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DIRECT LIMIT TOPOLOGIES OF QUASI-UNIFORM SPACES AND PARATOPOLOGICAL GROUPS

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Abstract. Given an increasing sequence $(X_n)_{n\in\omega}$ of quasi-uniform spaces and paratopological groups, we study the topology of the direct limits qu-$\varinjlim X_n$ and pg-$\varinjlim X_n$ of the sequence $(X_n)_{n\in\omega}$ in the categories of quasi-uniform spaces and paratopological groups, respectively. First, we prove that the quasi-uniformity of the quasi-uniform direct limit qu-$\varinjlim X_n$ is generated by some special family of quasi-pseudometrics. Then we discuss some properties of the direct limits pg-$\varinjlim X_n$. Finally, we give an explicit description of the topology of the direct limit pg-$\varinjlim X_n$ under certain conditions on the sequence of paratopological groups $(X_n)_{n\in\omega}$. Moreover, some questions about direct limits of qu-$\varinjlim X_n$ and pg-$\varinjlim X_n$ are posed.

1. Introduction

The concept of direct limit is a basic one in functional analysis, and it has become widely used in algebra, topology, and other areas of mathematics, such as algebraic geometry and complex analysis, including the fundamental notion that a stalk of a sheaf uses direct limits. An important special case of the direct limit is

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the direct limit of a directed family of mathematical structures of the same type. The problem of explicitly describing the topological structure of the direct limit has been discussed in [2], [3], [5], [8], [9], [10], and [13]. In this paper, we mainly discuss the topologies on a tower of quasi-uniform spaces and paratopological groups.

A *semitopological group* G is a group G with a topology such that the product map of $G \times G$ into G is separately continuous. If G is a semitopological group and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous, then G is called a *quasitopological group*. A *paratopological group* G is a group G with a topology such that the product map of $G \times G$ into G is jointly continuous. If G is a paratopological group and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous, then G is called a *topological group*. However, there exists a paratopological group which is not a topological group; the Sorgenfrey line (see [6, Example 1.2.2]) is such an example.

Given a tower

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n \subset \cdots$$

of topological spaces, the union $X = \bigcup_{n\in\omega} X_n$ endowed with the strongest topology making the inclusion maps $X_n \to X$, $n \in \omega$, continuous is called the *topological direct limit* of the tower $(X_n)_{n\in\omega}$ and is denoted by $\text{t-}\varinjlim X_n$.

By a *quasi-uniform space* we mean the natural analogue of a *uniform space* obtained by dropping the symmetry axiom. We shall denote the quasi-uniformity of a quasi-uniform space X by $\mathcal{QU}_X$. The direct limit $\text{qu-}\varinjlim X_n$ (resp., $\text{u-}\varinjlim X_n$) of a tower $(X_n)_{n\in\omega}$ of quasi-uniform spaces (resp., uniform spaces) is defined in a similar fashion as the countable union $X = \bigcup_{n\in\omega} X_n$ endowed with the strongest quasi-uniformity (resp., uniformity), making the inclusion maps $X_n \to X$, $n \in \omega$, quasi-uniformly continuous (resp., uniformly continuous).

The direct limit $\text{pg-}\varinjlim G_n$ (resp., $\text{g-}\varinjlim G_n$) of a tower $(G_n)_{n\in\omega}$ of paratopological groups (resp., topological groups) is defined as the countable union $G = \bigcup_{n\in\omega} G_n$ endowed with the strongest topology that turns G into a paratopological group (resp., topological group) and makes the inclusion maps $G_n \to G$, $n \in \omega$, continuous.

This paper is organized as follows.

In Section 3, we prove that the quasi-uniformity of the quasi-uniform direct limit $\text{qu-}\varinjlim X_n$ of the sequence $(X_n)_{n\in\omega}$ in the categories of quasi-uniform spaces is generated by some special family of quasi-pseudometrics. Moreover, for two towers $(X_n)_{n\in\omega}$ and $(Y_n)_{n\in\omega}$ of quasi-uniform spaces, we show that the topology of $\text{qu-}\varinjlim X_n \times \text{qu-}\varinjlim Y_n$ coincides with the topology of $\text{qu-}\varinjlim(X_n \times Y_n)$.

In Section 4, we discuss some properties of direct limits $\text{t-}\varinjlim G_n$ and $\text{pg-}\varinjlim G_n$ of the sequence $(G_n)_{n\in\omega}$ in the categories of paratopological groups. We show that if $\text{t-}\varinjlim G_n$ is a paratopological group, then $\text{t-}\varinjlim G_n^* = (\text{t-}\varinjlim G_n)^*$, where each G_n^* denotes the coarsest group topology on G_n which is finer than the original topology of G_n.

In Section 5, we give an explicit description of the topology of the direct limit $\text{pg-}\varinjlim G_n$ of the sequence $(G_n)_{n\in\omega}$ in the categories of paratopological groups under certain conditions on the sequence of paratopological groups $(G_n)_{n\in\omega}$. We prove that, under certain conditions on a tower of paratopological groups $(G_n)_{n\in\omega}$, the topology of the direct limit $\text{pg-}\varinjlim G_n$ coincides with one (or all) four simply described topologies on the group $G=\bigcup_{n\in\omega} G_n$. Moreover, we define the PPTA (passing through assumption in paratopological groups) property and other properties of the tower $(G_n)_{n\in\omega}$, guaranteeing that the topology of $\text{pg-}\varinjlim G_n$ coincides with some kind of topology.

2. Preliminaries

Let X be a set. The family of all subsets of $X\times X$ has an algebraic structure related to the operation

$$U\circ V=\big\{(x,z)\in X\times X : \text{there exists } y\in X \\ \text{such that } (x,y)\in U \text{ and } (y,z)\in V\big\}$$

for $U,V\subset X\times X$. Then the so-defined addition operation allows us to multiply subsets $U\subset X\times X$ by positive integers using the inductive formula $U=U^1$ and $U^{(n+1)}=U^n\circ U$ for $n>1$. Moreover, for a subset U of $X\times X$, denote

$$U^{-1}=\big\{(x,y) : (y,x)\in U\big\}.$$

For a sequence $(U_i)_{i\in\omega}$ of subsets of $X\times X$, put

$$\sum_{i\le n} U_i=U_0\circ\cdots\circ U_n$$

and

$$\sum_{i\in\omega} U_i=\bigcup_{n\in\omega}\sum_{i\le n} U_i.$$

For a point $x\in X$, a subset $A\subset X$, and $U\subset X\times X$, let

$$B(x,U)=\big\{y\in X : (x,y)\in U\big\}$$

and

$$B(A,U)=\bigcup_{x\in A} B(x,U)$$

be the U-ball around x and A, respectively.

Definition 2.1. A *quasi-uniformity* on a set X is a filter $\mathscr{U}$ on $X\times X$ such that (a) each member of $\mathscr{U}$ contains the diagonal of $X\times X$ and (b) if $U\in\mathscr{U}$, then $V\circ V\subset U$ for some $V\in\mathscr{U}$. The pair $(X,\mathscr{U})$ is called a *quasi-uniform space* and the members of $\mathscr{U}$ are called *entourages*.

By a tower of quasi-uniform spaces we shall understand any increasing sequence

$$X_0\subset X_1\subset X_2\subset\cdots\subset X_n\subset\cdots$$

of quasi-uniform spaces.

Definition 2.2. A function $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is called *quasi-uniformly continuous* (resp., *uniformly continuous*) if, for each $V \in \mathscr{V}$, there exists an $U \in \mathscr{U}$ such that $(f(x), f(y)) \in V$ whenever $(x, y) \in U$, where $\mathscr{U}$ and $\mathscr{V}$ are quasi-uniformities (resp., *uniformities*) for X and Y, respectively.

Definition 2.3. A *quasi-pseudometric* d on a set X is a function from $X \times X$ into the set of nonnegative real numbers such that, for $x, y, z \in X$: (a) $d(x, x) = 0$ and (b) $d(x, y) \leq d(x, z) + d(z, y)$.

Every quasi-pseudometric d on X generates a topology $\mathscr{F}(d)$ on X which has a base of the family of d-balls $\{B_d(x, r) : x \in X, r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$.

We denote by ω the first countable order and by $\mathbb{N}$ the set of all natural numbers. The letter e denotes the neutral element of a group. The readers may consult [1], [4] and [6] for notations and terminology not explicitly given here.

3. Topological direct limit of quasi-uniform spaces

A quasi-pseudometric on a quasi-uniform space X is called *quasi-uniform* if, for each $\varepsilon > 0$, the set

$$\{d < \varepsilon\} = \big\{(x, y) \in X \times X : d(x, y) < \varepsilon\big\}$$

belongs to the quasi-uniformity of X. By [7, p. 3], the family of $\mathcal{QPM}_X$ of all quasi-uniform quasi-pseudometrics on a quasi-uniform space X generates the quasi-uniformity $\mathcal{QU}_X$ of X in the sense that the sets $\{d < 1\}$, $d \in \mathcal{QPM}_X$, form a base of the quasi-uniformity $\mathcal{QU}_X$.

It follows from the definition of quasi-uniform direct limit that we have the following theorem.

Theorem 3.1. *Let $(X_n)_{n\in\omega}$ be a tower of quasi-uniform spaces. Then a quasi-pseudometric d on the quasi-uniform space* $\text{qu-}\varinjlim X_n$ *is quasi-uniform if and only if for each $n \in \omega$ the restriction $d|_{X_n \times X_n}$ is a quasi-uniform quasi-pseudometric on X_n.*

For a tower $(X_n)_{n\in\omega}$ of sets and points $x, y \in X = \bigcup_{n\in\omega} X_n$, let

$$|x| = \min\{n \in \omega : x \in X_n\}$$

and

$$|x, y| = \max\big\{|x|, |y|\big\}.$$

The $|x|$ is said to be the *height* of the point x in X.

