{}(j), \hat{g}(j)\big\rangle\big| \\ &\leq \|\lambda *_{B^*} f\|_{X^K_{E_2}} \|g\|_{X_{E_2}} \\ &\leq \|\lambda\|_{(X_{E_1}, X^K_{E_2})_{B^*}} \|f\|_{X_{E_1}} \|g\|_{X_{E_2}}.\end{aligned}$$

By linearity we can extend the result to finite linear combinations of $*_B$-products and, by continuity, to $X_{E_1} \circledast_B X_{E_2}$; that is,

$$\tilde{\lambda}(h) = \sum_n \tilde{\lambda}(f_n *_B g_n)$$

whenever $h = \sum_n f_n *_B g_n$ and $\sum_n \|f_n *_B g_n\|_B \leq \infty$. Therefore we conclude that $(X_{E_1}, X^K_{E_2})_{B^*} \subseteq (X_{E_1} \circledast_B X_{E_2})^K$.

For the other inclusion, consider $\gamma \in (X_{E_1} \circledast_B X_{E_2})^K$ and define $\tilde{\gamma}(f)\hat{}(j) \in E_2'$ by

$$\big\langle \tilde{\gamma}(f)\hat{}(j), y\big\rangle = \gamma(f *_B ye_j)\hat{}(j), \quad f \in X_{E_1}, y \in E_2, j \geq 0.$$

This gives

$$\big\langle \tilde{\gamma}(f)\hat{}(j), \hat{g}\big(j\big\rangle\big) = \gamma(f *_B g)\hat{}(j), \quad f \in X_{E_1}, g \in X_{E_2}, j \geq 0.$$

Let us see that, $\tilde{\gamma} \in (X_{E_1}, X^K_{E_2})_{B^*}$,

$$\begin{aligned}\big\|\tilde{\gamma}(f)\big\|_{X^K_{E_2}} &= \sup_{\|g\|_{X_{E_2}}=1} \sum_j \big|\gamma(f *_B g)\hat{}(j)\big| \\ &\leq \|\gamma\|_{(X_{E_1} \circledast_B X_{E_2})^K} \sup_{\|g\|_{X_{E_2}}=1} \|f *_B g\|_B \\ &\leq \|\gamma\|_{(X_{E_1} \circledast_B X_{E_2})^K} \|f\|_{X_{E_1}}.\end{aligned}$$

The argument to study the dual is similar: Let $\lambda \in (X_{E_1}, X'_{E_2})_{B^*}$, and define $\phi_\lambda(f *_B g) = \langle \lambda *_{B^*} f, g\rangle$. Note that X'_{E_2} is also $\mathcal{S}(E_2')$-admissible and that

$$\big|\phi_\lambda(f *_B g)\big| \leq \|\lambda\|_{(X_{E_1}, X'_{E_2})_{B^*}} \|f\|_{X_{E_1}} \|g\|_{X_{E_2}}.$$

By linearity we can extend the result to finite linear combinations of $*_B$-products and extend by continuity $X_{E_1} \circledast_B X_{E_2}$; that is,

$$\phi_\lambda(h) = \sum_n \phi_\lambda(f_n *_B g_n)$$

whenever $h = \sum_n f_n *_B g_n$ and $\sum_n \|f_n *_B g_n\|_B \leq \infty$. Therefore we conclude that $(X_{E_1}, X'_{E_2})_{B^*} \subseteq (X_{E_1} \circledast_B X_{E_2})'$.

For the other inclusion, consider $T \in (X_{E_1} \circledast_B X_{E_2})'$, and define

$$\lambda_T(f)(g) = T(f *_B g).$$

Then

$$\big\|\lambda_T(f)\big\|_{X'_{E_2}} = \sup_{\|g\|_{X_{E_2}}=1} \big|\lambda_T(f)(g)\big| \leq \sup_{\|g\|_{X_{E_2}}=1} \|T\| \|f *_B g\|_B \leq \|T\| \|f\|_{X_{E_1}}. \qquad \square$$

Theorem 4.7. *Let $X_{E_1}, X_{E_2}, X_{E_3}$ be admissible Banach spaces. Then*

$$(X_{E_1} \circledast X_{E_2}, X_{E_3}) = \big(X_{E_1}, (X_{E_2}, X_{E_3})\big).$$

