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WEYL-ALMOST PERIODIC SOLUTIONS AND ASYMPTOTICALLY WEYL-ALMOST PERIODIC SOLUTIONS OF ABSTRACT VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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To my brother Željko Kostić on the occasion of his 50th birthday

Communicated by P. E. T. Jorgensen

Abstract. The main purpose of this article is to investigate Weyl-almost periodic solutions and asymptotically Weyl-almost periodic solutions of abstract Volterra integro-differential equations and inclusions. The class of asymptotically Weyl-almost periodic functions that we introduce here seems not to have been considered elsewhere, even in the scalar-valued case. We analyze the Weyl-almost periodic and asymptotically Weyl-almost periodic properties of convolution products and various types of degenerate solution operator families subgenerated by multivalued linear operators.

1. Introduction and preliminaries

Periodicity and almost periodicity are phenomena that play a crucial role in various fields of mathematics and other sciences. The notion of an almost periodic function was introduced by Bohr in the mid 1920s and was later generalized by many other mathematicians (see, e.g., [15], [19], [29], [31]). The existence of various types of quasiperiodic solutions of abstract Volterra integro-differential equations is still a very active research topic.

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As in the abstract, the main aim of this article is to analyze the existence and uniqueness of Weyl-almost periodic solutions and asymptotically Weyl-almost periodic solutions of abstract Volterra integro-differential equations and inclusions. This is a very unexplored field in the theory of almost periodic abstract differential equations in Banach spaces (see, e.g., [7], [15], [19], [34], [39] and the references cited therein). Our results seem to be new even for abstract nondegenerate differential equations whose solutions are governed by strongly continuous semigroups of operators, while they seem to be completely new for abstract fractional differential equations and abstract (fractional) differential equations with almost sectorial linear operators (see [26], [32], [37], [38]). In this article, we go a step further by investigating the Weyl-almost periodic and asymptotically Weyl-almost periodic properties of various classes of degenerate solution operator families subgenerated by multivalued linear operators (see [26]), providing also a great number of illustrative examples and possible applications of our abstract results. The class of (asymptotically) Weyl $C^{(n)}$-almost periodic functions was recently introduced in [25], where the reader can find applications in the qualitative analysis of solutions to abstract Volterra integro-differential equations.

The organization of this article is as follows. After collecting the necessary material for our further investigations, in Section 2 we consider multivalued linear operators in Banach spaces and degenerate (a, k)-regularized C-resolvent families subgenerated by multivalued linear operators (see Section 2.1). Stepanov almost periodic functions and asymptotically Stepanov almost periodic functions are examined in Section 3, while the Weyl-almost periodic functions and asymptotically Weyl-almost periodic functions are examined in Section 4 (only Proposition 4.3 is new). Our most important original contributions are given in Section 4.1, where we introduce the class of asymptotically Weyl-almost periodic functions. To the author's best knowledge, this class has not yet been analyzed even in the scalar-valued case (it is worth noting that Abbas [2] has introduced the class of Weyl-p-pseudoergodic components, which is larger than the class of Weyl-p-vanishing functions considered here). For the sake of brevity and better exposition, we focus our attention on the analysis of asymptotically equi-Weyl-p-almost periodic functions and asymptotically Weyl-p-almost periodic functions, where $1 \leq p < \infty$ (without any doubt, the case $p = 1$ is most important in our analyses). The class of equi-Weyl-p-almost periodic functions is a subclass of the class of asymptotically Weyl-p-almost periodic functions, whereas any of these classes extend the well-known class of asymptotically Stepanov p-almost periodic functions, introduced by Henríquez [18]. Section 4 also introduces several new classes of “asymptotically almost periodic functions” and analyzes relations between them (we refer the reader to Andres, Bersani, and Grande [5] for an excellent survey of results about various classes of Stepanov and Weyl-almost periodic functions; see also Andres, Bersani, and Leśniak [6]). Section 5 is devoted to the study of Weyl-almost periodic and asympotically Weyl-almost periodic properties of finite and infinite convolution products; for applications, this section is essential. In Section 6, which is written without proofs of structural results, we continue our recent research studies [22], [23] of (Stepanov) almost periodic properties of abstract Volterra integro-differential equations and inclusions (Weyl-almost periodicity is now in focus). Finally, Section 7 is reserved for examples and applications of our established abstract results.

We use standard notation throughout. By X and Y we denote two nontrivial complex Banach spaces. The symbol $L(X,Y)$ designates the space consisting of all continuous linear mappings from X into Y; $L(X) \equiv L(X,X)$. Denote by $C_b([0,\infty) : X)$, $C_0([0,\infty) : X)$, and $BUC([0,\infty) : X)$ the vector spaces consisting of all bounded continuous functions from $[0,\infty)$ into X, all bounded continuous functions from $[0,\infty)$ into X vanishing at infinity, and all bounded uniformly continuous functions from $[0,\infty)$ into X, respectively. Equipped with the usual sup-norm, any of these spaces is a Banach one.

Fractional calculus and fractional differential equations are rapidly growing fields of research (see, e.g., [8], [16], [20], [21], [34], [36], [38]). We note that the most important special functions employed for seeking solutions of fractional differential equations are Mittag-Leffler functions and Wright functions. Suppose that $\gamma \in (0,1)$. Then the Wright function $\Phi_\gamma(\cdot)$ is defined by the formula

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\gamma-\gamma n)}, \quad z \in \mathbb{C}.$$

It is well known that $\Phi_\gamma(\cdot)$ is an entire function, as well as that $\Phi_\gamma(t) \geq 0$, $t \geq 0$, $\int_0^\infty t^r \Phi_\gamma(t)\,dt = \frac{\Gamma(1+r)}{\Gamma(1+\gamma r)}$, $r > -1$, and $\int_0^\infty e^{-zt}\Phi_\gamma(t)\,dt = E_\gamma(-z)$, $z \in \mathbb{C}$, where $E_\gamma(\cdot)$ denotes the Mittag-Leffler function. (For more details about Mittag-Leffler and Wright functions, we refer, for example, to the doctoral dissertation of Bazhlekova [8] and the references cited therein.)

For any $s \in \mathbb{R}$, we denote $\lfloor s \rfloor = \sup\{l \in \mathbb{Z} : s \geq l\}$ and $\lceil s \rceil = \inf\{l \in \mathbb{Z} : s \leq l\}$. If $\alpha > 0$, then we set $g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha)$, $t > 0$.

In this article, we use two different types of fractional derivatives. The Weyl–Liouville fractional derivative $D_{t,+}^\gamma u(t)$ of order $\gamma \in (0,1)$ is defined for those continuous functions $u : \mathbb{R} \to X$ such that $t \mapsto \int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\,ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^\gamma u(t) := \frac{d}{dt}\int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\,ds, \quad t \in \mathbb{R}.$$

Set $D_{t,+}^1 u(t) := -(d/dt)u(t)$. (For more details about Weyl–Liouville fractional derivatives, we refer the reader to Mu, Zhoa, and Peng [30].)

Suppose that $\alpha > 0$ and that $m = \lceil \alpha \rceil$. Then the Caputo fractional derivative $\mathbf{D}_t^\alpha u(t)$ is defined for those functions $u \in C^{m-1}([0,\infty) : X)$ for which $g_{m-\alpha} * (u - \sum_{k=0}^{m-1} u_k g_{k+1}) \in C^m([0,\infty) : X)$, by

$$\mathbf{D}_t^\alpha u(t) = \frac{d^m}{dt^m}\Big[g_{m-\alpha} * \Big(u - \sum_{k=0}^{m-1} u_k g_{k+1}\Big)\Big].$$

Various types of (abstract) fractional differential equations with Caputo derivatives have been investigated in [8], [16], [20], [21], and [26].