Definition 3.2. Let $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{QPM}_{X_n}$. Then we say that a sequence of quasi-pseudometrics $(d_n)_{n\in\omega}$ is *monotone* if $d_n \leq d_{n+1}|X_n \times X_n$ for any $n \in \omega$. By the direct limit $d_\infty = \varinjlim d_n$ of a monotone sequence of quasi-pseudometrics $(d_n)_{n\in\omega}$, we understand the quasi-pseudometric on $X = \bigcup_{n\in\omega} X_n$ defined by the following formula:

$$d_\infty(x, y) = \varinjlim d_n(x, y) = \inf\Big\{\sum_{i=1}^{n} d_{|x_{i-1}, x_i|}(x_{i-1}, x_i) : x = x_0, x_1, \dots, x_n = y\Big\}$$

on X.

Lemma 3.3. *Let $(X_n)_{n\in\omega}$ be a tower of quasi-uniform spaces, and let $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{QPM}_{X_n}$ be monotone. For arbitrary points $x, y \in X$ and $\varepsilon > 0$, there exists a chain of points $x = x_0, x_1, \dots, x_n = y$ such that*

$$\sum_{i=1}^{n} d_{|x_{i-1},x_i|}(x_{i-1}, x_i) < \varinjlim d_n(x, y) + \varepsilon \tag{1}$$

and $|x_i| < \max\{|x_{i-1}|, |x_i|\}$ for each $0 < i < n$. The latter condition implies that $|x_0| > |x_1| > \cdots > |x_k| \leq |x_{k+1}| < \cdots < |x_n|$ for some $0 \leq k < n$.

Proof. Let $x = x_0, x_1, \dots, x_n = y$ be a sequence satisfying (1) and having the smallest possible length n. Then the sequence $x = x_0, x_1, \dots, x_n = y$ has the desired property.

Indeed, suppose that $|x_i| \geq \max\{|x_{i-1}|, |x_i|\}$ for some $0 < i < n$. Since

$$\begin{aligned} d_{|x_{i-1},x_i|}(x_{i-1}, x_i) + d_{|x_i,x_{i+1}|}(x_i, x_{i+1}) &= d_{|x_i|}(x_{i-1}, x_i) + d_{|x_i|}(x_i, x_{i+1}) \\ &\geq d_{|x_i|}(x_{i-1}, x_{i+1}) \\ &\geq d_{|x_{i-1},x_{i+1}|}(x_{i-1}, x_{i+1}), \end{aligned}$$

we can delete the point x_i from the sequence $x_0, x_1, \dots, x_n$, which will not enlarge the sum in (1). However, it will diminish the length of the sequence, which is a contradiction.

Let k be the smallest number such that $|x_k| = \min_{i\leq n} |x_i|$. Then we have $|x_0| > |x_1| > \cdots > |x_k| \leq |x_{k+1}| < \cdots < |x_n|$. □

Lemma 3.4. *Let $(X_n)_{n\in\omega}$ be a tower of quasi-uniform spaces. For an arbitrary monotone sequence of quasi-pseudometrics $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{QPM}_{X_n}$, the direct limit quasi-pseudometrics $\varinjlim d_n$ on* qu-$\varinjlim X_n$ *is quasi-uniform.*

Proof. Let $d_\infty = \varinjlim d_n$. By Theorem 3.1, it is easy to see that the quasi-uniformity of the quasi-pseudometric d_∞ is equivalent to the quasi-uniform continuity of the identity map qu-$\varinjlim X_n \longrightarrow (X, d_\infty)$. It follows from the definition of quasi-uniform direct limit qu-$\varinjlim X_n$ that the quasi-uniform continuity of the identity map qu-$\varinjlim X_n \longrightarrow (X, d_\infty)$ is equivalent to the quasi-uniform continuity of the inclusion embedding $X_n \longrightarrow (X, d_\infty)$ for each $n \in \omega$. For each $n \in \omega$, since d_n is a quasi-uniform quasi-pseudometric and $(d_n)_{n\in\omega}$ is monotone, the inclusion embedding $X_n \longrightarrow (X, d_\infty)$ is a quasi-uniform continuity. □

Lemma 3.5. *Let $(X, \mathcal{QU}_X)$ be a quasi-uniform space, and let M be a subset of X. Then every bounded quasi-pseudometric ρ on the set M, which is quasi-uniform with respect to $\mathcal{QU}_M$, is extendable to a bounded quasi-pseudometric σ on the set X, which is quasi-uniform with respect to $\mathcal{QU}_X$.*

Proof. Without loss of generality, we may assume that $\rho < \frac{1}{2}$ for all $x, y \in M$. For each $i \in \mathbb{N}$, take a $V_i \in \mathcal{QU}_X$ such that

$$V_i \cap (M \times M) \subset \Big\{(x, y) \in M \times M : \rho(x, y) < \frac{1}{2^i}\Big\},$$

and take a quasi-uniform quasi-pseudometric σ_i on the set X bounded by 1 such that

$$\Big\{(x,y) \in X \times X : \sigma_i(x,y) < \frac{1}{4}\Big\} \subset V_i$$

(see [7, p. 3] or [11]). Put

$$\sigma'(x,y) = 8\sum_{i=1}^{\infty} \frac{1}{2^i}\sigma_i(x,y).$$

Then it is easy to see that σ' is a quasi-pseudometric on X, which is quasi-uniform with respect to $\mathcal{QU}_X$. Next we shall show that $\rho(x,y) \leq \sigma'(x,y)$ for all $x,y \in M$. Indeed, for each $x,y \in M$, if $\rho(x,y) = 0$, then it is obvious. Hence we may assume that $\rho(x,y) \neq 0$. Then there exists an $i \in \mathbb{N}$ such that $\frac{1}{2^{i+1}} \leq \rho(x,y) < \frac{1}{2^i}$; hence $(x,y) \notin V_{i+1}$, which implies that $\sigma_{i+1}(x,y) \geq \frac{1}{4}$. Therefore, $\sigma'(x,y) \geq 8 \times \frac{1}{2^{i+1}}\sigma_{i+1}(x,y) \geq 8 \times \frac{1}{2^{i+1}} \times \frac{1}{4} = \frac{1}{2^i} > \rho(x,y)$.

For $x,y \in X$, let

$$\sigma''(x,y) = \inf\big\{\sigma'(x,a) + \rho(a,b) + \sigma'(b,y) : a,b \in M\big\}$$

and

$$\sigma(x,y) = \min\big\{\sigma'(x,y), \sigma''(x,y)\big\}.$$

Since σ' is a quasi-uniform quasi-pseudometric on X, σ is a quasi-uniform quasi-pseudometric σ on the set X. Finally, we shall check $\sigma|_{M\times M} = \rho$. Indeed, it suffices to check that $\sigma'' = \rho$ on M. Fix arbitrary $x,y \in M$. Then we have

$$\sigma''(x,y) \leq \sigma'(x,x) + \rho(x,y) + \sigma'(y,y) = \rho(x,y).$$

Moreover, for arbitrary $a,b \in M$, we have

$$\begin{aligned}\rho(x,y) &\leq \rho(x,a) + \rho(a,b) + \rho(b,y)\\ &\leq \sigma'(x,a) + \rho(a,b) + \sigma'(b,y),\end{aligned}$$

which implies that $\rho(x,y) \leq \sigma''(x,y)$. Therefore, $\sigma'' = \rho$ on M. $\square$

Theorem 3.6. *The quasi-uniformity of the quasi-uniform direct limit* qu-$\varinjlim X_n$ *of a tower of quasi-uniform spaces* $(X_n)_{n\in\omega}$ *is generated by the family of quasi-pseudometrics*

$$\Big\{\varinjlim d_n : (d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{QPM}_{X_n} \textit{ is monotone}\Big\}.$$

Proof. Let $U \in \mathcal{QPM}_X$ be an entourage of the diagonal of the quasi-uniform space qu-$\varinjlim X_n$. By Lemma 3.4, for each monotone sequence of quasi-pseudometrics $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{QPM}_{X_n}$, the direct limit quasi-pseudometric $\varinjlim d_n$ on qu-$\varinjlim X_n$ is quasi-uniform. Therefore, we need to find a monotone sequence of quasi-pseudometrics $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{QPM}_{X_n}$ such that $\{\varinjlim d_n < 1\} \subset U$.

Choose a sequence of entourages $(U_n)_{n\in\omega} \in (\mathcal{QPM}_X)^\omega$ such that $U_0^4 \subset U$ and $U_{n+1}^2 \subset U_n$ for each $n \in \omega$.

Claim 1. *There exists a monotone sequence of quasi-pseudometrics* $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{QPM}_{X_n}$ *such that* $\{d_n < 1\} \subset U_n$ *for each* $n \in \omega$.

Indeed, it follows from [7, p. 3] or [11] that, for each $k \in \omega$, there is a bounded quasi-uniform quasi-pseudometric ρ_k on X_k such that $\{\rho_k < 1\} \subset U_k$. By Lemma 3.5, for each $n \geq k$, the quasi-pseudometric ρ_k can be extended to a quasi-uniform quasi-pseudometric $\widetilde{\rho}_{k,n}$ on the quasi-uniform space $X_n \supset X_k$. For each $n \in \omega$, let $d_n = \sum_{k\leq n} \widetilde{\rho}_{k,n}$. Then $(d_n)_{n\in\omega}$ is a required monotone sequence of quasi-uniform quasi-pseudometrics.

By Claim 1, there exists a monotone sequence of quasi-pseudometrics $(d_n)_{n\in\omega} \in \prod_{n\in\omega} \mathcal{QPM}_{X_n}$ such that $\{d_n < 1\} \subset U_n$ for each $n \in \omega$. Let $d_\infty = \underrightarrow{\lim}\, d_n$. Then we have $\{d_\infty < 1\} \subset U$.