Proof. Due to the identification between $\mathcal{L}(E_1 \hat{\otimes}_\pi E_2, E_3)$ and $\mathcal{L}(E_1, \mathcal{L}(E_2, E_3))$, where the correspondence was given by $\phi(x \otimes y) = T_\phi(x)(y)$, we obtain, in our case, that each $\lambda \in \mathcal{S}(\mathcal{L}(E_1 \hat{\otimes}_\pi E_2, E_3))$ can be identified with $\tilde{\lambda} \in \mathcal{S}(\mathcal{L}(E_1, \mathcal{L}(E_2, E_3))$ satisfying

$$\hat{\lambda}(j)\big(\hat{f}(j) \otimes \hat{g}(j)\big) = \widehat{\tilde{\lambda}}(j)\big(\hat{f}(j)\big)\big(\hat{g}(j)\big).$$

Let $\lambda \in (X_{E_1} \circledast X_{E_2}, X_{E_3})$. For each $f \in X_{E_1}$ and $g \in X_{E_2}$, we have

$$\lambda *_1 (f *_\pi g) = (\tilde{\lambda} *_2 f) *_3 g, \tag{4.3}$$

where $*_1$ is used for multipliers in $\mathcal{S}(\mathcal{L}(E_1 \hat{\otimes}_\pi E_2), E_3)$, $*_2$ is used for multipliers in $\mathcal{S}(\mathcal{L}(E_1, \mathcal{L}(E_2, E_3)))$, and $*_3$ is used for multipliers in $\mathcal{S}(\mathcal{L}(E_2, E_3))$.

Let us now show that $\tilde{\lambda} \in (X_{E_1}, (X_{E_2}, X_{E_3}))$.

We use (4.3) to get

$$\begin{aligned}\big\|(\tilde{\lambda} *_2 f) *_3 g\big\|_{X_{E_3}} &\le \|\lambda\|_{(X_{E_1} \circledast X_{E_2}, X_{E_3})}\big\|(f *_\pi g)\big\| \\ &= \|\lambda\|_{(X_{E_1} \circledast X_{E_2}, X_{E_3})}\|f\|_{X_{E_1}}\|g\|_{X_{E_2}}.\end{aligned}$$

Therefore $\|\tilde{\lambda}\|_{(X_{E_1},(X_{E_2},X_{E_3}))} \le \|\lambda\|_{(X_{E_1} \circledast X_{E_2}, X_{E_3})}$.

For the converse, take $\tilde{\lambda} \in (X_{E_1}, (X_{E_2}, X_{E_3}))$ and $h \in X_{E_1} \circledast X_{E_2}$. Assume that $h = \sum_n f_n *_\pi g_n$ with $\sum_n \|f_n\|_{X_{E_1}}\|g_n\|_{X_{E_2}} < \infty$. Hence

$$\begin{aligned}\|\lambda *_1 h\|_{X_{E_3}} &\le \sum_n \big\|\lambda *_1 (f_n *_\pi g_n)\big\|_{X_{E_3}} \\ &= \sum_n \big\|(\tilde{\lambda} *_2 f_n)\big\|_{(X_{E_2},X_{E_3})}\|g_n\|_{X_{E_2}} \\ &\le \sum_n \|\tilde{\lambda}\|_{(X_{E_1},(X_{E_2},X_{E_3}))}\|f_n\|_{X_{E_1}}\|g_n\|_{X_{E_2}} \\ &\le \|\tilde{\lambda}\|_{(X_{E_1},(X_{E_2},X_{E_3}))}\sum_n \|f_n\|_{X_{E_1}}\|g_n\|_{X_{E_2}},\end{aligned}$$

which gives $\|\lambda\|_{(X_{E_1} \circledast X_{E_2}, X_{E_3})} \le \|\tilde{\lambda}\|_{(X_{E_1},(X_{E_2},X_{E_3}))}$. □

5. Examples and applications

In this section we use Theorem 4.7 in both directions; that is, we compute multiplier spaces and Hadamard tensor products.