2. Multivalued linear operators

The main aim of this section is to give a brief overview of the theory of multivalued linear operators in Banach spaces. In Section 2.1, we consider multivalued linear operators as subgenerators of various classes of degenerate (a,k)-regularized C-resolvent families. (For further information regarding the theory of multivalued linear operators, we refer the reader to monographs by Cross [13] and Favini and Yagi [17].)

A multivalued map (multimap) $\mathcal{A} : X \to P(Y)$ is said to be a *multivalued linear operator* (MLO) if and only if the following conditions hold:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;

(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x, y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

If $X = Y$, then we say that $\mathcal{A}$ is an MLO in X.

Let us recall that if $x, y \in D(\mathcal{A})$ and $\lambda, \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, then $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. If $\mathcal{A}$ is an MLO, then $\mathcal{A}0$ is a linear submanifold of Y and $\mathcal{A}x = f + \mathcal{A}0$ for any $x \in D(\mathcal{A})$ and $f \in \mathcal{A}x$. Put $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. Then the set $\mathcal{A}^{-1}0 = \{x \in D(\mathcal{A}) : 0 \in \mathcal{A}x\}$ is called the *kernel* of $\mathcal{A}$ and it is denoted by $N(\mathcal{A})$. The inverse $\mathcal{A}^{-1}$ of an MLO is defined by $D(\mathcal{A}^{-1}) := R(\mathcal{A})$ and $\mathcal{A}^{-1}y := \{x \in D(\mathcal{A}) : y \in \mathcal{A}x\}$. It can be easily shown that $\mathcal{A}^{-1}$ is an MLO in X, as well as that $N(\mathcal{A}^{-1}) = \mathcal{A}0$ and $(\mathcal{A}^{-1})^{-1} = \mathcal{A}$; $\mathcal{A}$ is said to be *injective* if and only if $\mathcal{A}^{-1}$ is single-valued.

Let $\mathcal{A}, \mathcal{B} : X \to P(Y)$ be two MLOs. Then its sum $\mathcal{A} + \mathcal{B}$ is defined by $D(\mathcal{A}+\mathcal{B}) := D(\mathcal{A}) \cap D(\mathcal{B})$ and $(\mathcal{A}+\mathcal{B})x := \mathcal{A}x + \mathcal{B}x$, $x \in D(\mathcal{A}+\mathcal{B})$. It can be easily checked that $\mathcal{A}+\mathcal{B}$ is an MLO.

Let $\mathcal{A} : X \to P(Y)$ and $\mathcal{B} : Y \to P(Z)$ be two MLOs, where Z is also a complex Banach space. The product of operators $\mathcal{A}$ and $\mathcal{B}$ is defined by $D(\mathcal{B}\mathcal{A}) := \{x \in D(\mathcal{A}) : D(\mathcal{B}) \cap \mathcal{A}x \neq \emptyset\}$ and $\mathcal{B}\mathcal{A}x := \mathcal{B}(D(\mathcal{B}) \cap \mathcal{A}x)$. It can be easily proved that $\mathcal{B}\mathcal{A} : X \to P(Z)$ is an MLO and $(\mathcal{B}\mathcal{A})^{-1} = \mathcal{A}^{-1}\mathcal{B}^{-1}$.

It is well known that the class of MLOs is closed under the action of taking closures. It is said that an MLO operator $\mathcal{A} : X \to P(Y)$ is *closed* if and only if for any sequences (x_n) in $D(\mathcal{A})$ and (y_n) in Y such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$, we have that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply that $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

Let $\mathcal{A}$ be an MLO in X, let $C \in L(X)$ be injective, and let $C\mathcal{A} \subseteq \mathcal{A}C$. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;

(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is called the *C-resolvent* of $\mathcal{A}$ ($\lambda \in \rho_C(\mathcal{A})$); the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ ($\lambda \in \rho(\mathcal{A})$). The basic properties of C-resolvents of single-valued linear operators continue to hold in the multivalued linear setting (see [17], [26]).

Concerning the abstract degenerate Volterra integro-differential equations and abstract degenerate fractional differential equations, the reader may consult the author's forthcoming monograph [26]. For fractional powers and interpolation spaces of MLOs, we refer the interested reader to [17].

2.1. Degenerate (a, k)-regularized C-resolvent families. Let $0 < \tau \leq \infty$, $\alpha > 0$, $a \in L^1_{\text{loc}}([0, \tau))$, $a \neq 0$, $\mathcal{F} : [0, \tau) \to P(Y)$, and let $\mathcal{A} : X \to P(Y)$, $\mathcal{B} : X \to P(Y)$ be two given mappings (possibly nonlinear). We refer the reader to [26] for the notions of various types of solutions to the abstract degenerate inclusion

$$\mathcal{B}u(t) \subseteq \mathcal{A} \int_0^t a(t-s)u(s)\,ds + \mathcal{F}(t), \quad t \in [0, \tau). \tag{2.1}$$

In [26], we recently analyzed the following notions of solution operator families for the abstract Cauchy problem (2.1).

Definition 2.1. Suppose that $0 < \tau \leq \infty$, that $k \in C([0, \tau))$, that $k \neq 0$, that $a \in L^1_{\text{loc}}([0, \tau))$, that $a \neq 0$, $\mathcal{A} : X \to P(X)$ is an MLO, that $C_1 \in L(Y, X)$, and that $C_2 \in L(X)$.

(i) Then it is said that $\mathcal{A}$ is a *subgenerator* of a (local, if $\tau < \infty$) mild (a, k)-regularized (C_1, C_2)-existence and uniqueness family $(R_1(t), R_2(t))_{t \in [0,\tau)} \subseteq L(Y, X) \times L(X)$ if and only if the mappings $t \mapsto R_1(t)y$, $t \geq 0$ and $t \mapsto R_2(t)x$, $t \in [0, \tau)$ are continuous for every fixed $x \in X$ and $y \in Y$, and the following conditions hold:

$$\Big(\int_0^t a(t-s)R_1(s)y\,ds, R_1(t)y - k(t)C_1 y\Big) \in \mathcal{A}, \\ t \in [0, \tau), y \in Y \tag{2.2}$$

and

$$\int_0^t a(t-s)R_2(s)y\,ds = R_2(t)x - k(t)C_2 x, \\ \text{whenever } t \in [0, \tau) \text{ and } (x, y) \in \mathcal{A}. \tag{2.3}$$

(ii) Let $(R_1(t))_{t \in [0,\tau)} \subseteq L(Y, X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a, k)-regularized C_1-existence family $(R_1(t))_{t \in [0,\tau)}$ if and only if (2.2) holds.

(iii) Let $(R_2(t))_{t \in [0,\tau)} \subseteq L(X)$ be strongly continuous. Then it is said that $\mathcal{A}$ is a subgenerator of a (local, if $\tau < \infty$) mild (a, k)-regularized C_2-uniqueness family $(R_2(t))_{t \in [0,\tau)}$ if and only if (2.3) holds.

The notion of an (a, k)-regularized C-resolvent family is introduced as follows.

Definition 2.2. Suppose that $0 < \tau \leq \infty$, that $k \in C([0, \tau))$, that $k \neq 0$, that $a \in L^1_{\text{loc}}([0, \tau))$, that $a \neq 0$, that $\mathcal{A} : X \to P(X)$ is an MLO, that $C \in L(X)$ and that $C\mathcal{A} \subseteq \mathcal{A}C$. Then it is said that a strongly continuous operator family $(R(t))_{t \in [0,\tau)} \subseteq L(X)$ is an *(a, k)-regularized C-resolvent family* with a subgenerator $\mathcal{A}$ if and only if $(R(t))_{t \in [0,\tau)}$ is a mild (a, k)-regularized C-uniqueness family having $\mathcal{A}$ as subgenerator, $R(t)C = CR(t)$, and $R(t)\mathcal{A} \subseteq \mathcal{A}R(t)$ $(t \in [0, \tau))$.

Throughout the article, it will be assumed that any (a, k)-regularized C-resolvent family is also a mild (a, k)-regularized C-existence family; the condition $0 \in \operatorname{supp}(a)$ will also be assumed to be true.