Indeed, take any points $x, y \in X$ with $d_\infty(x, y) < 1$. By Lemma 3.3, there exists a sequence of points $x = x_0, x_1, \ldots, x_n = y$ such that

$$\sum_{i=1}^{n} d_{|x_{i-1},x_i|}(x_{i-1}, x_i) < 1$$

and

$$|x_0| > |x_1| > \cdots > |x_k| \leq |x_{k+1}| < \cdots < |x_n|$$

for some $0 \leq k < n$. Note that, for each $i \leq k$, we get $|x_{i-1}| \geq |x_i|$ and then

$$d_{|x_{i-1}|}(x_{i-1}, x_i) = d_{|x_{i-1},x_i|}(x_{i-1}, x_i) < 1.$$

Therefore, the choice of the quasi-pseudometric $d_{|x_{i-1}|}$ guarantees that $(x_{i-1}, x_i) \in \{d_{|x_{i-1}|} < 1\} \subset U_{|x_{i-1}|}$.

For $i > k$, since $|x_{i-1}| \leq |x_i|$, we have

$$d_{|x_i|}(x_{i-1}, x_i) = d_{|x_{i-1},x_i|}(x_{i-1}, x_i) < 1,$$

which implies that $(x_{i-1}, x_i) \in U_{|x_i|}$. It follows from

$$|x_0| > |x_1| > \cdots > |x_k| \leq |x_{k+1}| < \cdots < |x_n|$$

that

$$\begin{aligned}(x, y) &\in U_{|x_0|} \circ \cdots \circ U_{|x_{k-1}|} \circ U_{|x_{k+1}|} \circ \cdots \circ U_{|x_n|}\\ &\subset U_{k-1} \circ \cdots \circ U_0 \circ U_0 \circ \cdots \circ U_{n-k}\\ &\subset U_0^4\\ &\subset U.\end{aligned}$$

□

Question 3.7. For towers $(X_n)_{n\in\omega}$, $(Y_n)_{n\in\omega}$ of quasi-uniform spaces, is the identity map id : qu-$\underrightarrow{\lim}(X_n \times Y_n) \longrightarrow$ qu-$\underrightarrow{\lim}\, X_n \times$ qu-$\underrightarrow{\lim}\, Y_n$ a homeomorphism?

Let $(X, \mathcal{QU}_X)$ be a quasi-uniform space. Denote $\mathcal{QU}_X^*$ be the coarsest uniformity on X which is finer than $\mathcal{QU}_X$.

Theorem 3.8. *Let* $(X_n, \mathcal{QU}_{X_n})$ *be a tower of quasi-uniform spaces, and let* $\mathcal{QU}_X$ *be the quasi-uniformity of* qu-$\underrightarrow{\lim}\, X_n$. *Then* $\mathcal{QU}_X^*$ *is the uniformity of* u-$\underrightarrow{\lim}\, Y_n$, *where each* Y_n *is the uniform space* $(X_n, \mathcal{QU}_{X_n}^*)$.

Proof. Let $\mathcal{U}_X$ be the uniformity of u-$\varinjlim Y_n$. Obviously, we have $\mathcal{QU}_X^* \subset \mathcal{U}_X$. Next we shall prove that $\mathcal{U}_X \subset \mathcal{QU}_X^*$.

Let $U \in \mathcal{U}_X$. Obviously, we have $U \cap (X_n \times X_n) \in \mathcal{QU}_{X_n}^*$ for each $n \in \omega$. Next, we shall construct a sequence $\{V_n : n \in \omega\}$ of subsets of $X \times X$ such that the following conditions hold:

(1) for each $n \in \omega$, $V_n \in \mathcal{QU}_{X_n}$;
(2) for each $n \in \omega$, $V_n = V_{n+1} \cap (X_n \times X_n)$;
(3) for each $n \in \omega$, $V_n \cap V_n^{-1} \subset U \cap (X_n \times X_n)$.

Indeed, since $U \cap (X_0 \times X_0) \in \mathcal{QU}_{X_0}^*$, there exists a $V_0 \in \mathcal{QU}_{X_0}$ such that $V_0 \cap V_0^{-1} \subset U \cap (X_0 \times X_0)$. Now, suppose we have defined $V_0, V_1, \dots, V_n$. Since $V_n \cap V_n^{-1} \subset (X_{n+1} \times X_{n+1}) \cap U$ and $V_n \cap V_n^{-1} \in \mathcal{QU}_{X_n}^*$, there exists a symmetric subset $W \in \mathcal{QU}_{X_{n+1}}^*$ such that $V_n \cap V_n^{-1} = W \cap (X_n \times X_n)$ and $W \subset (X_{n+1} \times X_{n+1}) \cap U$. Moreover, since $V_n \in \mathcal{QU}_{X_n}$, there exists a $L \in \mathcal{QU}_{X_{n+1}}$ such that $L \cap (X_n \times X_n) = V_n$. Therefore, there exists an $O \subset L$ and $O \in \mathcal{QU}_{X_{n+1}}$ such that $O \cap O^{-1} \subset W$. Put $V_{n+1} = O \cup V_n$. Then $V_{n+1} \in \mathcal{QU}_{X_{n+1}}$ and $V_n = V_{n+1} \cap (X_n \times X_n)$. Moreover,

$$
\begin{aligned}
&V_{n+1} \cap V_{n+1}^{-1} \\
&\quad = (O \cup V_n) \cap (O^{-1} \cup V_n^{-1}) \\
&\quad = (O \cap O^{-1}) \cup (O \cap V_n^{-1}) \cup (O^{-1} \cap V_n) \cup (V_n \cap V_n^{-1}) \\
&\quad = (O \cap O^{-1}) \cup \big(O \cap (X_n \times X_n) \cap V_n^{-1}\big) \\
&\qquad \cup \big(O^{-1} \cap (X_n \times X_n) \cap V_n\big) \cup (V_n \cap V_n^{-1}) \\
&\quad \subset (O \cap O^{-1}) \cup \big(L \cap (X_n \times X_n) \cap V_n^{-1}\big) \\
&\qquad \cup \big(L^{-1} \cap (X_n \times X_n) \cap V_n\big) \cup (V_n \cap V_n^{-1}) \\
&\quad \subset (O \cap O^{-1}) \cup (V_n \cap V_n^{-1}) \\
&\quad \subset W \cup (U \cap U_n) \\
&\quad \subset U \cap (X_{n+1} \times X_{n+1}).
\end{aligned}
$$

Therefore, $\{V_n : n \in \omega\}$ satisfies (1)–(3).

Put $V = \bigcup_{n\in\omega} V_n$. Obviously, the set $V \in \mathcal{U}_X$, and hence $V \cap V^{-1} \in \mathcal{U}_X^*$. Obviously, we have

$$V \cap (-V) = \bigcup_{n\in\omega} (V_n \cap V_n^{-1}) \subset \bigcup_{n\in\omega} \big(U \cap (X_n \times X_n)\big) = U.$$

Therefore, we have $\mathcal{U}_X^* \subset \mathcal{QU}_X$. Then $\mathcal{U}_X^* = \mathcal{QU}_X$. □

4. Some properties of topological direct limit of paratopological groups

Obviously, we have the following two theorems.

Theorem 4.1. *Let $(G_n)_{n\in\mathbb{N}}$ be a tower of paratopological groups. Then the topological direct limit* t-$\varinjlim G_n$ *is a semitopological group.*

Remark 4.2. However, the multiplication $\text{t-}\varinjlim G_n \times \text{t-}\varinjlim G_n \ni (g,h) \longmapsto gh \in \text{t-}\varinjlim G_n$ is not necessarily jointly continuous (see [12, Example 1.2]).

Theorem 4.3. *Let $(G_n)_{n\in\mathbb{N}}$ be a tower of paratopological groups, and let H be a paratopological group. If φ is an algebraic homomorphism of the group $G = \bigcup_{n\in\omega} G_n$ into H, then the following conditions are equivalent:*

(1) *φ is continuous as a map from $\text{pg-}\varinjlim G_n$ to H;*
(2) *φ is continuous as a map from $\text{t-}\varinjlim G_n$ to H.*

Let (G,τ) be a paratopological group. Denote G^* by the coarsest group topology on the abstract group G which is finer than τ.

Theorem 4.4. *Let $(G_n)_{n\in\mathbb{N}}$ be a tower of paratopological groups. If the topological direct limit $\text{t-}\varinjlim G_n$ is a paratopological group, then $\text{t-}\varinjlim G_n^* = (\text{t-}\varinjlim G_n)^*$.*

Proof. Let τ, δ be the topologies of $\text{t-}\varinjlim G_n$ and $\text{t-}\varinjlim G_n^*$, respectively. Obviously, we have $\tau^* \subset \delta$ by the definition of the topological direct limit $\text{t-}\varinjlim G_n^*$. Next, we shall prove that $\delta \subset \tau^*$. Let U be an open neighborhood of e in δ. Then each $U \cap G_n$ is open in G_n^*. Next, we shall construct a sequence $\{V_n : n \in \omega\}$ of subsets of G such that the following conditions hold:

(1) for each $n \in \omega$, V_n is a neighborhood of e in G_n;
(2) for each $n \in \omega$, $V_n = V_{n+1} \cap G_n$;
(3) for each $n \in \omega$, $V_n \cap V_n^{-1} \subset G_n \cap U$.