We first start with a characterization of $\mathcal{S}(E)$-solid spaces in terms of Hadamard tensor products.

Proposition 5.1. *Let X_E be admissible. Then $\ell^\infty \circledast_{B_0} X_E$ is the smallest $\mathcal{S}(E)$-solid space which contains X_E.*

In particular, X_E is $\mathcal{S}(E)$-solid if and only if $X_E = \ell^\infty \circledast_{B_0} X_E$.

Proof. Of course, $X_E \subseteq \ell^\infty \circledast_{B_0} X_E$, and $\ell^\infty \circledast_{B_0} X_E$ is solid (due to Remark 4.3).

Let Y_E be a solid space with $X_E \subset Y_E$. We shall see that $\ell^\infty \circledast_{B_0} X_E \subset Y_E$. Let $h \in \ell^\infty \circledast_{B_0} X_E$ be given by $h = \sum_n f_n * g_n$, where $f_n \in \ell^\infty$, $g_n \in X_E$, and $\sum_n \|f_n\|_\infty \|g_n\|_{X_E} < \infty$. Note that $f_n * g_n \in Y_E$ and $\|f_n * g_n\|_{Y_E} \leq \|f_n\|_\infty \|g_n\|_{Y_E}$ for each n because Y_E is solid. Hence

$$\sum_n \|f_n * g_n\|_{Y_E} \leq \sum_n \|f_n\|_\infty \|g_n\|_{Y_E} \leq C \sum_n \|f_n\|_\infty \|g_n\|_{X_E} < \infty,$$

and then $h \in Y_E$. □

Proposition 5.2. *Let* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\ell^p(E_1) \circledast \ell^q(E_2) = \ell^1(E_1 \hat{\otimes}_\pi E_2).$$

Proof. Let $f \in \ell^p(E_1)$ and $g \in \ell^q(E_2)$. Since $\widehat{f *_\pi g}(j) = \hat{f}(j) \otimes \hat{g}(j)$ and

$$\big\|\widehat{f *_\pi g}(j)\big\|_{E_1 \hat{\otimes}_\pi E_2} \leq \big\|\hat{f}(j)\big\|_{E_1} \big\|\hat{g}(j)\big\|_{E_2},$$

we have, using Hölder's inequality,

$$\|f *_\pi g\|_{\ell^1(E_1 \hat{\otimes}_\pi E_2)} \leq \|f\|_{\ell^p(E_1)} \|g\|_{\ell^q(E_2)}. \tag{5.1}$$

Let $h \in \ell^p(E_1) \circledast \ell^q(E_2)$. Let $\epsilon > 0$, and take $h = \sum_n f_n *_\pi g_n$ with $f_n \in \ell^p(E_1)$ and $g_n \in \ell^q(E_2)$ and $\sum_n \|f_n\|_{\ell^p(E_1)} \|g_n\|_{\ell^q(E_2)} \leq \|h\|_{B_\pi} + \epsilon$.

From (5.1) we have that $h = \sum_n f_n *_\pi g_n$ converges in $\ell^1(E_1 \hat{\otimes}_\pi E_2)$ and $\|h\|_{\ell^1(E_1 \hat{\otimes}_\pi E_2)} \leq \|h\|_{B_\pi} + \epsilon$. This implies that $\ell^p(E_1) \circledast \ell^q(E_2) \subseteq \ell^1(E_1 \hat{\otimes}_\pi E_2)$ with inclusion of norm 1.

Take now $h \in \ell^1(E_1 \hat{\otimes}_\pi E_2)$. In particular, for each $j \geq 0$ and $\epsilon > 0$, there exists $x_n^j \in E_1$ and $y_n^j \in E_2$ such that $\hat{h}(j) = \sum_n x_n^j \otimes y_n^j$ and

$$\sum_n \|x_n^j\|_{E_1} \|y_n^j\|_{E_2} < \big\|\hat{h}(j)\big\|_{E_1 \hat{\otimes}_\pi E_2} + \frac{\epsilon}{2^j}.$$