We say that an (a,k)-regularized C-resolvent family $(R(t))_{t\geq 0}$ is *exponentially bounded* (bounded) if and only if there exists $\omega \in \mathbb{R}$ ($\omega = 0$) such that the family $\{e^{-\omega t}R(t) : t \geq 0\} \subseteq L(X)$ is bounded. If $k(t) = g_{\alpha+1}(t)$, where $\alpha \geq 0$, then we also say that $(R(t))_{t\in[0,\tau)}$ is an α-times integrated (a,C)-resolvent family; a 0-times integrated (a,C)-resolvent family is further abbreviated to (a,C)-resolvent family. We pay special attention to the case $a(t) \equiv 1$ (resp., $a(t) \equiv t$) when we say that $(R(t))_{t\geq 0}$ is an α-times integrated C-semigroup (C-semigroup, if $\alpha = 0$) (resp., an α-times integrated C-cosine function (C-cosine function, if $\alpha = 0$)).

By $\chi(R)$ we denote the set consisting of all subgenerators of $(R(t))_{t\in[0,\tau)}$. It is clear that for each subgenerator $\mathcal{A} \in \chi(R)$, we have $\overline{\mathcal{A}} \in \chi(R)$. The set $\chi(R)$ can have infinitely many elements; furthermore, if $\mathcal{A} \in \chi(R)$, then $\mathcal{A} \subseteq \mathcal{A}_{\text{int}}$, where the integral generator of $(R(t))_{t\in[0,\tau)}$ is defined by

$$\mathcal{A}_{\text{int}} := \Big\{(x,y) \in X \times X : R(t)x - k(t)Cx = \int_0^t a(t-s)R(s)y\,ds \text{ for all } t \in [0,\tau)\Big\}.$$

The integral generator $\mathcal{A}_{\text{int}}$ of $(R(t))_{t\in[0,\tau)}$ is always a closed subgenerator of $(R(t))_{t\in[0,\tau)}$, provided that $\tau = \infty$. Assuming $\mathcal{A}$ and $\mathcal{B}$ are two subgenerators of $(R(t))_{t\in[0,\tau)}$ and $\alpha, \beta \in \mathbb{C}$ with $\alpha + \beta = 1$, then $C(D(\mathcal{A})) \subseteq D(\mathcal{B})$, $\mathcal{A}_{\text{int}} \subseteq C^{-1}\mathcal{A}C$ and $\alpha\mathcal{A} + \beta\mathcal{B}$ is also a subgenerator of $(R(t))_{t\in[0,\tau)}$; furthermore, if C is injective, then $\mathcal{A}_{\text{int}} = C^{-1}\mathcal{A}C$. The notion of integral generator of a mild (a,k)-regularized C_2-uniqueness family $(R_2(t))_{t\in[0,\tau)}$ is defined similarly. (We direct the reader to [26] for various characterizations of the above classes of degenerate solution operator families in terms of vector-valued Laplace transform identities.)

3. Stepanov almost periodic functions and asymptotically Stepanov almost periodic functions

Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an *ϵ-period* for $f(\cdot)$ if and only if $\|f(t+\tau) - f(t)\| \leq \epsilon$, $t \in I$. The set consisting of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is *almost periodic*, a.p. for short, if and only if for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in I, which means that there exists $l > 0$ such that any subinterval of I of length l meets $\vartheta(f,\epsilon)$. The space consisting of all almost periodic functions from the interval I into X will be denoted by $AP(I : X)$.

The notion of an asymptotically almost periodic function was introduced by Fréchet in 1941 (for further information concerning the vector-valued asymptotically almost periodic functions, see [10], [15], [31] and the references cited therein). A function $f \in C_b([0,\infty) : X)$ is considered asymptotically almost periodic if and only if for every $\epsilon > 0$ we can find numbers $l > 0$ and $M > 0$ such that every subinterval of $[0,\infty)$ of length l contains at least one number τ such that $\|f(t+\tau) - f(t)\| \leq \epsilon$ for all $t \geq M$. The space consisting of all asymptotically almost periodic functions from $[0,\infty)$ into X is denoted by $AAP([0,\infty) : X)$. For a function $f \in C([0,\infty) : X)$, the following statements are equivalent (see [35]).

(i) We have that $f \in AAP([0,\infty) : X)$.

(ii) There exist uniquely determined functions $g \in AP([0,\infty) : X)$ and $\phi \in C_0([0,\infty) : X)$ such that $f = g + \phi$.

(iii) The set $H(f) := \{f(\cdot + s) : s \geq 0\}$ is relatively compact in $C_b([0,\infty) : X)$.

Let $1 \leq p < \infty$, let $l > 0$, and let $f, g \in L^p_{\text{loc}}(I : X)$, where $I = \mathbb{R}$ or $I = [0,\infty)$. We define the Stepanov "metric" by

$$D^p_{S_l}\big[f(\cdot), g(\cdot)\big] := \sup_{x \in I}\Big[\frac{1}{l}\int_x^{x+l} \big\|f(t) - g(t)\big\|^p \, dt\Big]^{1/p}. \tag{3.1}$$

Then we know that, for every two numbers $l_1, l_2 > 0$, there exist two positive real constants $k_1, k_2 > 0$ independent of f, g, such that

$$k_1 D^p_{S_{l_1}}\big[f(\cdot), g(\cdot)\big] \leq D^p_{S_{l_2}}\big[f(\cdot), g(\cdot)\big] \leq k_2 D^p_{S_{l_1}}\big[f(\cdot), g(\cdot)\big], \tag{3.2}$$

as well as that (see, e.g., [9, pp. 72–73]) in the scalar-valued case there exists

$$D^p_W\big[f(\cdot), g(\cdot)\big] := \lim_{l \to \infty} D^p_{S_l}\big[f(\cdot), g(\cdot)\big] \tag{3.3}$$

in $[0,\infty]$. The distance appearing in (3.3) is called the *Weyl distance* of $f(\cdot)$ and $g(\cdot)$. The Stepanov and Weyl "norms" of $f(\cdot)$ are defined by

$$\|f\|_{S^p_l} := D^p_{S_l}\big[f(\cdot), 0\big] \qquad \text{and} \qquad \|f\|_{W^p} := D^p_W\big[f(\cdot), 0\big],$$

respectively.

Taking into account (3.2), in the rest of this section it will be appropriate to assume that $l_1 = l_2 = 1$. We say that a function $f \in L^p_{\text{loc}}(I : X)$ is *Stepanov p-bounded (S^p-bounded)* if and only if

$$\|f\|_{S^p} := \sup_{t \in I}\Big(\int_t^{t+1} \big\|f(s)\big\|^p \, ds\Big)^{1/p} < \infty.$$

The space $L^p_S(I : X)$ consisting of all S^p-bounded functions becomes a Banach space when equipped with the above norm. A function $f \in L^p_S(I : X)$ is said to be *Stepanov p-almost periodic*, S^p-almost periodic for short, if and only if the function $\hat{f} : I \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I, s \in [0,1],$$

is almost periodic (see [4] for more details). It is said that $f \in L^p_S([0,\infty) : X)$ is *asymptotically Stepanov p-almost periodic (asymptotically S^p-almost periodic)* if and only if $\hat{f} : [0,\infty) \to L^p([0,1] : X)$ is asymptotically almost periodic.

It is a well-known fact that if $f(\cdot)$ is an almost periodic function, then $f(\cdot)$ is also S^p-almost periodic (resp., asymptotically S^p-almost periodic) for $1 \leq p < \infty$. The converse statement is false, however.