Indeed, since $U \cap G_0$ is open in G_0^*, there exists an open neighborhood V_0 of e in G_0 such that $V_0 \cap V_0^{-1} \subset G_0 \cap U$. Now, suppose we have defined $V_0, V_1, \dots, V_n$. Since $V_n \cap V_n^{-1} \subset G_{n+1} \cap U$ and $V_n \cap V_n^{-1}$ is open in G_n^*, there exists a symmetric open neighborhood W of e in G_{n+1}^* such that $V_n \cap V_n^{-1} = W \cap G_n$ and $W \subset G_{n+1} \cap U$. Moreover, since V_n is open in G_n, there exists an open neighborhood L of e in G_{n+1} such that $L \cap G_n = V_n$. Therefore, there exists an open neighborhood $O \subset L$ of e in G_{n+1} such that $O \cap O^{-1} \subset W$. Put $V_{n+1} = O \cup V_n$. Then V_{n+1} is a neighborhood of e in G_{n+1} and $V_n = V_{n+1} \cap G_n$. Then we have

$$\begin{aligned}
V_{n+1} \cap V_{n+1}^{-1} &= (O \cup V_n) \cap (O^{-1} \cup V_n^{-1}) \\
&= (O \cap O^{-1}) \cup (O \cap V_n^{-1}) \cup (O^{-1} \cap V_n) \cup (V_n \cap V_n^{-1}) \\
&= (O \cap O^{-1}) \cup (O \cap G_n \cap V_n^{-1}) \cup (O^{-1} \cap G_n \cap V_n) \cup (V_n \cap V_n^{-1}) \\
&\subset (O \cap O^{-1}) \cup (L \cap G_n \cap V_n^{-1}) \cup (L^{-1} \cap G_n \cap V_n) \cup (V_n \cap V_n^{-1}) \\
&\subset (O \cap O^{-1}) \cup (V_n \cap V_n^{-1}) \\
&\subset W \cup (U \cap U_n) \\
&\subset U \cap G_{n+1}.
\end{aligned}$$

Therefore, $\{V_n : n \in \omega\}$ satisfies (1)–(3).

Put $V = \bigcup_{n\in\omega} V_n$. Obviously, the set V is a neighborhood of e in τ; hence $V \cap V^{-1}$ is a neighborhood of e in τ^*. Obviously, we have

$$V \cap V^{-1} = \bigcup_{n\in\omega} (V_n \cap V_n^{-1}) \subset \bigcup_{n\in\omega} (U \cap G_n) = U.$$

Therefore, we have $\delta \subset \tau^*$. Then $\delta = \tau^*$; that is, $\text{t-}\varinjlim G_n^* = (\text{t-}\varinjlim G_n)^*$. $\square$

Lemma 4.5 (see [5, Theorem Y])**.** *Let $(G_n)_{n\in\mathbb{N}}$ be a tower of metrizable topological groups, where each G_n is closed in G_{n+1}. Then $\text{t-}\varinjlim G_n$ is a topological group if and only if each G_n is locally compact or some G_n is open in all G_m, $m \geq n$.*

Corollary 4.6. *Let $(G_n)_{n\in\mathbb{N}}$ be a tower of T_0 first-countable paratopological groups, where each G_n is closed in G_{n+1}. If the topological direct limit $\text{t-}\varinjlim G_n$ is a paratopological group, then each G_n^* is locally compact or some G_n^* is open in all G_m^*, $m \geq n$.*

Proof. Obviously, each G_n^* is metrizable and each G_n^* is closed in G_{n+1}^*. By Lemma 4.5 and Theorem 4.4, the theorem holds. $\square$

Example 4.7. There exists a tower of T_0 first-countable paratopological groups $(G_n)_{n\in\mathbb{N}}$ such that each G_n is closed in G_{n+1} and the topological direct limit $\text{t-}\varinjlim G_n$ is not a paratopological group.

Proof. For each $n \in \omega$, let $G_n = \underbrace{Q \times \cdots \times Q}_{n}$, where the rational number $\mathbb{Q}$ endows with usual topology. Obviously, we can identify each G_n as a subset of G_{n+1}. Then each G_n is a metrizable topological groups and each G_n is closed in G_{n+1}. However, it follows from Lemma 4.5 that $\text{t-}\varinjlim G_n$ is not a topological group, and hence it is not a paratopological group by Theorem 4.4. $\square$

Question 4.8. Let $(G_n)_{n\in\mathbb{N}}$ be a tower of paratopological groups. If the topological direct limit $\text{t-}\varinjlim G_n^*$ is a topological group, is the topological direct limit $\text{t-}\varinjlim G_n$ a paratopological group?

Question 4.9. Let $(G_n)_{n\in\mathbb{N}}$ be a tower of T_0 first-countable paratopological groups, where each G_n is closed in G_{n+1}. If the topological direct limit $\text{t-}\varinjlim G_n^*$ is a topological group, is the topological direct limit $\text{t-}\varinjlim G_n$ a paratopological group?

Finally, we discuss the direct product of two towers of paratopological groups. The following lemma is obvious.

Lemma 4.10. *Let $(G_n)_{n\in\omega}$ be a tower of paratopological groups, and let L be a paratopological group. For each $n \in \omega$, if the map $\psi_n : G_n \longrightarrow L$ is a continuous homomorphism and $\psi_{n+1}|_{G_n} = \psi_n$, then the following homomorphism map*

$$\psi : \text{pg-}\varinjlim G_n \longrightarrow L, \qquad X \ni x \mapsto \psi_n(x),$$

where $n = \min\{n \in \omega : x \in G_n\}$, is continuous.

Theorem 4.11. *Let $(G_n)_{n\in\omega}$, $(H_n)_{n\in\omega}$ be two towers of paratopological groups. Then the identity map* $\mathrm{id} : (\text{pg-}\varinjlim G_n) \times (\text{pg-}\varinjlim H_n) \longrightarrow \text{pg-}\varinjlim (G_n \times H_n)$ *is a homeomorphism.*

Proof. Obviously, the map $\mathrm{id}^{-1} : \text{pg-}\varinjlim(G_n \times H_n) \longrightarrow (\text{pg-}\varinjlim G_n) \times (\text{pg-}\varinjlim H_n)$ is continuous. Let us prove the continuity of id.

We assert that the homomorphism $(\text{pg-}\varinjlim G_n) \ni g \longmapsto (g, e_H) \in \text{pg-}\varinjlim(G_n \times H_n)$ is continuous, where e_H denotes the neutral element of $\text{pg-}\varinjlim H_n$.

To prove this, we apply Lemma 4.10 for $L = \text{pg-}\varinjlim(G_n \times H_n)$. Indeed, for each $n \in \omega$, the canonical map $G_n \times H_n \ni (g_n, h_n) \longmapsto (g_n, h_n) \in L = \text{pg-}\varinjlim(G_n \times H_n)$ is continuous by the definition of the topology of $\text{pg-}\varinjlim(G_n \times H_n)$, and the embedding map $G_n \ni g_n \longmapsto (g_n, e_{H_n}) \in G_n \times H_n$ is also continuous. Therefore, the homomorphism $G_n \ni g_n \longmapsto (g_n, e_H) \in L$ is continuous. Hence, by Lemma 4.10, we get the asserted continuity.

Similarly, the homomorphism $(\text{pg-}\varinjlim H_n) \ni h \longmapsto (e_G, h) \in \text{pg-}\varinjlim(G_n \times H_n)$ is continuous, where e_G denotes the neutral element of $\text{pg-}\varinjlim G_n$. Then the map

$$\phi : \text{pg-}\varinjlim G_n \times \text{pg-}\varinjlim H_n \to \text{pg-}\varinjlim(G_n \times H_n) \times \text{pg-}\varinjlim(G_n \times H_n),$$

$(g, h) \longmapsto ((g, e_H), (e_G, h))$, is continuous. Moreover, the product map

$$\varphi : \text{pg-}\varinjlim(G_n \times H_n) \times \text{pg-}\varinjlim(G_n \times H_n) \to \text{pg-}\varinjlim(G_n \times H_n),$$

$((g, h), (g', h')) \longmapsto (gg', hh')$, is continuous since the topology of $\text{pg-}\varinjlim(G_n \times H_n)$ is a paratopological group topology. Therefore, the product of maps $\psi \cdot \varphi = \mathrm{id}$ is continuous. $\square$

5. Topological direct limit of paratopological groups

In this section, given a tower of paratopological groups

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots,$$

we define topologies on the group $G = \bigcup_{n \in \omega} G_n$.

Given a sequence of subsets $(U_n)_{n \in \omega}$ of the group G, where $e \in U_n$, for each $m \in \omega$, let

$$\overrightarrow{\prod_{0 \le n \le m}} U_n = U_0 U_1 \cdots U_m;$$

$$\overleftarrow{\prod_{0 \le n \le m}} U_n = U_m U_{m-1} \cdots U_0;$$

$$\overleftrightarrow{\prod_{0 \le n \le m}} U_n = U_m U_{m-1} \cdots U_0 U_0 U_1 \cdots U_m.$$

Then we consider the following direct product in G:

$$\overrightarrow{\prod_{n \in \omega}} U_n = \bigcup_{m \in \omega} \overrightarrow{\prod_{0 \le n \le m}} U_n;$$

$$\overleftarrow{\prod_{n\in\omega}} U_n = \bigcup_{m\in\omega} \overleftarrow{\prod_{0\le n\le m}} U_n;$$
$$\overleftrightarrow{\prod_{n\in\omega}} U_n = \bigcup_{m\in\omega} \overleftrightarrow{\prod_{0\le n\le m}} U_n.$$

Note that

$$\Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big)^{-1} = \overleftarrow{\prod_{n\in\omega}} U_n^{-1} \qquad \text{and} \qquad \overleftrightarrow{\prod_{n\in\omega}} U_n = \Big(\overleftarrow{\prod_{n\in\omega}} U_n\Big)\cdot\Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big).$$

In each paratopological group G_n, fix a base $\mathscr{B}_n$ of open neighborhoods of the neutral element e. The topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overleftrightarrow{\tau}$, $\overleftrightarrows{\tau}$ on the group G are generated by the bases, respectively:

$$\overrightarrow{\mathscr{B}} = \Big\{\Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big)\cdot x : x\in G, (U_n)_{n\in\omega}\in\prod_{n\in\omega}\mathscr{B}_n\Big\};$$
$$\overleftarrow{\mathscr{B}} = \Big\{x\cdot\Big(\overleftarrow{\prod_{n\in\omega}} U_n\Big) : x\in G, (U_n)_{n\in\omega}\in\prod_{n\in\omega}\mathscr{B}_n\Big\};$$
$$\overleftrightarrow{\mathscr{B}} = \Big\{x\cdot\Big(\overleftrightarrow{\prod_{n\in\omega}} U_n\Big)\cdot y : x,y\in G, (U_n)_{n\in\omega}\in\prod_{n\in\omega}\mathscr{B}_n\Big\};$$
$$\overleftrightarrows{\mathscr{B}} = \Big\{x\cdot\Big(\overleftarrow{\prod_{n\in\omega}} U_n\Big)\cap\Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big)\cdot y : x,y\in G, (U_n)_{n\in\omega}\in\prod_{n\in\omega}\mathscr{B}_n\Big\}.$$