Define F_n and G_n by the formulas

$$\hat{F}_n(j) = \big(\|x_n^j\|_{E_1} \|y_n^j\|_{E_2}\big)^{1/p} \frac{x_j^n}{\|x_j^n\|_{E_1}}, \qquad \hat{G}_n(j) = \big(\|x_n^j\|_{E_1} \|y_n^j\|_{E_2}\big)^{1/q} \frac{y_j^n}{\|y_j^n\|_{E_2}}.$$

Note that

$$\|F_n\|_{\ell^p(E_1)} = \Big(\sum_j \|x_n^j\|_{E_1} \|y_n^j\|_{E_2}\Big)^{1/p}, \qquad \|G_n\|_{\ell^q(E_2)} = \Big(\sum_j \|x_n^j\|_{E_1} \|y_n^j\|_{E_2}\Big)^{1/q},$$

and

$$\sum_n \|F_n\|_{\ell^p(E_1)} \|G_n\|_{\ell^q(E_2)} = \sum_{n,j} \|x_n^j\|_{E_1} \|y_n^j\|_{E_2} \leq \|h\|_{\ell^1(E_1 \hat{\otimes}_\pi E_2)} + \epsilon.$$

In such a way we have $h = \sum_n F_n *_\pi G_n \in \ell^p(E_1) \circledast \ell^q(E_2)$ with $\|h\|_{B_\pi} \leq \|h\|_{\ell^1(E_1 \hat{\otimes}_\pi E_2)}$. □

To analyze the other values of p, we shall make use of the following result of multipliers (see [1, Proposition 2.2]):

$$\big(\ell^{p_1}(E_1), \ell^{p_2}(E_2)\big) = \ell^{p_3}\big(\mathcal{L}(E_1, E_2)\big), \tag{5.2}$$

where $0 < \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} < 1$.

Proposition 5.3. *Let* $1 \le p, q \le \infty$ *with* $0 < \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$. *Then*

$$\ell^p(E_1) \circledast \ell^q(E_2) = \ell^r(E_1 \hat{\otimes}_\pi E_2).$$

Proof. Note that the same argument as in Proposition 5.2 gives $\ell^p(E_1) \circledast \ell^q(E_2) \subseteq \ell^r(E_1 \hat{\otimes}_\pi E_2)$ with inclusion of norm 1.

Indeed, as above, if $f \in \ell^p(E_1)$ and $g \in \ell^q(E_2)$, then

$$\big\|\widehat{f *_\pi g}(j)\big\|_{E_1 \hat{\otimes}_\pi E_2} \le \big\|\hat{f}(j)\big\|_{E_1} \big\|\hat{g}(j)\big\|_{E_2}.$$

Hence

$$\|f *_\pi g\|_{\ell^r(E_1 \hat{\otimes}_\pi E_2)} \le \|f\|_{\ell^p(E_1)} \|g\|_{\ell^q(E_2)}. \tag{5.3}$$

For a general $h = \sum_n f_n *_\pi g_n \in \ell^p(E_1) \circledast \ell^q(E_2)$, where f_n, g_n are chosen such that $f_n \in \ell^p(E_1)$ and $g_n \in \ell^q(E_2)$ and $\sum_n \|f_n\|_{\ell^p(E_1)} \|g_n\|_{\ell^q(E_2)} \le \|h\|_{B_\pi} + \epsilon$, we have from (5.3) that $\sum_n \|f_n *_\pi g_n\|_{\ell^r(E_1 \hat{\otimes}_\pi E_2)} < \infty$. Then $h = \sum_n f_n *_\pi g_n$ converges in $\ell^r(E_1 \hat{\otimes}_\pi E_2)$ and $\|h\|_{\ell^r(E_1 \hat{\otimes}_\pi E_2)} \le \|h\|_{B_\pi} + \epsilon$.