Denote by $APS^p(I : X)$ the space consisting of all S^p-almost periodic functions $I \mapsto X$. For any S^p-almost periodic function $f(\cdot)$ and for any real number $\delta \in (0,1)$, we define the function

$$f_\delta(t) := \frac{1}{\delta}\int_t^{t+\delta} f(s) \, ds, \quad t \in I.$$

Arguing as in scalar-valued case (see [9]), we can prove that the function $f_\delta(\cdot)$ is almost periodic ($0 < \delta < 1$) as well as that $\|f_\delta - f\|_{S^p} \to 0$ as $\delta \to 0+$. Hereafter we will also use the Bochner theorem, which asserts that any uniformly continuous function that is Stepanov p-almost periodic needs to be almost periodic ($1 \leq p < \infty$). In the case where the value of p is irrelevant, then we simply say that the function under our consideration is (asymptotically) Stepanov almost periodic. In the rest of this section, we will use the following lemma (see, e.g., [9, p. 70] for the scalar-valued case).

Lemma 3.1. *Let $-\infty < a < b < \infty$, let $1 \leq p' < p'' < \infty$, and let $f \in L^{p''}([a,b] : X)$. Then $f \in L^{p'}([a,b] : X)$ and*

$$\Big[\frac{1}{b-a}\int_a^b \big\|f(s)\big\|^{p'}\,ds\Big]^{1/p'} \leq \Big[\frac{1}{b-a}\int_a^b \big\|f(s)\big\|^{p''}\,ds\Big]^{1/p''}.$$

4. Weyl-almost periodic functions and asymptotically Weyl-almost periodic functions

Unless specified otherwise, in this section it will always be assumed that $I = \mathbb{R}$ or $I = [0, \infty)$. The pivot Banach space will be denoted by X.

The notion of an (equi-)Weyl-almost periodic function is given as follows (see also (3.1)).

Definition 4.1. Let $1 \leq p < \infty$ and $f \in L^p_{\mathrm{loc}}(I : X)$.

(i) We say that the function $f(\cdot)$ is *equi-Weyl-p-almost periodic*, $f \in e - W^p_{\mathrm{ap}}(I : X)$ for short, if and only if for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\sup_{x\in I}\Big[\frac{1}{l}\int_x^{x+l}\big\|f(t+\tau)-f(t)\big\|^p\,dt\Big]^{1/p} \leq \epsilon,$$
$$\text{i.e., } D^p_{S_l}\big[f(\cdot+\tau), f(\cdot)\big] \leq \epsilon.$$

(ii) We say that the function $f(\cdot)$ is *Weyl-p-almost periodic*, $f \in W^p_{\mathrm{ap}}(I : X)$ for short, if and only if for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I' \subseteq I$ of length L contains a point $\tau \in I'$ such that

$$\lim_{l\to\infty}\sup_{x\in I}\Big[\frac{1}{l}\int_x^{x+l}\big\|f(t+\tau)-f(t)\big\|^p\,dt\Big]^{1/p} \leq \epsilon,$$
$$\text{i.e., } \lim_{l\to\infty} D^p_{S_l}\big[f(\cdot+\tau), f(\cdot)\big] \leq \epsilon.$$

Let us recall that

$$APS^p(I : X) \subseteq e - W^p_{\mathrm{ap}}(I : X) \subseteq W^p_{\mathrm{ap}}(I : X)$$

in the set-theoretical sense and that any of these two inclusions can be strict (see [5]). For example, the scalar-valued function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) := \chi_{(0,1/2)}$, $x \in \mathbb{R}$ is not Stepanov 1-almost periodic but it is equi-Weyl-almost-1-periodic (see, e.g., [5, Example 4.27]), and the scalar-valued function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) := \chi_{(0,\infty)}$, $x \in \mathbb{R}$ is not equi-Weyl-almost-1-periodic but it is Weyl-almost-1-

periodic (see, e.g., [6, Example 1]); here, $\chi(\cdot)$ denotes the characteristic function. We also want to point out that the space of scalar-valued functions $W^p_{\mathrm{ap}}(\mathbb{R} : \mathbb{R})$ seems to have been first defined and analyzed by Kovanko [28] in 1944 (according to the information given in the survey paper [5]).

It is well known that for any function $f \in L^p_{\mathrm{loc}}(I : X)$, its Stepanov boundedness is equivalent to its Weyl boundedness, that is,

$$\|f\|_{S^p} < \infty \qquad \text{iff} \qquad \|f\|_{W^p} < \infty.$$

Let $1 \leq p < \infty$ and $f \in L^p_{\mathrm{loc}}(I : X)$. Then $f \in e - W^p_{\mathrm{ap}}(I : X)$ (see [5]) if and only if for every $\epsilon > 0$ there exists a trigonometric X-valued polynomial $P_\epsilon(\cdot)$ such that

$$D^p_W\big[P_\epsilon(\cdot), f(\cdot)\big] < \epsilon.$$

A Bochner-type theorem holds for Weyl-almost periodic functions, as well (see [11]). (For some other notions of Weyl-almost periodicity, like equi-W^p-normality and W^p-normality, we refer the reader to [5, Section 4].)

Theorem 4.2. *Let $1 \leq p < \infty$ and let $f \in W^p_{\mathrm{ap}}(I : X)$ be uniformly continuous. Then $f \in AP(I : X)$.*

In the rest of this section, we use abbreviations $e{-}W_{\mathrm{ap}}(I : X)$ and $W_{\mathrm{ap}}(I : X)$ to denote the spaces $e{-}W^1_{\mathrm{ap}}(I : X) \subseteq W^1_{\mathrm{ap}}(I : X)$, respectively. Similarly, we say that a function is *(equi)-Weyl-almost periodic* if and only if it is (equi)-Weyl-1-almost periodic. It can be easily proved that the limit of any uniformly convergent sequence of bounded continuous functions that are (asymptotically) almost periodic (resp., (asymptotically) Stepanov almost periodic) has again this property. The following result holds for Weyl-almost periodic functions.

Proposition 4.3. *Let (f_n) be a uniformly convergent sequence of functions from $e - W^p(I : X) \cap C_b(I : X)$ (resp., $W^p(I : X) \cap C_b(I : X)$), where $1 \leq p < \infty$. If $f(\cdot)$ is the corresponding limit function, then $f \in e - W^p(I : X) \cap C_b(I : X)$ (resp., $f \in W^p(I : X) \cap C_b(I : X)$).*

Proof. We will prove the statement of the proposition only for the equi-Weyl-p-almost periodic functions. It is clear that $f \in C_b(I : X)$. Let $\epsilon > 0$ be given in advance. Then there exists an integer $n_0(\epsilon)$ such that for each $n \geq n_0(\epsilon)$, we have that

$$\big\|f_n(t) - f(t)\big\| \leq \epsilon, \quad t \in I. \tag{4.1}$$

By definition, we know that there exist two real numbers $l_{n_0} > 0$ and $L_{n_0} > 0$ such that any interval $I' \subseteq I$ of length L_{n_0} contains a point $\tau_{n_0} \in I'$ such that

$$\sup_{x \in I}\Big[\frac{1}{l_{n_0}} \int_x^{x+l_{n_0}} \big\|f_{n_0}(t + \tau_{n_0}) - f_{n_0}(t)\big\|^p \, dt\Big]^{1/p} \leq \epsilon. \tag{4.2}$$

Then, for the proof of equi-Weyl-p-almost periodicity of function $f(\cdot)$, we can choose the same $l := l_{n_0} > 0$ and $L := L_{n_0} > 0$, and the same $\tau := \tau_{n_0}$ from any subinterval $I' \subseteq I$; strictly speaking, we have

$$\big\|f(t+\tau) - f(t)\big\| \leq \big\|f(t+\tau) - f_{n_0}(t+\tau)\big\| + \big\|f_{n_0}(t+\tau) - f_{n_0}(t)\big\| + \big\|f(t) - f_{n_0}(t)\big\|$$

for all $t \in I$, so that a simple calculation involving (4.1) gives the existence of a finite constant $c_p > 0$ such that

$$\sup_{x\in I}\Big[\frac{1}{l}\int_x^{x+l}\big\|f(t+\tau)-f(t)\big\|^p\,dt\Big]^{1/p}$$
$$\leq c_p\Big[\epsilon+\sup_{x\in I}\Big[\frac{1}{l_{n_0}}\int_x^{x+l_{n_0}}\big\|f_{n_0}(t+\tau)-f_{n_0}(t)\big\|^p\,dt\Big]^{1/p}\Big]\leq 2c_p\epsilon.$$