By $\overrightarrow{G}$, $\overleftarrow{G}$, $\overleftrightarrow{G}$, $\overleftrightarrows{G}$ we denote the group G endowed with the topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overleftrightarrow{\tau}$, $\overleftrightarrows{\tau}$, respectively. Obviously, $\overrightarrow{G}$, $\overleftarrow{G}$, $\overleftrightarrow{G}$, $\overleftrightarrows{G}$ are semitopological groups having the families

$$\overrightarrow{\mathscr{B}}_e = \Big\{\overrightarrow{\prod_{n\in\omega}} U_n : (U_n)_{n\in\omega}\in\prod_{n\in\omega}\mathscr{B}_n\Big\},$$
$$\overleftarrow{\mathscr{B}}_e = \Big\{\overleftarrow{\prod_{n\in\omega}} U_n : (U_n)_{n\in\omega}\in\prod_{n\in\omega}\mathscr{B}_n\Big\},$$
$$\overleftrightarrow{\mathscr{B}}_e = \Big\{\overleftrightarrow{\prod_{n\in\omega}} U_n : (U_n)_{n\in\omega}\in\prod_{n\in\omega}\mathscr{B}_n\Big\},$$
$$\overleftrightarrows{\mathscr{B}}_e = \Big\{\Big(\overleftarrow{\prod_{n\in\omega}} U_n\Big)\cap\Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big) : (U_n)_{n\in\omega}\in\prod_{n\in\omega}\mathscr{B}_n\Big\},$$

and are neighborhood bases at the neutral element e, respectively.

Theorem 5.1. *The identity map* $\overleftrightarrow{G} \longrightarrow \text{pg-}\varinjlim G_n$ *is continuous.*

Proof. Obviously, $\overleftrightarrow{G}$ and $\text{pg-}\varinjlim G_n$ are semitopological groups. Therefore, it suffices to prove the continuity of the identity map $\overleftrightarrow{G} \longrightarrow \text{pg-}\varinjlim G_n$ at the neutral element e.

Take a neighborhood U of e in $\text{pg-}\varinjlim G_n$. By induction, construct a sequence of open neighborhoods V_n of e in $\text{pg-}\varinjlim G_n$ such that $V_0^4 \subset U$ and $V_{n+1}^2 \subset V_n$ for each $n \in \omega$. Then, for each $m \in \omega$, we have

$$V_m^2 \cdot \Big(\overleftarrow{\prod_{0 \le n < m}} V_n \Big) \cdot \Big(\overrightarrow{\prod_{0 \le n < m}} V_n \Big) \cdot V_m^2 \subset U.$$

Indeed, for $m = 0$, this inclusion holds according to the choice of V_0. Suppose that for some $m \in \omega$ the inclusion holds. Then we have

$$\begin{aligned} & V_{m+1}^2 \cdot \Big(\overleftarrow{\prod_{0 \le n \le m}} V_n \Big) \cdot \Big(\overrightarrow{\prod_{0 \le n \le m}} V_n \Big) \cdot V_{m+1}^2 \\ & = V_{m+1}^2 \cdot V_m \cdot \Big(\overleftarrow{\prod_{0 \le n < m}} V_n \Big) \cdot \Big(\overrightarrow{\prod_{0 \le n < m}} V_n \Big) \cdot V_m \cdot V_{m+1}^2 \\ & \subset V_m \cdot V_m \cdot \Big(\overleftarrow{\prod_{0 \le n < m}} V_n \Big) \cdot \Big(\overrightarrow{\prod_{0 \le n < m}} V_n \Big) \cdot V_m \cdot V_m \\ & \subset U. \end{aligned}$$

Therefore, we have

$$\Big(\overleftarrow{\prod_{n \in \omega}} V_n \Big) \cdot \Big(\overrightarrow{\prod_{n \in \omega}} V_n \Big) = \Big(\bigcup_{m \in \omega} \overleftarrow{\prod_{0 \le n \le m}} V_n \Big) \cdot \Big(\bigcup_{m \in \omega} \overrightarrow{\prod_{0 \le n \le m}} V_n \Big) \subset U.$$

Since $\overleftrightarrow{\prod}_{n \in \omega} V_n = (\overleftarrow{\prod}_{n \in \omega} V_n) \cdot (\overrightarrow{\prod}_{n \in \omega} V_n)$, we have $\overleftrightarrow{\prod}_{n \in \omega} V_n \subset U$.

Therefore, the identity map $\overleftrightarrow{G} \longrightarrow \text{pg-}\varinjlim G_n$ is continuous. $\square$

Obviously, the following five semitopological groups are linked by continuous identity homomorphisms:

$$\begin{tikzcd} & \overleftarrow{G} \arrow[dr] & & \\ \overset{\rightleftarrows}{G} \arrow[ur] \arrow[dr] & & \overleftrightarrow{G} \arrow[r] & \text{pg-}\varinjlim G_n. \\ & \overrightarrow{G} \arrow[ur] & & \end{tikzcd}$$

Theorem 5.2. *The following conditions are equivalent:*

(1) $\overrightarrow{G}$ *and* $\overleftarrow{G}$ *are paratopological groups;*

(2) $\overset{\rightleftarrows}{G}$ *is a paratopological group;*

(3) *the identity map* $\overleftrightarrow{G} \longrightarrow \overset{\rightleftarrows}{G}$ *is continuous;*

(4) *the identity map* $\overset{\rightleftarrows}{G} \longrightarrow \text{pg-}\varinjlim G_n$ *is a homeomorphism.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\overrightarrow{G}$ and $\overleftarrow{G}$ are paratopological groups. Next we shall prove that $\overset{\rightleftarrows}{G}$ is a paratopological group.

Indeed, take a $(\overleftarrow{\prod}_{n\in\omega}U_n) \cap (\overrightarrow{\prod}_{n\in\omega}U_n) \in \overset{\rightleftarrows}{\mathscr{B}}_e$. For each $m \in \omega$, since $\overrightarrow{G}$ and $\overleftarrow{G}$ are paratopological groups, there exists a $V_m \in \mathscr{B}_m$ such that $(\overrightarrow{\prod}_{n\in\omega}V_n)^2 \subset \overrightarrow{\prod}_{n\in\omega}U_n$ and $(\overleftarrow{\prod}_{n\in\omega}V_n)^2 \subset \overleftarrow{\prod}_{n\in\omega}U_n$, and hence

$$\Big(\overrightarrow{\prod_{n\in\omega}} V_n \cap \overleftarrow{\prod_{n\in\omega}} V_n\Big)^2 \subset \Big(\overrightarrow{\prod_{n\in\omega}} V_n\Big)^2 \cap \Big(\overleftarrow{\prod_{n\in\omega}} V_n\Big)^2 \subset \Big(\overleftarrow{\prod_{n\in\omega}} U_n\Big) \cap \Big(\overrightarrow{\prod_{n\in\omega}} U_n\Big).$$

Therefore, $\overset{\rightleftarrows}{G}$ is a paratopological group.

(2) $\Rightarrow$ (4). Obviously, all the inclusion homomorphisms $G_n \longrightarrow \overset{\rightleftarrows}{G}$, $n \in \omega$, are continuous. Since $\overset{\rightleftarrows}{G}$ is a paratopological group, it follows from the definition of $\text{pg-}\varinjlim G_n$ that the identity map $\text{pg-}\varinjlim G_n \longrightarrow \overset{\rightleftarrows}{G}$ is continuous. By Theorem 5.1, the identity map $\overset{\rightleftarrows}{G} \longrightarrow \text{pg-}\varinjlim G_n$ is also continuous. Then the identity map $\overset{\rightleftarrows}{G} \longrightarrow \text{pg-}\varinjlim G_n$ is a homeomorphism.

(4) $\Rightarrow$ (3). Let the identity map $\overset{\rightleftarrows}{G} \longrightarrow \text{pg-}\varinjlim G_n$ be a homeomorphism. Then the composition of two continuous maps

$$\overleftrightarrow{G} \longrightarrow \text{pg-}\varinjlim G_n \qquad \text{and} \qquad \text{pg-}\varinjlim G_n \longrightarrow \overset{\rightleftarrows}{G}$$

is continuous, and hence the identity map $\overleftrightarrow{G} \longrightarrow \overset{\rightleftarrows}{G}$ is continuous.

(3) $\Rightarrow$ (1). Let the identity map $\overleftrightarrow{G} \longrightarrow \overset{\rightleftarrows}{G}$ be a continuous map. Then $\overleftrightarrow{G} \longrightarrow \overset{\rightleftarrows}{G}$ is a homeomorphism. Therefore, the identity maps between semitopological groups $\overrightarrow{G}$, $\overleftarrow{G}$, $\overleftrightarrow{G}$, $\overset{\rightleftarrows}{G}$ are homeomorphisms. Moreover, the map $\overleftarrow{G} \times \overrightarrow{G} \longrightarrow \overset{\rightleftarrows}{G}$, $(x, y) \mapsto xy$ is continuous. Therefore, it is easy to see that the multiplication map $\overrightarrow{G} \times \overrightarrow{G} \longrightarrow \overrightarrow{G}$, $(x, y) \to xy$ is continuous. $\square$

Theorem 5.3. *The semitopological group* $\overleftrightarrow{G}$ *is a paratopological group if and only if the identity map* $\overleftrightarrow{G} \longrightarrow \text{pg-}\varinjlim G_n$ *is a homeomorphism.*

Proof. Sufficiency. Let the identity map $\overleftrightarrow{G} \longrightarrow \text{pg-}\varinjlim G_n$ be a homeomorphism. Then $\overleftrightarrow{G}$ is a paratopological group because pg-$\varinjlim G_n$ is a paratopological group by the definition of pg-$\varinjlim G_n$.