To see that they coincide it suffices to show that $(\ell^p(E_1) \circledast \ell^q(E_2))' = (\ell^r(E_1 \hat{\otimes}_\pi E_2))'$. It is well known that, for $\frac{1}{r'} = 1 - \frac{1}{r}$,

$$\big(\ell^r(E_1 \hat{\otimes}_\pi E_2)\big)' = \ell^{r'}\big(\mathcal{L}(E_1, E_2')\big).$$

On the other hand, using Proposition 4.6 and (5.2) we have

$$\big(\ell^p(E_1) \circledast \ell^q(E_2)\big)' = \big(\ell^p(E_1), \ell^{q'}(E_2')\big) = \ell^{r'}\big(\mathcal{L}(E_1, E_2')\big),$$

where $\frac{1}{q'} = 1 - \frac{1}{q}$. $\square$

We now compute the Hadamard tensor product in some particular cases of spaces of analytic functions. We shall analyze the case H^1 and $H^1(\mathbb{D}, E)$, at least for particular Banach spaces E, following the ideas developed in [7].

We need some notions and lemmas before the statement of the result. Given an E-valued analytic function, $F(z) = \sum_{j=0}^\infty x_j z^j$, we define

$$DF(z) = \sum_{j=0}^\infty (j+1) x_j z^j.$$

Lemma 5.4. *Let E be a complex Banach space,* $1 \le p \le \infty$.

(i) *There exist* $A_1, A_2 > 0$ *such that*

$$A_1 r^m \|f\|_{H^p(\mathbb{D},E)} \le M_p(f, r) \le A_2 r^n \|f\|_{H^p(\mathbb{D},E)}, \quad 0 < r < 1,$$

for $f \in \mathcal{P}(E)$ *given by* $f(z) = \sum_{j=n}^m x_j z^j$, $x_j \in E$, $n, m \in \mathbb{N}$, *and where* $M_p(f, r) = (\int_0^1 \|f(re^{it})\|^p \frac{dt}{2\pi})^{1/p}$.

(ii) *If* $P(z) = \sum_{k=2^{n-1}}^{2^{n+1}} \hat{P}(k) z^k$, $\hat{P}(k) \in \mathbb{C}$, *then there exist constants* B_1 *and* B_2 *such that*

$$B_1 2^n \|P *_{B_0} f\|_{H^p(\mathbb{D},E)} \le \|P *_{B_0} Df\|_{H^p(\mathbb{D},E)} \le B_2 2^n \|P *_{B_0} f\|_{H^p(\mathbb{D},E)} \tag{5.4}$$

for any $f \in H^p(\mathbb{D}, E)$.

Proof. It is well known (see Lemma 3.1 [19]) that

$$r^m \|\phi\|_\infty \le M_\infty(\phi, r) \le r^n \|\phi\|_\infty, \quad 0 < r < 1,$$

for each scalar-valued polynomial $\phi(z) = \sum_{j=n}^m \alpha_j z^j$, where $\|\phi\|_\infty = \sup_{|z|=1} |\phi(z)|$ and $M_\infty(\phi, r) = \sup_{|z|=1} |\phi(rz)|$.

This allows us to conclude, composing with elements in the unit ball of the dual space,

$$r^m \|F\|_\infty \le M_\infty(F, r) \le r^n \|F\|_\infty, \quad 0 < r < 1,$$

for any $F(z) = \sum_{j=n}^m y_j z^j$ where $y_j \in Y$ and where Y is a complex Banach space.

For $f : \mathbb{D} \to \mathbb{K}$ an analytic function, define f_w to be $f_w(z) = f(wz)$. Now select $Y = H^p(\mathbb{D}, E)$ and $F(z) = f_z$; that is to say

$$F(z)(w) = \sum_{j=n}^m x_j w^j z^j.$$

Using that

$$\|F\|_\infty = \sup_{|z|=1} \|f_z\|_{H^p(\mathbb{D},E)} = \|f\|_{H^p(\mathbb{D},E)}$$

and $M_\infty(F, r) = M_p(f, r)$, we obtain the result.