Then the final result simply follows from (4.2). □

4.1. Asymptotically Weyl-almost periodic functions. For the start, we need to introduce the following notion. If $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$, then we define the function $\mathbf{q}(\cdot,\cdot) : [0,\infty)\times[0,\infty) \to X$ by

$$\mathbf{q}(t,s) := q(t+s), \quad t,s \geq 0.$$

Definition 4.4. It is said that $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$ is *Weyl-p-vanishing* if and only if

$$\lim_{t\to\infty}\big\|\mathbf{q}(t,\cdot)\big\|_{W^p}=0, \quad \text{i.e.,}\ \lim_{t\to\infty}\lim_{l\to\infty}\sup_{x\geq 0}\Big[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p}=0. \tag{4.3}$$

It is clear that for any function $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$ we can replace the limits in (4.3). We say that $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$ is *equi-Weyl-p-vanishing* if and only if

$$\lim_{l\to\infty}\lim_{t\to\infty}\sup_{x\geq 0}\Big[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p}=0. \tag{4.4}$$

Since the second limit in (4.3) always exists in $[0,\infty]$ (on account of (3.3)) and the second limit in (4.4) always exists in $[0,\infty]$ (on account of the fact that the mapping $t \mapsto \sup_{x\geq 0}\frac{1}{l}\int_x^{x+l}\|q(t+s)^p\|\,ds^{1/p}$, $t \geq 0$ is monotonically decreasing), the condition (4.3) is equivalent with

$$\forall\epsilon>0\ \exists t_0(\epsilon)>0\ \forall t\geq t_0(\epsilon)\ \exists l_t>0\ \forall l>l_t:$$
$$\sup_{x\geq 0}\Big[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p}\leq\epsilon, \tag{4.5}$$

while the condition (4.4) is equivalent with

$$\forall\epsilon>0\ \exists l_0(\epsilon)>0\ \forall l\geq l_0(\epsilon)\ \exists t_l>0\ \forall t>t_l:$$
$$\sup_{x\geq 0}\Big[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p}\leq\epsilon. \tag{4.6}$$

Before proceeding further, we would like to observe that there is a great number of very simple examples showing that, for a function $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$, the situation in which $\|\mathbf{q}(t,\cdot)\|_{W^p} \neq \|\mathbf{q}(t',\cdot)\|_{W^p}$ for all $t \neq t'$ can occur. Consider, for instance, the function $q(t) := 2^{-1}(t+1)^{(-1)/2}$, $t \geq 0$ and the case in which $p = 1$; then a direct computation yields that $\|\mathbf{q}(t,\cdot)\|_{W^p} = (t+1)^{(-1)/2}$, $t \geq 0$.

(1) Assume that $q \in L^p([0,\infty) : X)$. Then for each $\epsilon > 0$ there exists $t_0(\epsilon) > 0$ such that $\int_t^\infty \|q(s)\|^p\,ds \leq \epsilon^p$, $t \geq t_0(\epsilon)$. In particular, $\int_t^{t+1}\|q(s)\|^p\,ds \leq \epsilon^p$, $t \geq t_0(\epsilon)$ and the function $\hat{q} : [0,\infty) \to L^p([0,1] : X)$ belongs to the class

$C_0([0,\infty) : L^p([0,1] : X))$. The converse statement is not true, however, since the scalar-valued function $q(t) = t^{(-1)/2p}$, $t > 0$ satisfies that $\hat{q} \in C_0([0,\infty) : L^p([0,1] : X))$ and $q \notin L^p([0,\infty) : X)$.

(2) If $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$ and $\hat{q} \in C_0([0,\infty) : L^p([0,1] : X))$, then the computation

$$\begin{aligned}&\sup_{x\geq 0}\Big[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p}\\ &\quad\leq\Big[\frac{1}{l}\Big(\int_{x+t}^{x+t+1}+\cdots+\int_{x+t+\lceil l\rceil-1}^{x+t+\lceil l\rceil}\Big)\big\|q(s)\big\|^p\,ds\Big]^{1/p}\leq\Big(\epsilon\frac{\lceil l\rceil}{l}\Big)^{1/p}\leq 2^p\epsilon,\end{aligned}$$

holding for any $t \geq 0$, shows that the function $q(\cdot)$ is equi-Weyl-p-vanishing, with $l_0(\epsilon) = 1$ and $t_l = t_0(\epsilon)$ chosen so that $\int_t^{t+1}\|q(s)\|^p\,ds \leq \epsilon^p$, $t \geq t_0(\epsilon)$ $(l > l_0(\epsilon))$. As the following simple counterexample shows, the converse statement does not hold in general.

Example 4.5. Define

$$q(t) := \sum_{n=0}^{\infty}\chi_{[n^2,n^2+1]}(t), \quad t \geq 0.$$

Since $\int_{n^2}^{n^2+1}\|q(s)\|^p\,ds = 1$, $n \in \mathbb{N}$, it is clear that $\hat{q} \notin C_0([0,\infty) : L^p([0,1] : X))$. On the other hand, the interval $[t, t+l]$ contains at most $\sqrt{t+l} - \sqrt{t} + 2$ squares of nonnegative integers, so that

$$\begin{aligned}\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds &\leq \sup_{x\geq t}\frac{1}{l}\int_t^{t+l}\big\|q(s)\big\|^p\,ds\\ &\leq\frac{1}{l}(\sqrt{t+l}-\sqrt{t}+2)\leq\frac{1}{l}\Big(2+\frac{l}{\sqrt{t}+\sqrt{l}}\Big),\quad x\geq 0, t\geq 0,\end{aligned}$$

so that (4.6) holds with $l_0(\epsilon) > 0$ sufficiently large and $t_l = l$ $(l \geq l_0(\epsilon))$.

(3) If $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$ and $q(\cdot)$ is equi-Weyl-p-vanishing, then $q(\cdot)$ is Weyl-p-vanishing. To see this, assume that (4.6) holds with $l_0(\epsilon) > 0$ and put after that $t_0(\epsilon) := t_{l_0(\epsilon)}$. Therefore,

$$\sup_{x\geq 0}\Big[\frac{1}{l_0(\epsilon)}\int_x^{x+l_0(\epsilon)}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p}\leq\epsilon,\quad t\geq t_0(\epsilon). \tag{4.7}$$

For any fixed $t \geq t_0(\epsilon)$, we set $l_t := l_0(\epsilon)$. Then it suffices to show that for any $l > l_t$, we have

$$\sup_{x\geq 0}\Big[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p}\leq 2\epsilon.$$

This follows from (4.7) and a simple analysis involving the second inequality in [6, Proposition 1(i)]:

$$\sup_{x\geq 0}\Big[\frac{1}{l}\int_x^{x+l}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p}\leq 2^{1/p}\sup_{x\geq 0}\Big[\frac{1}{l_0(\epsilon)}\int_x^{x+l_0(\epsilon)}\big\|q(t+s)\big\|^p\,ds\Big]^{1/p},$$

which is valid for any $l > l_t = l_0(\epsilon)$. Again, the converse statement does not hold in general, and a Weyl-p-vanishing function need not be equi-Weyl-p-vanishing.