The proof of necessity is similar to that of (2) ⇒ (4) in Theorem 5.2. □

Remark 5.4. Obviously, the equivalent conditions (1) through (4) in Theorem 5.2 imply that $\overleftrightarrow{G}$ is a paratopological group.

Corollary 5.5. *Let $(G_n)_{n\in\mathbb{N}}$ be a tower of abelian paratopological groups. Then the following conditions are equivalent:*

(1) $\overrightarrow{G}$ *is a paratopological group;*
(2) $\overleftarrow{G}$ *is a paratopological group;*
(3) $\overleftrightarrows{G}$ *is a paratopological group;*
(4) *the identity map* $\overleftrightarrow{G} \longrightarrow \overleftrightarrows{G}$ *is continuous;*
(5) *the identity map* $\overleftrightarrows{G} \longrightarrow \text{pg-}\varinjlim G_n$ *is a homeomorphism.*

Next we shall introduce the properties of PPTA, balanced and bi-balanced, guaranteeing that the topology of the direct limit pg-$\varinjlim G_n$ of a tower of paratopological groups $(G_n)_{n\in\omega}$ coincides with the topologies $\overrightarrow{G}$, $\overleftarrow{G}$, $\overleftrightarrow{G}$, and $\overleftrightarrows{G}$.

Definition 5.6. A tower of paratopological groups $(G_n)_{n\in\omega}$ is said to satisfy PPTA if each paratopological group G_n has a neighborhood base $\mathscr{B}_n$ at the neutral element e, consisting of open neighborhoods of e in G_n, such that, for each $U \in \mathscr{B}_n$ and each neighborhood V of e in G_m with $m \geq n$, there is a neighborhood W of e in G_m such that $WU \subset UV$ and $UW \subset VU$.

Theorem 5.7. *If a tower of paratopological groups $(G_n)_{n\in\omega}$ satisfies PPTA, then semitopological groups $\overrightarrow{G}$ and $\overleftarrow{G}$ are paratopological groups and hence conditions (1)–(4) of Theorem 5.2 hold. In particular, the topology of* pg-$\varinjlim G_n$ *coincides with any of the topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overleftrightarrow{\tau}$, $\overleftrightarrows{\tau}$.*

Proof. Since the tower of paratopological groups $(G_n)_{n\in\omega}$ satisfies PPTA, each paratopological group G_n has an open neighborhood base $\mathscr{B}_n$ at e, consisting of an open neighborhood U in G_n, such that, for each $m \geq n$ and a neighborhood V of e in G_m, there is a neighborhood W of e in G_m such that $WU \subset UV$ and $UW \subset VU$.

By Theorem 5.2, we shall show that $\overrightarrow{G}$ and $\overleftarrow{G}$ are paratopological groups. Therefore, it suffices to check the continuity of the multiplication at the neutral element e for semitopological groups $\overrightarrow{G}$ and $\overleftarrow{G}$, respectively.

Let $\overrightarrow{\prod}_{n\in\omega} W_n \in \overrightarrow{\mathscr{B}}_e$ and $\overleftarrow{\prod}_{n\in\omega} W_n \in \overleftarrow{\mathscr{B}}_e$. Next, we find a neighborhood $\overrightarrow{\prod}_{n\in\omega} V_n \in \overrightarrow{\mathscr{B}}_e$ and $\overleftarrow{\prod}_{n\in\omega} V_n \in \overleftarrow{\mathscr{B}}_e$ such that

$$\Big(\overrightarrow{\prod_{n\in\omega}} V_n\Big)^2 \subset \overrightarrow{\prod_{n\in\omega}} W_n \qquad \text{and} \qquad \Big(\overleftarrow{\prod_{n\in\omega}} V_n\Big)^2 \subset \overleftarrow{\prod_{n\in\omega}} W_n.$$

For each $n\in\omega$, we can find a neighborhood U_n of e in G_n such that $U_n^2 \subset W_n$. Put $V_n^{(0)} = U_n$ and, using PPTA, for each $0 < i \le n$ find a neighborhood V_n^i of e in G_n such that

(a) $V_n^i \subset U_n$, and
(b) $V_n^i U_{n-i} \subset U_{n-i} V_n^{i-1}$ and $U_{n-i} V_n^i \subset V_n^{i-1} U_{n-i}$.
Note that, for $i = n-k$, the latter inclusion yields
(c) $V_n^{n-k} U_k \subset U_k V_n^{n-k-1}$ and $U_k V_n^{n-k} \subset V_n^{n-k-1} U_k$.
By induction on k we can deduce from (c) the following two inclusions:
(d) $(\overrightarrow{\prod}_{k<n\le m} V_n^{(n-k)}) \cdot U_k \subset U_k \cdot (\overrightarrow{\prod}_{k<n\le m} V_n^{(n-k-1)})$;
(e) $U_k \cdot (\overleftarrow{\prod}_{k<n\le m} V_n^{(n-k)}) \subset (\overleftarrow{\prod}_{k<n\le m} V_n^{(n-k-1)}) \cdot U_k$.

Claim 2. *For each $m > 0$, we have*

$$\Big(\overrightarrow{\prod_{n\le m}} V_n^{(n)}\Big) \cdot \Big(\overrightarrow{\prod_{n\le m}} U_n\Big) \subset \overrightarrow{\prod_{n\le m}} W_n \qquad \text{and} \qquad \Big(\overleftarrow{\prod_{n\le m}} U_n\Big) \cdot \Big(\overleftarrow{\prod_{n\le m}} V_n^{(n)}\Big) \subset \overleftarrow{\prod_{n\le m}} W_n.$$

Indeed, for each nonnegative integer $k \le m+1$, put

$$\overrightarrow{\prod_{k}} = \Big(\overrightarrow{\prod_{0\le n<k}} W_n\Big) \cdot \Big(\overrightarrow{\prod_{k\le n\le m}} V_n^{(n-k)}\Big) \cdot \Big(\overrightarrow{\prod_{k\le n\le m}} U_n\Big)$$

and

$$\overleftarrow{\prod_{k}} = \Big(\overleftarrow{\prod_{k\le n\le m}} U_n\Big) \cdot \Big(\overleftarrow{\prod_{k\le n\le m}} V_n^{(n-k)}\Big) \cdot \Big(\overleftarrow{\prod_{0\le n<k}} W_n\Big)$$

of paratopological group G_m. Then it is easy to see that Claim 2 holds if and only if $\overrightarrow{\prod}_0 \subset \overrightarrow{\prod}_{m+1}$ and $\overleftarrow{\prod}_0 \subset \overleftarrow{\prod}_{m+1}$. Next we shall show that $\overrightarrow{\prod}_k \subset \overrightarrow{\prod}_{k+1}$ and $\overleftarrow{\prod}_k \subset \overleftarrow{\prod}_{k+1}$ for each $k \le m$.

Since $V_k^{(0)} U_k = U_k U_k \subset W_k$ and $U_k V_k^{(0)} = U_k U_k \subset W_k$, it follows from (d) and (e) that

$$\begin{aligned}\overrightarrow{\prod_{k}} &= \Big(\overrightarrow{\prod_{0\le n<k}} W_n\Big) \cdot \Big(\overrightarrow{\prod_{k\le n\le m}} V_n^{(n-k)}\Big) \cdot \Big(\overrightarrow{\prod_{k\le n\le m}} U_n\Big)\\ &= \Big(\overrightarrow{\prod_{0\le n<k}} W_n\Big) \cdot V_k^{(0)} \cdot \Big(\overrightarrow{\prod_{k<n\le m}} V_n^{(n-k)}\Big) \cdot U_k \cdot \Big(\overrightarrow{\prod_{k<n\le m}} U_n\Big)\end{aligned}$$

$$
\begin{aligned}
&\subset \Big(\overrightarrow{\prod_{0\le n<k}} W_n\Big)\cdot V_k^{(0)}\cdot \Big(U_k\cdot \overrightarrow{\prod_{k<n\le m}} V_n^{(n-k-1)}\Big)\cdot \Big(\overrightarrow{\prod_{k<n\le m}} U_n\Big)\\
&\subset \Big(\overrightarrow{\prod_{0\le n<k}} W_n\Big)\cdot W_k\cdot \Big(\overrightarrow{\prod_{k<n\le m}} V_n^{(n-k-1)}\Big)\cdot \Big(\overrightarrow{\prod_{k<n\le m}} U_n\Big)\\
&= \overrightarrow{\prod_{k+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
\overleftarrow{\prod_{k}} &= \Big(\overleftarrow{\prod_{k\le n\le m}} U_n\Big)\cdot \Big(\overleftarrow{\prod_{k\le n\le m}} V_n^{(n-k)}\Big)\cdot \Big(\overleftarrow{\prod_{0\le n<k}} W_n\Big)\\
&= \Big(\overleftarrow{\prod_{k<n\le m}} U_n\Big)\cdot U_k\cdot \Big(\overleftarrow{\prod_{k<n\le m}} V_n^{(n-k)}\Big)\cdot V_k^{(0)}\cdot \Big(\overleftarrow{\prod_{0\le n<k}} W_n\Big)\\
&\subset \Big(\overleftarrow{\prod_{k<n\le m}} U_n\Big)\cdot \Big(\overleftarrow{\prod_{k<n\le m}} V_n^{(n-k-1)}\Big)\cdot U_k\cdot V_k^{(0)}\cdot \Big(\overleftarrow{\prod_{0\le n<k}} W_n\Big)\\
&\subset \Big(\overleftarrow{\prod_{k<n\le m}} U_n\Big)\cdot \Big(\overleftarrow{\prod_{k<n\le m}} V_n^{(n-k-1)}\Big)\cdot W_k\cdot \Big(\overleftarrow{\prod_{0\le n<k}} W_n\Big)\\
&= \overleftarrow{\prod_{k+1}}.
\end{aligned}
$$

Therefore, Claim 2 holds.