To see (ii) we first use [7, Lemma 7.2], which guarantees the existence of constants B_1, B_2 such that

$$B_1 2^n \|P *_{B_0} \phi\|_\infty \le \|P *_{B_0} D\phi\|_\infty \le B_2 2^n \|P *_{B_0} \phi\|_\infty$$

for any $\phi \in H^\infty(\mathbb{D})$. Now apply the same argument as above to extend it to $H^p(\mathbb{D}, E)$. $\square$

Theorem 5.5. *Let* $\mathfrak{B}^1(\mathbb{D}, E)$ *denote the space of* E*-valued analytic functions* $F(z) = \sum_{j=0} x_j z^j$ *such that* $DF(z) \in A^1(\mathbb{D}, E)$ *with the norm given by*

$$\|F\|_{\mathfrak{B}^1(\mathbb{D},E)} = \big\|F(0)\big\|_E + \int_{\mathbb{D}} \big\|DF(z)\big\|_E \, dA(z).$$

Let $E = L^p(\mu)$ *for any measure* μ *and* $1 \le p \le 2$. *Then*

$$\big(H^1(\mathbb{D}) \circledast_{B_0} H^1\big(\mathbb{D}, L^p(\mu)\big)\big) = \mathfrak{B}^1\big(\mathbb{D}, L^p(\mu)\big).$$

Proof. Let us first show that $\mathfrak{B}^1(\mathbb{D}, E) \subseteq (H^1(\mathbb{D}) \circledast_{B_0} H^1(\mathbb{D}, E))$ for any Banach space E. We argue similarly to [7, Theorem 7.1].

Let $\{W_n\}_0^\infty$ be a sequence of polynomials such that

$$\operatorname{supp}(\hat{W}_n) \subset [2^{n-1}, 2^{n+1}] \quad (n \ge 1), \qquad \operatorname{supp}(\hat{W}_0) \subset [0, 1], \qquad \sup_n \|W_n\|_1 < \infty,$$

and

$$g=\sum_{n=0}^{\infty} W_n *_{B_0} g, \quad g\in\mathcal{H}(\mathbb{D},E).$$

Let $f\in\mathfrak{B}^1(\mathbb{D},E)$. Note that

$$\big\|(W_n *_{B_0} f)_r\big\|_{H^1(\mathbb{D},E)} \le \|W_n\|_1\|f_r\|_{H^1(\mathbb{D},E)} \le C\|f\|_{H^1(\mathbb{D},E)}.$$

Hence, $\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)} \le C\|f\|_{H^1(\mathbb{D},E)}$.

Denoting $Q_n=W_{n-1}+W_n+W_{n+1}$ we can write

$$f=\sum_{n=0}^{\infty} Q_n *_{B_0} W_n *_{B_0} f.$$

Note now that Lemma 5.4 allows us to conclude that

$$\begin{aligned}
\sum_{n=0}^{\infty}\|Q_n\|_1\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)} &\le K\sum_{n=0}^{\infty}\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)}\\
&\le K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} 2^n r^{2^n}\|W_n *_{B_0} f\|_{H^1(\mathbb{D},E)}\,dr\\
&\le K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} r^{2^n}\|W_n *_{B_0} Df\|_{H^1(\mathbb{D},E)}\,dr\\
&\le K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} M_1(W_n *_{B_0} Df,r)\,dr\\
&\le K\sum_{n=0}^{\infty}\int_{1-2^{-n}}^{1-2^{-(n+1)}} M_1(Df,r)\,dr\\
&= K\int_0^1 M_1(Df,r)\,dr\\
&\le K\|f\|_{\mathfrak{B}^1(\mathbb{D},E)}.
\end{aligned}$$

To show the other inclusion between these spaces we shall use that $E=L^p(\mu)$ for $1\le p\le 2$ satisfies the following vector-valued extension of a Hardy–Littlewood theorem (see [14]):

$$\Big[\int_0^1(1-r)M_1^2(Df,r)\,dr\Big]^{1/2}\le A\|f\|_{H^1(\mathbb{D},E)}, \tag{5.5}$$

for some constant $A>0$ (see [6, Definition 3.5, Proposition 4.4]).