Example 4.6. Define

$$q(t) := \sum_{n=0}^{\infty} \sqrt{n}\chi_{[n^2,n^2+1]}(t), \quad t \geq 0.$$

Then it is clear that

$$\frac{1}{l}\int_x^{x+l} \|q(t+s)\|^p \, ds \leq \sup_{x\geq t} \frac{1}{l}\int_t^{t+l} \|q(s)\|^p \, ds \leq \sqrt{\frac{t+l}{l^2}}, \quad x \geq 0, t \geq 0,$$

so that (4.5) holds with $t_0(\epsilon) > 0$ chosen so that $\sqrt{1/(t+1)} \leq \epsilon$ for $t \geq t_0(\epsilon)$ and $l_t = (t+1)^2$. Hence, $q(\cdot)$ is Weyl-p-vanishing. On the other hand, $q(\cdot)$ cannot be equi-Weyl-p-vanishing because for each number $l > 1$ there does not exist a finite limit

$$\lim_{t\to\infty} \sup_{x\geq 0} \Big[\frac{1}{l}\int_x^{x+l} \|q(t+s)\|^p \, ds\Big]^{1/p}.$$

To see this, it suffices to observe that for each $t > 0$ and $n \in \mathbb{N}$ such that $n^2 > t$, we have

$$\sup_{x\geq 0} \frac{1}{l}\int_x^{x+l} \|q(t+s)\|^p \, ds \geq \frac{\sqrt{n}}{l} > \frac{t}{l}.$$

Before proceeding further, we would like to note that an equi-Weyl-p-vanishing function $q(\cdot)$ need not be bounded as $t \to +\infty$ (this, certainly, implies that the notion of asymptotically equi-Weyl-almost periodicity is very general compared with the usually considered notion of asymptotically almost periodicity).

Example 4.7. Define

$$q(t) := \sum_{n=0}^{\infty} n^{1/4p}\chi_{[n^4,n^4+1]}(t), \quad t \geq 0.$$

Then, similarly as in Example 4.5, we can prove that

$$\begin{aligned} \frac{1}{l}\int_x^{x+l} \|q(t+s)\|^p \, ds &\leq \sup_{x\geq t} \frac{1}{l}\int_t^{t+l} \|q(s)\|^p \, ds \\ &\leq \frac{1}{l}\Big(2 + \frac{l}{\sqrt{t}+\sqrt{l}}\Big), \quad x \geq 0, t \geq 0, \end{aligned}$$

which implies the required conclusions.

Denote by $W_0^p([0,\infty) : X)$ and $e - W_0^p([0,\infty) : X)$ the sets consisting of all Weyl-p-vanishing functions and equi-Weyl-p-vanishing functions, respectively. The symbol $S_0^p([0,\infty) : X)$ will be used to denote the set of all functions $q \in L^p_{\rm loc}([0,\infty) : X)$ such that $\hat{q} \in C_0([0,\infty) : L^p([0,1] : X))$. By our considerations in points (1)–(3), Example 4.5, and Example 4.6, we have the following result.

Theorem 4.8. *The inclusions*

$$L^p\big([0,\infty):X\big) \subseteq S_0^p\big([0,\infty):X\big) \subseteq e-W_0^p\big([0,\infty):X\big) \subseteq W_0^p\big([0,\infty):X\big)$$

hold, and any of them can be strict.

We introduce the following function spaces:

$$\begin{aligned} AAPW^p\big([0,\infty):X\big) &:= AP\big([0,\infty):X\big) + W_0^p\big([0,\infty):X\big),\\ e-AAPW^p\big([0,\infty):X\big) &:= AP\big([0,\infty):X\big) + e-W_0^p\big([0,\infty):X\big),\\ AAPSW^p\big([0,\infty):X\big) &:= APS^p\big([0,\infty):X\big) + W_0^p\big([0,\infty):X\big),\\ e-AAPSW^p\big([0,\infty):X\big) &:= APS^p\big([0,\infty):X\big) + e-W_0^p\big([0,\infty):X\big),\\ e-W^p_{\mathrm{aap}}\big([0,\infty):X\big) &:= e-W^p_{\mathrm{ap}}\big([0,\infty):X\big) + W_0^p\big([0,\infty):X\big),\\ ee-W^p_{\mathrm{aap}}\big([0,\infty):X\big) &:= e-W^p_{\mathrm{ap}}\big([0,\infty):X\big) + e-W_0^p\big([0,\infty):X\big),\\ W^p_{\mathrm{aap}}\big([0,\infty):X\big) &:= W^p_{\mathrm{ap}}\big([0,\infty):X\big) + W_0^p\big([0,\infty):X\big),\\ W^p_{\mathrm{eaap}}\big([0,\infty):X\big) &:= W^p_{\mathrm{ap}}\big([0,\infty):X\big) + e-W_0^p\big([0,\infty):X\big). \end{aligned}$$

Then it is clear that

$$\begin{aligned} AAPW^p\big([0,\infty):X\big) &\subseteq AAPSW^p\big([0,\infty):X\big)\\ &\subseteq e-W^p_{\mathrm{aap}}\big([0,\infty):X\big) \subseteq W^p_{\mathrm{aap}}\big([0,\infty):X\big) \end{aligned}$$

and

$$\begin{aligned} e-AAPW^p\big([0,\infty):X\big) &\subseteq e-AAPSW^p\big([0,\infty):X\big)\\ &\subseteq ee-W^p_{\mathrm{aap}}\big([0,\infty):X\big) \subseteq W^p_{\mathrm{eaap}}\big([0,\infty):X\big) \end{aligned}$$

and that any of these inclusions can be strict.

By the analysis contained in [5, Example 4.27], the function $f:[0,\infty)\to\mathbb{C}$ defined by $f(t):=\chi_{(0,1/2)}(t)$, $t\geq 0$ is equi-Weyl-almost periodic; since this function is also in class $e-W_0^1([0,\infty):X)$, we have that the sums defining $e-W^p_{\mathrm{aap}}([0,\infty):X)$, $ee-W^p_{\mathrm{aap}}([0,\infty):X)$, $W^p_{\mathrm{aap}}([0,\infty):X)$, and $W^p_{\mathrm{eaap}}([0,\infty):X)$ cannot be direct. For the first four spaces $AAPW^p([0,\infty):X)$, $e-AAPW^p([0,\infty):X)$, $AAPSW^p([0,\infty):X)$, and $e-AAPSW^p([0,\infty):X)$, the sums in their definitions are direct, which follows from the following proposition.

Proposition 4.9. *Let* $1\leq p<\infty$. *Then* $W_0^p([0,\infty):X)\cap APS^p([0,\infty):X)=\{0\}$.

Proof. Assume that $q\in W_0^p([0,\infty):X)\cap APS^p([0,\infty):X)$. In order to see that $q(t)=0$ for almost everywhere $t\geq 0$, it suffices to show that $\hat{q}(t)=0$, $t\geq 0$ in $L^p([0,1]:X)$. Since $\hat{q}(\cdot)$ is almost periodic, we only need to prove that any Bohr–Fourier coefficient of $\hat{q}(\cdot)$ is equal to zero, that is, that

$$\lim_{t\to\infty}\Big(\int_0^1\Big\|\frac{1}{t}\int_0^t e^{-irs}q(s+v)\,ds\Big\|^p\,dv\Big)^{1/p}=0,\quad r\in\mathbb{R}. \tag{4.8}$$

To see that (4.8) holds good, observe first that

$$\Big(\int_0^1\Big\|\frac{1}{t}\int_0^t e^{-irs}q(s+v)\,ds\Big\|^p\,dv\Big)^{1/p}\leq\frac{1}{t}\Big(\int_0^1\Big[\int_0^t\|q(s+v)\|\,ds\Big]^p\,dv\Big)^{1/p},$$

which can be further majorized by using Lemma 3.1:

$$\leq\frac{1}{t}\Big(\int_0^1 t^{p-1}\int_0^t\|q(s+v)\|^p\,ds\,dv\Big)^{1/p}=t^{(-1)/p}\Big(\int_0^1\int_0^t\|q(s+v)\|^p\,ds\,dv\Big)^{1/p}.$$

Hence, we need to prove that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^1\int_0^t\|q(s+v)\|^p\,ds\,dv=\lim_{t\to\infty}\frac{1}{t}\int_0^1\int_s^{s+t}\|q(r)\|^p\,dr\,dv=0. \tag{4.9}$$