Since $V_n^{(n)} \subset U_n$, it follows from Claim 2 that the following Claim 3 holds.

Claim 3. $(\overrightarrow{\prod}_{n\in\omega} V_n^{(n)})^2 \subset \overrightarrow{\prod}_{n\in\omega} W_n$ *and* $(\overleftarrow{\prod}_{n\in\omega} V_n^{(n)})^2 \subset \overleftarrow{\prod}_{n\in\omega} W_n$.

By Claim 3 and Theorem 5.2, semitopological groups $\overrightarrow{G}$ and $\overleftarrow{G}$ are paratopological groups and hence conditions (1) through (4) of Theorem 5.2 hold. In particular, the topology of pg-$\varinjlim G_n$ coincides with any of the topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overleftrightarrow{\tau}$, $\overset{\rightleftarrows}{\tau}$ by Theorems 5.2 and 5.3. $\square$

Definition 5.8. A subset U of a group G is said to be H-*invariant* for a subgroup $H \subset G$ if $xUx^{-1} = U$ for all $x \in H$.

Note that, for any subset $U \subset G$, the set

$$\sqrt[H]{U} = \{x \in G : x^H \subset U\}$$

is the largest H-invariant subset of U, where $x^H = \{hxh^{-1} : h \in H\}$ denotes the conjugacy class of a point $x \in G$.

Definition 5.9. A triple (G, H, L) of paratopological groups $L \subset H \subset G$ is called *balanced* if, for any neighborhood V of e in H and neighborhood U of e in G, the products $V \cdot \sqrt[L]{U}$ and $\sqrt[L]{U} \cdot V$ are neighborhoods of e in G.

Definition 5.10. A tower of paratopological groups $(G_n)_{n\in\omega}$ is said to be *balanced* if each triple (G_{n+2}, G_{n+1}, G_n) is balanced for each $n \in \omega$.

Theorem 5.11. *If a tower of paratopological groups $(G_n)_{n\in\omega}$ is balanced, then semitopological groups $\overrightarrow{G}$ and $\overleftarrow{G}$ are paratopological groups and hence conditions (1)–(4) of Theorem 5.2 hold. In particular, the topology of* pg-$\varinjlim G_n$ *coincides with any of the topologies $\overrightarrow{\tau}, \overleftarrow{\tau}, \overleftrightarrow{\tau}, \overleftrightarrows{\tau}$.*

Proof. Obviously, it suffices to check the continuity of the multiplication at the neutral element e for semitopological groups $\overrightarrow{G}$ and $\overleftarrow{G}$, respectively.

Let $\overrightarrow{\prod}_{n\in\omega} U_n \in \overrightarrow{\mathscr{B}}_e$ and $\overleftarrow{\prod}_{n\in\omega} U_n \in \overleftarrow{\mathscr{B}}_e$. For each $n \in \omega$, take a neighborhood W_n of e in G_n such that $W_n \cdot W_n \subset U_n$. Let

$$Z_n = \sqrt[G_{n-2}]{W_n} = \{x \in G_n : x^{G_{n-2}} \subset W_n\}$$

be the largest G_{n-2}-invariant subset of W_n, where we assume that $G_k = \{e\}$ for $k < 0$.

Put $V_0 = U_0 \cap W_1$, and let V_1 be a neighborhood of e in G_1 such that $V_1^2 \subset W_1$. Then, for each $n \geq 2$, by induction take a neighborhood V_n of e in G_n so that

(f) $V_n^2 \subset V_{n-1} \cdot Z_n \cap Z_n \cdot V_{n-1}$, and
(g) $V_n \subset W_{n+1}$.

For each $n \in \omega$, since (G_n, G_{n-1}, G_{n-2}) is balanced, condition (f) can be satisfied.

Claim 4. *For each $m \geq 2$, we have*

$$\Big(\overrightarrow{\prod_{n\le m}} V_n\Big) \cdot \Big(\overrightarrow{\prod_{n\le m}} V_n\Big) \subset \overrightarrow{\prod_{n\le m}} U_n \qquad \text{and} \qquad \Big(\overleftarrow{\prod_{n\le m}} V_n\Big) \cdot \Big(\overleftarrow{\prod_{n\le m}} V_n\Big) \subset \overleftarrow{\prod_{n\le m}} U_n.$$

Indeed, for each nonnegative integer $k \leq m$, put

$$\overrightarrow{\prod_k} = \Big(\overrightarrow{\prod_{n\le m-k}} V_n\Big) \cdot V_{m-k+1}^2 \cdot \Big(\overrightarrow{\prod_{n\le m-k}} V_n\Big) \cdot \Big(\overrightarrow{\prod_{m-k<n<m}} Z_{n+1}V_n\Big) \cdot V_m$$

and

$$\overleftarrow{\prod_k} = V_m \cdot \Big(\overleftarrow{\prod_{m-k<n<m}} V_n Z_{n+1}\Big) \cdot \Big(\overleftarrow{\prod_{n\le m-k}} V_n\Big) \cdot V_{m-k+1}^2 \cdot \Big(\overleftarrow{\prod_{n\le m-k}} V_n\Big)$$

of paratopological group G_m. Next we shall show that $\overrightarrow{\prod}_k \subset \overrightarrow{\prod}_{k+1}$ and $\overleftarrow{\prod}_k \subset \overleftarrow{\prod}_{k+1}$ for each $k \leq m$.

Then it follows from (f) and (g) that we have

$$\begin{aligned}
\overrightarrow{\prod_k} &= \Big(\overrightarrow{\prod_{n\le m-k}} V_n\Big) \cdot V_{m-k+1}^2 \cdot \Big(\overrightarrow{\prod_{n\le m-k}} V_n\Big) \cdot \Big(\overrightarrow{\prod_{m-k<n<m}} Z_{n+1}V_n\Big) \cdot V_m \\
&\subset \Big(\overrightarrow{\prod_{n\le m-k}} V_n\Big) \cdot V_{m-k} \cdot Z_{m-k+1} \cdot \Big(\overrightarrow{\prod_{n<m-k}} V_n\Big) \cdot V_{m-k} \cdot \Big(\overrightarrow{\prod_{m-k<n<m}} Z_{n+1}V_n\Big) \cdot V_m
\end{aligned}$$

$$= \Big(\overrightarrow{\prod_{n<m-k}} V_n\Big) \cdot V_{m-k}^2 \cdot \Big(\overrightarrow{\prod_{n<m-k}} V_n\Big) \cdot Z_{m-k+1} \cdot V_{m-k} \cdot \Big(\overrightarrow{\prod_{m-k<n<m}} Z_{n+1}V_n\Big) \cdot V_m$$
$$= \Big(\overrightarrow{\prod_{n<m-k}} V_n\Big) \cdot V_{m-k}^2 \cdot \Big(\overrightarrow{\prod_{n<m-k}} V_n\Big) \cdot \Big(\overrightarrow{\prod_{m-k\le n<m}} Z_{n+1}V_n\Big) \cdot V_m$$
$$= \overrightarrow{\prod_{k+1}}$$

and

$$\overleftarrow{\prod_{k}} = V_m \cdot \Big(\overleftarrow{\prod_{m-k<n<m}} V_nZ_{n+1}\Big) \cdot \Big(\overleftarrow{\prod_{n\le m-k}} V_n\Big) \cdot V_{m-k+1}^2 \cdot \Big(\overleftarrow{\prod_{n\le m-k}} V_n\Big)$$
$$\subset V_m \cdot \Big(\overleftarrow{\prod_{m-k<n<m}} V_nZ_{n+1}\Big) \cdot V_{m-k} \cdot \Big(\overleftarrow{\prod_{n<m-k}} V_n\Big) \cdot Z_{m-k+1} \cdot V_{m-k} \cdot \Big(\overleftarrow{\prod_{n\le m-k}} V_n\Big)$$
$$= V_m \cdot \Big(\overleftarrow{\prod_{m-k<n<m}} V_nZ_{n+1}\Big) \cdot V_{m-k} \cdot Z_{m-k+1} \cdot \Big(\overleftarrow{\prod_{n<m-k}} V_n\Big) \cdot V_{m-k}^2 \cdot \Big(\overleftarrow{\prod_{n<m-k}} V_n\Big)$$
$$= V_m \cdot \Big(\overleftarrow{\prod_{m-k\le n<m}} V_nZ_{n+1}\Big) \cdot \Big(\overleftarrow{\prod_{n<m-k}} V_n\Big) \cdot V_{m-k}^2 \cdot \Big(\overleftarrow{\prod_{n<m-k}} V_n\Big)$$
$$= \overleftarrow{\prod_{k+1}}.$$