It suffices to see that $\phi *_{B_0} g\in\mathfrak{B}^1(\mathbb{D},L^p(\mu))$ for each $\phi\in H^1(\mathbb{D})$ and $g\in H^1(\mathbb{D},L^p(\mu))$. Now taking into account that $D^2(\phi *_{B_0} g)=D\phi *_{B_0} Dg$ and

$$rD(\phi *_{B_0} g)(re^{it})=\sum_{j=0}^{\infty}(j+1)\hat{\phi}(j)\hat{g}(j)r^{j+1}e^{itj}=\int_0^r D^2(\phi *_{B_0} g)(se^{it})\,ds,$$

we have

$$\begin{aligned}\int_0^1 M_1\big(D(\phi *_{B_0} g), r\big) r\,dr &\le \int_0^1 \Big[\int_0^r M_1\big(D^2(\phi *_{B_0} g), s\big)\,ds\Big] r\,dr \\ &= \int_0^1 (1-s) M_1\big(D^2(\phi *_{B_0} g), s\big)\,ds \\ &\le 2\int_0^1 (1-r^2) M_1(r, D\phi) M_1(Dg, r) r\,dr.\end{aligned}$$

Now, from Cauchy–Schwarz and (5.5), we obtain

$$\begin{aligned}\int_0^1 (1-r^2) M_1(D\phi, r) M_1(Dg, r) r\,dr &\le \Big[\int_0^1 (1-r^2) M_1^2(D\phi, r) r\,dr\Big]^{1/2} \\ &\quad \cdot \Big[\int_0^1 (1-r^2) M_1^2(Dg, r) r\,dr\Big]^{1/2} \\ &\le K\|\phi\|_{H^1}\|g\|_{H^1(\mathbb{D}, L^p(\mu))}. \qquad \square\end{aligned}$$

It is known, by Fefferman's duality result, that $(H^1)' = \mathrm{BMOA}$ (see [12], [23]). In the vector-valued case, using L^p as an unconditional martingale difference space for $1 < p < \infty$, we have

$$\big(H^1\big(\mathbb{T}, L^p(\mu)\big)\big)' = \mathrm{BMOA}\big(\mathbb{T}, L^{p'}(\mu)\big), \quad 1 < p < \infty$$

(see [4]). It is also well known that $(\mathfrak{B}^1)' = \mathcal{B}\mathrm{loch}$ (see [3]) and that, for the vector-valued case, $(\mathfrak{B}^1(\mathbb{D}, E))' = \mathcal{B}\mathrm{loch}(\mathbb{D}, E')$ for any complex Banach space E (see [5, Corollary 2.1]) under the pairing

$$\langle F, G\rangle = \int_{\mathbb{D}} \big\langle DF(z), G(z)\big\rangle\,dA(z).$$

Using now Proposition 4.6, we recover the following result.

Corollary 5.6 ([6, Corollary 8.4])**.** *Let $1 \le p_1 \le 2$ and $2 \le p_2 < \infty$. Then we have*

$$\big(H^1(\mathbb{T}, L^{p_1}), \mathrm{BMOA}(\mathbb{T})\big)_{B_{\mathcal{L}}} = \mathcal{B}\mathrm{loch}\big(\mathbb{D}, \mathcal{L}(L^{p_1'}, L^{p_1'})\big);$$
$$\big(H^1(\mathbb{T}), \mathrm{BMOA}(\mathbb{T}, L^{p_2})\big)_{B_{\mathcal{L}}} = \mathcal{B}\mathrm{loch}\big(\mathbb{D}, \mathcal{L}(L^{p_2}, L^{p_2})\big).$$

Acknowledgments. The work of both authors was partially supported by MECC Spain projects BMF2011-231674, MTM2011-23164, and MTM2014-53009-P. Carme Zaragoza-Berzosa's work was also supported by the predoctoral grant VALi+d from the Generalitat Valenciana (expedient no. ACIF/2012/107).

References

1. J. L. Arregui and O. Blasco, *Bergman and Bloch spaces of vector-valued functions*, Math. Nach. **261/262** (2003), 3–22. MR2020384. DOI 10.1002/mana.200310109. 77, 91
2. J. L. Arregui and O. Blasco, *(p,q)-summing sequences of operators*, Quaest. Math. **26** (2003), no. 4, 441–452. MR2046149. DOI 10.2989/16073600309486074. 72, 83
3. J. M. Anderson, J. Clunie, and C. Pommerenke, *On Bloch functions and normal functions*, J. Reine Anew. Math. **270** (1974), 12–37. Zbl 0292.30030. MR0361090. 81, 94