Let $\epsilon>0$ be given in advance. Since $q\in W_0^p([0,\infty):X)$, we know that there exist two finite numbers $t_0(\epsilon)>0$ and $l_0(\epsilon)>0$ such that, for every $l>l_0(\epsilon)$, we have

$$\sup_{x\geq 0}\Big[\frac{1}{l}\int_x^{x+l}\big\|q\big(t_0(\epsilon)+s\big)\big\|^p\,ds\Big]^{1/p}\leq\epsilon. \tag{4.10}$$

Let $T_0(\epsilon)>0$ be such that, for each $t>T_0(\epsilon)$, we have

$$t\geq t_0(\epsilon)^2\qquad\text{and}\qquad t-\sqrt{t}\geq l_0(\epsilon). \tag{4.11}$$

The validity of (4.11) clearly implies by (4.10) that

$$\frac{1}{t-\sqrt{t}}\int_{s+\sqrt{t}}^{s+t}\|q(s)\|^p\,ds\leq\epsilon,\quad s\in[0,1]. \tag{4.12}$$

Since

$$\begin{aligned}&\frac{1}{t}\int_s^{s+t}\|q(r)\|^p\,dr\\&=\frac{1}{t}\Big(\int_s^{s+1}+\int_{s+1}^{s+2}+\cdots+\int_{s+\lceil\sqrt{t}\rceil-1}^{s+\lceil\sqrt{t}\rceil}\Big)\|q(r)\|^p\,dr\\&\quad+\frac{1}{t}\Big(\int_{s+\lceil\sqrt{t}\rceil}^{s+\lceil\sqrt{t}\rceil+1}+\cdots+\int_{s+\lfloor t\rfloor}^{s+t}\Big)\|q(r)\|^p\,dr\\&\leq\frac{\lceil\sqrt{t}\rceil}{t}\|q\|_{S^p}+\frac{t-\lfloor t\rfloor}{t}\epsilon\end{aligned}$$

by S^p-boundedness of $q(\cdot)$ and (4.12), the equation (4.9) holds true. The proof of the proposition is thereby complete. □

It is very simple to prove that $W_0^p([0,\infty):X)$ and $W_0^p([0,\infty):X)$ are vector spaces, so that the introduced eight function spaces have a linear vector structure. Disregarding the term $([0,\infty):X)$, and taking into consideration the previously defined spaces AAP and $AAPS^p$, we have the following inclusion diagram of "asymptotically almost periodic function spaces" (see Theorem 4.8):

$$\begin{array}{ccccc} AAP & \subseteq & e-AAPW^p & \subseteq & AAPW^p \\ \cap\!\!\!| & & \cap\!\!\!| & & \cap\!\!\!| \\ AAPS^p & \subseteq & e-AAPSW^p & \subseteq & AAPSW^p \\ & & \cap\!\!\!| & & \cap\!\!\!| \\ & & ee-W^p_{\mathrm{aap}} & \subseteq & e-W^p_{\mathrm{aap}} \\ & & \cap\!\!\!| & & \cap\!\!\!| \\ & & W^p_{\mathrm{eaap}} & \subseteq & W^p_{\mathrm{aap}}. \end{array}$$

By the foregoing, any inclusion of this diagram can be strict. Furthermore, for any two function spaces A and B belonging to this diagram and satisfying additionally that there is no transitive sequence of inclusions connecting either A and B, or B and A, we have that $A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$ (the diagram can be expanded by constructing the sums of spaces of (equi-)Weyl-almost periodic functions with $S^p_0([0,\infty) : X)$ and the space $W_pPAA_0(\mathbb{R} : X)$ defined below, which will not be examined here). Abbas [2] introduced the notions of a Weyl-p-pseudo-almost automorphic function and a Weyl-p-pseudoergodic component.

Definition 4.10. Let $p \geq 1$. Then we say that a function $q \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ is a *Weyl-p-pseudoergodic component* if and only if it satisfies

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \Big[\lim_{l\to+\infty} \frac{1}{2l} \int_{x-l}^{x+l} \|q(t)\|^p \, dt \Big]^{1/p} dx = 0. \tag{4.13}$$

The set of all such functions is denoted by $W_pPAA_0(\mathbb{R} : X)$.

Now we will prove that the class of Weyl-p-vanishing functions (extended by zero outside $[0,\infty)$) is contained in $W_pPAA_0(\mathbb{R} : X)$.

Proposition 4.11. *Let $1 \leq p < \infty$, and let $q \in L^p_{\mathrm{loc}}([0,\infty) : X)$ be a Weyl-p-vanishing function. Let $q_e \in L^p_{\mathrm{loc}}(\mathbb{R} : X)$ be given by $q_e(t) := q(t)$, $t \geq 0$ and $q_e(t) := 0$, $t < 0$. Then $q_e \in W_pPAA_0(\mathbb{R} : X)$.*

Proof. We only need to prove that (4.13) holds with $q(\cdot)$ replaced therein with $q_e(\cdot)$, that is, that

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{0}^{T} \Big[\lim_{l\to+\infty} \frac{1}{2l} \int_{0}^{x+l} \|q(t)\|^p \, dt \Big]^{1/p} dx = 0.$$

Let $x \in [0,T]$ be fixed. It suffices to show that

$$\lim_{l\to+\infty} \frac{1}{2l} \int_{0}^{x+l} \|q(t)\|^p \, dt = 0. \tag{4.14}$$

To this end, fix a number $\epsilon > 0$. Owing to the fact that $q(\cdot)$ is Weyl-p-vanishing, that is, that

$$\lim_{t\to\infty} \lim_{l\to\infty} \sup_{x\geq t} \Big[\frac{1}{l} \int_{x}^{x+l} \|q(s)\|^p \, ds \Big]^{1/p} = 0,$$

we have the existence of numbers $l_0(\epsilon) > 0$ and $t_0(\epsilon) > 0$ such that

$$\frac{1}{2l}\int_x^{x+l}\|q(s)\|^p\,ds \leq \epsilon, \quad x \geq t_0(\epsilon), l \geq l_0(\epsilon). \tag{4.15}$$

Then we have

$$\frac{1}{2l}\int_0^{x+l}\|q(t)\|^p\,dt \leq \frac{1}{2l}\Big[\int_0^{x}\|q(s)\|^p\,ds + \int_x^{x+l}\|q(s)\|^p\,ds\Big], \quad l > 0.$$

If $x \geq t_0(\epsilon)$, then the addend $(1/2l)\int_x^{x+l}\|q(s)\|^p\,ds$ is less than or equal to ϵ by (4.15), which clearly implies the existence of a number $l_1(\epsilon) > 0$ such that for each $l \geq l_1(\epsilon)$, we have

$$\frac{1}{2l}\int_0^{x+l}\|q(t)\|^p\,dt \leq 2\epsilon.$$

If $x < t_0(\epsilon)$, then we have

$$\begin{aligned}\frac{1}{2l}\int_x^{x+l}\|q(t)\|^p\,dt &\leq \frac{1}{2l}\Big[\int_x^{t_0(\epsilon)}\|q(s)\|^p\,ds + \int_{t_0(\epsilon)}^{t_0(\epsilon)+l}\|q(s)\|^p\,ds\Big] \\ &\leq \frac{1}{2l}\int_x^{t_0(\epsilon)}\|q(s)\|^p\,ds + \epsilon, \quad l \geq l_1(\epsilon),\end{aligned}$$

which clearly implies the existence of a number $l_2(\epsilon) > l_1(\epsilon)$ such that for each $l \geq l_2(\epsilon)$, we have

$$\frac{1}{2l}\int_0^{x+l}\|q(t)\|^p\,dt \leq 2\epsilon.$$

This yields (4.14) and completes the proof of Proposition 4.11. □

We round off the section by introducing the following definition.