Therefore, we have

$$\Big(\overrightarrow{\prod_{n\le m}} V_n\Big)^2 \subset \Big(\overrightarrow{\prod_{n\le m-1}} V_n\Big) \cdot V_m^2 \cdot \Big(\overrightarrow{\prod_{n\le m}} V_n\Big)$$
$$= \overrightarrow{\prod_{1}}$$
$$\subset \overrightarrow{\prod_{m}}$$
$$= V_0V_1^2V_0 \cdot \Big(\overrightarrow{\prod_{0<n<m}} Z_{n+1}V_n\Big) \cdot V_m$$
$$\subset U_0W_1W_1 \cdot \Big(\overrightarrow{\prod_{0<n<m}} W_{n+1}W_{+1}\Big) \cdot W_{m+1}$$
$$\subset U_0U_1 \cdot \Big(\overrightarrow{\prod_{0<n<m}} U_{n+1}\Big) \cdot U_{m+1} = \overrightarrow{\prod_{n\le m+1}} U_n$$

and

$$
\begin{aligned}
\Big(\overleftarrow{\prod_{n\le m}} V_n\Big)^2 &\subset \Big(\overleftarrow{\prod_{n\le m}} V_n\Big)\cdot V_m^2\cdot\Big(\overleftarrow{\prod_{n\le m-1}} V_n\Big)\\
&= \overleftarrow{\prod_{1}}\\
&\subset \overleftarrow{\prod_{m}}\\
&= V_m\cdot\Big(\overleftarrow{\prod_{0<n<m}} V_n Z_{n+1}\Big)\cdot V_0V_1^2V_0\\
&\subset W_{m+1}\cdot\Big(\overleftarrow{\prod_{0<n<m}} W_{n+1}W_{+1}\Big)\cdot W_1W_1U_0\\
&\subset U_{m+1}\cdot\Big(\overleftarrow{\prod_{0<n<m}} U_{n+1}\Big)\cdot U_1U_0\\
&= \overleftarrow{\prod_{n\le m+1}} U_n.
\end{aligned}
$$

By Claim 4, it is easy to see that $(\overrightarrow{\prod}_{n\in\omega}V_n)\cdot(\overrightarrow{\prod}_{n\in\omega}V_n)\subset\overrightarrow{\prod}_{n\in\omega}U_n$ and $(\overleftarrow{\prod}_{n\in\omega}V_n)\cdot(\overleftarrow{\prod}_{n\in\omega}V_n)\subset\overleftarrow{\prod}_{n\in\omega}U_n$.

Therefore, semitopological groups $\overrightarrow{G}$ and $\overleftarrow{G}$ are paratopological groups and hence conditions (1)–(4) of Theorem 5.2 hold. In particular, the topology of $\text{pg-}\varinjlim G_n$ coincides with any of the topologies $\overrightarrow{\tau}$, $\overleftarrow{\tau}$, $\overleftrightarrow{\tau}$, $\overset{\rightleftarrows}{\tau}$ by Theorems 5.2 and 5.3. $\square$

Definition 5.12. A triple (G,H,L) of paratopological groups $L\subset H\subset G$ is called *bi-balanced* if, for each neighborhood V of e in H and U of e in G, the product $\sqrt[L]{U}\cdot V\cdot\sqrt[L]{U}$ is a neighborhood of e in G.

A tower of paratopological groups $(G_n)_{n\in\omega}$ is called *bi-balanced* if each triple (G_{n+2}, G_{n+1}, G_n) is bi-balanced for each $n\in\omega$.

Theorem 5.13. *If a tower of paratopological groups $(G_n)_{n\in\omega}$ is bi-balanced, then the identity map $\overleftrightarrow{G}\longrightarrow\text{pg-}\varinjlim G_n$ is a homeomorphism and hence the topology of $\text{pg-}\varinjlim G_n$ coincides with the topology $\overleftrightarrow{\tau}$.*

Proof. By Theorem 5.3, it suffices to show that $\overleftrightarrow{G}$ is a paratopological group. Therefore, it suffices to check the continuity of multiplication of the neutral element. Let $\overleftrightarrow{\prod}_{n\in\omega}W_n\in\overleftrightarrow{\mathscr{B}}_e$. Then we shall find a $\overleftrightarrow{\prod}_{n\in\omega}V_n\in\overleftrightarrow{\mathscr{B}}_e$ such that $(\overleftrightarrow{\prod}_{n\in\omega}V_n)^2\subset\overleftrightarrow{\prod}_{n\in\omega}W_n$.

For each $n \in \omega$, there exists a neighborhood U_n of e in G_n such that $U_n^2 \subset W_n$, and let $Z_n = {}^{G_{n-2}}\!\sqrt{U_n}$ be the largest G_{n-2}-invariant subset of U_n, where we assume that $G_k = \{e\}$ for $k < 0$. Let $O_0 = W_0$. By induction for each $n \in \mathbb{N}$, since each set $Z_n O_{n-1} Z_n$ is a neighborhood of e in G_n, we can take a neighborhood $O_n \subset U_n$ of e in G_n such that $O_n^3 \subset Z_n O_{n-1} Z_n$. For each $n \in \omega$, put $V_n = G_n \cap O_{n+1}$.

Claim 5. *For each $m \in \omega$, we have $(\overleftrightarrow{\prod}_{n<m} V_n)^2 \subset \overleftrightarrow{\prod}_{n \le m} W_n$.*

For $m = 0$, this inclusion is obvious. Suppose that the inclusion in Claim 5 has been proved for some $m = l \in \omega$. Next, we shall prove it for $m = l + 1$. For each nonnegative $k < m = l + 1$, put

$$\prod_k = \Big(\overleftarrow{\prod_{k \le n < m}} V_n Z_{n+1} \Big) \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n \Big) \cdot O_k \cdot \Big(\overleftrightarrow{\prod_{n<k}} V_n \Big) \cdot \Big(\overrightarrow{\prod_{k \le n < m}} Z_{n+1} V_n \Big)$$

of paratopological group G_m. Then we have $\prod_{k+1} \subset \prod_k$. Indeed, we have

$$\begin{aligned}
\prod_{k+1} &= \Big(\overleftarrow{\prod_{k < n < m}} V_n Z_{n+1} \Big) \cdot \Big(\overleftrightarrow{\prod_{n \le k}} V_n \Big) \cdot O_{k+1} \cdot \Big(\overleftrightarrow{\prod_{n \le k}} V_n \Big) \cdot \Big(\overrightarrow{\prod_{k < n < m}} Z_{n+1} V_n \Big) \\
&\subset \Big(\overleftarrow{\prod_{k < n < m}} V_n Z_{n+1} \Big) \cdot V_k \cdot \Big(\overleftrightarrow{\prod_{n \le k}} V_n \Big) \cdot V_k \cdot O_{k+1} \cdot V_k \cdot \Big(\overleftrightarrow{\prod_{n < k}} V_n \Big) \cdot V_k \\
&\quad \cdot \Big(\overrightarrow{\prod_{k < n < m}} Z_{n+1} V_n \Big) \\
&\subset \Big(\overleftarrow{\prod_{k < n < m}} V_n Z_{n+1} \Big) \cdot V_k \cdot \Big(\overleftrightarrow{\prod_{n \le k}} V_n \Big) \cdot O_{k+1}^3 \cdot \Big(\overleftrightarrow{\prod_{n < k}} V_n \Big) \cdot V_k \cdot \Big(\overrightarrow{\prod_{k < n < m}} Z_{n+1} V_n \Big) \\
&\subset \Big(\overleftarrow{\prod_{k < n < m}} V_n Z_{n+1} \Big) \cdot V_k \cdot \Big(\overleftrightarrow{\prod_{n \le k}} V_n \Big) \cdot Z_{k+1} \cdot O_k \cdot Z_{k+1} \cdot \Big(\overleftrightarrow{\prod_{n < k}} V_n \Big) \cdot V_k \\
&\quad \cdot \Big(\overrightarrow{\prod_{k < n < m}} Z_{n+1} V_n \Big) \\
&= \Big(\overleftarrow{\prod_{k < n < m}} V_n Z_{n+1} \Big) \cdot V_k \cdot Z_{k+1} \cdot \Big(\overleftrightarrow{\prod_{n \le k}} V_n \Big) \cdot O_k \cdot \Big(\overleftrightarrow{\prod_{n < k}} V_n \Big) \cdot Z_{k+1} \cdot V_k \\
&\quad \cdot \Big(\overrightarrow{\prod_{k < n < m}} Z_{n+1} V_n \Big) \\
&= \Big(\overleftarrow{\prod_{k \le n < m}} V_n Z_{n+1} \Big) \cdot \Big(\overleftrightarrow{\prod_{n \le k}} V_n \Big) \cdot O_k \cdot \Big(\overleftrightarrow{\prod_{n < k}} V_n \Big) \cdot \Big(\overrightarrow{\prod_{k \le n < m}} Z_{n+1} V_n \Big) \\
&= \prod_k .
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\Big(\overset{\longleftrightarrow}{\prod_{n<m}} V_n\Big)^2 &\subset \Big(\overset{\longleftrightarrow}{\prod_{n<m}} V_n\Big) \cdot O_m \cdot \Big(\overset{\longleftrightarrow}{\prod_{n<m}} V_n\Big) \\
&= \prod_m \\
&\subset \prod_0 \\
&= \Big(\overset{\longleftarrow}{\prod_{n<m}} V_n Z_{n+1}\Big) \cdot O_0 \cdot \Big(\overset{\longrightarrow}{\prod_{n<m}} Z_{n+1} V_n\Big) \\
&\subset \Big(\overset{\longleftarrow}{\prod_{n<m}} U_{n+1}^2\Big) \cdot O_0 \cdot \Big(\overset{\longrightarrow}{\prod_{n<m}} U_{n+1}^2\Big) \\
&\subset \Big(\overset{\longleftarrow}{\prod_{n<m}} W_{n+1}\Big) \cdot W_0^2 \cdot \Big(\overset{\longrightarrow}{\prod_{n<m}} W_{n+1}\Big) \\
&= \overset{\longleftrightarrow}{\prod_{n<m}} W_n.
\end{aligned}$$

By Claim 5, it is easy to see that $(\overset{\longleftrightarrow}{\prod}_{n\in\omega} V_n)\cdot(\overset{\longleftrightarrow}{\prod}_{n\in\omega} V_n) \subset \overset{\longleftrightarrow}{\prod}_{n\in\omega} W_n$. Therefore, $\overleftrightarrow{G}$ is a paratopological group. □

Remark 5.14. (1) In [3], the authors proved that the properties of PPTA and balanced are independent.

(2) It is clear that each balanced triple of paratopological groups is bi-balanced. The converse implication is not true (see [3, Example 5.3]).

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