4. O. Blasco, *Hardy spaces of vector-valued functions: Duality*, Trans. Amer. Math. Soc. **308** (1988), 495–507. Zbl 0658.46021. MR0951618. DOI 10.2307/2001088. 94
5. O. Blasco, "Vector-valued Hardy spaces from operator theory" in *Analysis at Urbana, Vol. II (Urbana, Ill., 1986–1987)*, London Math. Soc. Lecture Note Ser. **138**, Cambridge Univ. Press, Cambridge, 1989, 52–60. MR1009185. 94
6. O. Blasco, *Vector-valued analytic functions of bounded mean oscillation and geometry of Banach spaces*, Illinois J. Math. **41** (1997), no. 4, 532–558. Zbl 0915.46009. MR1468865. 72, 73, 93, 94
7. O. Blasco and M. Pavlović, *Coefficient multipliers on Banach spaces of analytic functions*, Rev. Mat. Iberoam. **27** (2011), no. 2, 415–447. Zbl 1235.42004. MR2848526. DOI 10.4171/RMI/642. 72, 91, 92
8. Q. Bu and J. Diestel, *Observations about the projective tensor product of Banach spaces, I: $\ell_p \hat{\otimes} X$*, Quaest. Math. (4) **24** (2001), no. 4, 519–533. Zbl 1038.46061. MR1871895. DOI 10.1080/16073606.2001.9639238. 83
9. J. Diestel, H. Jarchow, and A. Tongue, *Absolutely Summing Operators*, Cambridge Univ. Press, 1995. MR1342297. DOI 10.1017/CBO9780511526138. 76, 81
10. J. Diestel and J. J. Uhl, *Vector Measures*, Amer. Math. Soc., Providence, 1977. MR0453964. 73
11. P. L. Duren, *Theory of H^p Spaces*, Pure Appl. Math. **38**, Academic Press, New York, 1970. MR0268655. 72, 78
12. C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), nos. 3–4, 137–193. MR0447953. 94
13. J. H. Fourie and I. Röntgen, *Banach space sequences and projective tensor products*, J. Math. Anal. Appl. **277** (2003), no. 2, 629–644. Zbl 1043.46053. MR1961250. DOI 10.1016/S0022-247X(02)00635-2. 83
14. G. H. Hardy and J. E. Littlewood, *Theorems concerning mean values of analytic or harmonic functions*, Quart. J. Math. (Oxford Ser.) **12** (1941), 221–256. MR0006581. 93
15. H. Hendenmaln, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Grad. Texts in Math. **199**, Springer, New York, 2000. MR1758653. DOI 10.1007/978-1-4612-0497-8. 72
16. M. Jevtić and I. Jovanović, *Coefficient multipliers of mixed norm spaces*, Canad. Math. Bull. **36** (1993), no. 3, 283–285. Zbl 0801.30002. MR1245812. DOI 10.4153/CMB-1993-040-2. 71
17. M. Jevtić and J. Pavlović, *Coefficient multipliers on spaces of analytic functions*, Acta Sci. Math. **64** (1998), nos. 3–4, 531–545. Zbl 0926.46022. MR1666039. 71
18. M. Jevtić, D. Vukotić, and M. Arsenović, *Taylor coefficients and coefficient multipliers of Hardy and Bergman-type spaces*, in preparation. 71
19. M. Mateljević and M. Pavlović, *L^p-behaviour of the integral means of analytic functions*, Studia Math. **77** (1984), no. 3, 219–237. MR0745278. 92
20. B. Osikiewicz, *Multipliers of Hardy spaces*, Quaest. Math. **27** (2004), no. 1, 57–73. Zbl 1065.42011. MR2060908. DOI 10.2989/16073600409486084. 71
21. M. Pavlović, *Function Classes on the Disk: An Introduction*, De Gruyter Stud. Math. **52**, De Gruyter, Berlin, 2013. MR3154590. 72
22. R. A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer Monogr. Math., Springer, London, 2002. MR1888309. DOI 10.1007/978-1-4471-3903-4. 73
23. E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. **43**, Princeton Univ. Press, Princeton, 1993. MR1232192. 94
24. K. Zhu, *Operator Theory in Function Spaces*, Dekker, New York, 1990. MR1074007. 72

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