Definition 4.12. Let $I = \mathbb{R}$ or $I = [0, \infty)$, let $(R(t))_{t\in I} \subseteq L(X)$ be a strongly continuous operator family, and let $\oplus$ denote any of ((equi-)Weyl, Stepanov, asymptotically) almost periodic properties considered above. Then we say that $(R(t))_{t\in I}$ is $\oplus$ ((equi-)Weyl, Stepanov, asymptotically) almost periodic if and only if the mapping $t \mapsto R(t)x$, $t \in I$ is $\oplus$ (asymptotically) almost periodic for all $x \in X$.

5. Weyl-almost periodic and asymptotically Weyl-almost periodic properties of convolution products

The main aim of this section is to investigate the (asymptotically) Weyl-almost periodic properties of finite and infinite convolution products. We first state the following result.

Proposition 5.1.

(i) *Suppose that* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $\int_0^\infty \|R(s)\|\,ds < \infty$. *If* $g : \mathbb{R} \to X$ *is bounded and (equi)-Weyl-almost periodic, then the function* $G(\cdot)$, *given by*

$$G(t) := \int_{-\infty}^{t} R(t-s)g(s)\,ds, \quad t \geq 0, \tag{5.1}$$

is bounded and (equi)-Weyl-almost periodic, as well.

(ii) *Suppose that* $(R(t))_{t>0} \subseteq L(X,Y)$ *is a strongly continuous operator family satisfying that* $M = \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^\infty[k,k+1]} < \infty$. *If* $g : \mathbb{R} \to X$ *is equi-Weyl-almost periodic, then the function* $G(\cdot)$, *given by* (5.1), *is bounded and equi-Weyl-almost periodic, as well.*

Proof. We will prove (i) only in the case in which $g : \mathbb{R} \to Y$ is bounded and Weyl-almost periodic. It is clear that, for every $t \in \mathbb{R}$, we have that $G(t)$ is well defined and

$$\big\|G(t)\big\| \leq \|g\|_\infty \int_0^\infty \big\|R(s)\big\|\,ds.$$

In particular, $G(\cdot)$ belongs to the space $L^p_{\mathrm{loc}}(\mathbb{R} : X)$ with all its translations, so that (3.3) implies the existence of the limit

$$\lim_{l\to\infty} D_{S_l}\big[G(\cdot+\tau), G(\cdot)\big]$$

for any $\tau \in \mathbb{R}$. Let a number $\epsilon > 0$ be given in advance. Then we can find two finite numbers $l_\epsilon > 0$ and $L_\epsilon > 0$ such that any subinterval I of $\mathbb{R}$ of length L_ϵ contains a number $\tau \in I$ such that

$$\sup_{x\in\mathbb{R}} \frac{1}{l} \int_x^{x+l} \big\|g(t+\tau) - g(t)\big\|\,dt \leq \epsilon, \quad l \geq l_\epsilon. \tag{5.2}$$

It remains to prove that for any such τ we have

$$\lim_{l\to\infty} D_{S_l}\big[G(\cdot+\tau), G(\cdot)\big] \leq \epsilon,$$

whose validity immediately follows if we prove that

$$\sup_{x\in\mathbb{R}} \frac{1}{l} \int_x^{x+l} \big\|G(t+\tau) - G(t)\big\|\,dt \leq \text{Const.}\epsilon, \quad l \geq l_\epsilon. \tag{5.3}$$

To see that (5.3) holds, we can argue as follows. Applying Fubini's theorem and (5.2), we get that, for every $x \in \mathbb{R}$ and $l \geq l_\epsilon$,

$$\begin{aligned}
&\frac{1}{l} \int_x^{x+l} \big\|G(t+\tau) - G(t)\big\|\,dt \\
&\quad \leq \frac{1}{l} \int_x^{x+l} \Big[\int_0^\infty \big\|R(s)\big\| \big\|g(t+\tau-s) - g(t-s)\big\|\,ds\Big]\,dt \\
&\quad \leq \int_0^\infty \Big[\big\|R(s)\big\| \frac{1}{l} \int_x^{x+l} \big\|g(t+\tau-s) - g(t-s)\big\|\,dt\Big]\,ds
\end{aligned}$$

$$\le \epsilon \int_0^{\infty} \|R(s)\| \, ds.$$

This completes the proof of (i). For the proof of (ii), we first recall that any equi-Weyl-almost periodic function needs to be Stepanov (Weyl, equivalently) bounded, so that our assumption $M < \infty$, taken together with the proof of [27, Proposition 2.6.11], shows that the function $G(\cdot)$ is well defined and bounded on the real line. The remaining part of the proof is essentially the same as that in part (i). □

Remark 5.2. It is not clear how to consider the case in which $g : \mathbb{R} \to Y$ is (equi)-Weyl-p-almost periodic for some $p > 1$.

For any locally integrable function $q \in L^1_{\text{loc}}(\mathbb{R} : X)$ and for any strongly continuous operator family $(R(t))_{t>0} \subseteq L(X, Y)$ satisfying $\int_0^\infty \|R(s)\| \, ds < \infty$, we formally set

$$J(t,l) := \sup_{x \ge 0} \Big\{ \int_0^{x+t} \Big[\frac{1}{l} \int_{x+t-r}^{x+t-r+l} \|R(v)\| \, dv \Big] \|q(r)\| \, dr \Big\}, \quad t > 0, l > 0.$$

Consider the conditions

$$\lim_{t\to\infty} \lim_{l\to\infty} J(t,l) = 0 \tag{5.4}$$

and

$$\lim_{l\to\infty} \lim_{t\to\infty} J(t,l) = 0. \tag{5.5}$$

The main purpose of the following proposition is to investigate the asymptotically Weyl-almost periodic properties of finite convolution products (Weyl-p-vanishing functions and equi-Weyl-p-vanishing functions behave here much better than Weyl-p-pseudoergodic components).

Proposition 5.3.

(i) *Suppose that $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying that $\int_0^\infty \|R(s)\| \, ds < \infty$. If $g : \mathbb{R} \to X$ is bounded and Weyl-almost periodic and $q \in W^1_0([0, \infty) : X)$ (resp., $q \in e - W^1_0([0, \infty) : X)$) satisfies (5.4) (resp., (5.5)), then the function $F(\cdot)$, given by*

$$F(t) := \int_0^t R(t-s) \big[g(s) + q(s) \big] \, ds, \quad t \ge 0, \tag{5.6}$$

is in class $W^1_{\text{aap}}([0, \infty) : Y)$ (resp., $W^1_{\text{eaap}}([0, \infty) : Y)$).

(ii) *Let the requirements of part (i) hold with $g : \mathbb{R} \to X$ being bounded and equi-Weyl-almost periodic as well as with the function $q(\cdot)$ satisfying the same conditions as in (i). Then the function $F(\cdot)$, given by (5.6), is in class $e - W^1_{\text{aap}}([0, \infty) : X)$ (resp., $ee - W^1_{\text{aap}}([0, \infty) : X)$).*

(iii) *Suppose that $(R(t))_{t>0} \subseteq L(X, Y)$ is a strongly continuous operator family satisfying that $M = \sum_{k=0}^\infty \|R(\cdot)\|_{L^\infty[k,k+1]} < \infty$. If $g : \mathbb{R} \to X$ is equi-Weyl-almost periodic and the function $q(\cdot)$ satisfies the same conditions as in (i), then the function $F(\cdot)$, given by (5.6), is in class $e - W^1_{\text{aap}}([0, \infty) : X)$ (resp., $ee - W^1_{\text{aap}}([0, \infty) : X)$).*

We will only prove part (i). By Proposition 5.1, $G(\cdot)$ is bounded and Weyl-almost periodic. Define

$$F(t) := \int_0^t R(t-s)q(s)\,ds - \int_t^\infty R(s)g(t-s)\,ds, \quad t \geq 0. \tag{5.7}$$

The measurability of integrand functions and the local integrability of convolution products in (5.7) follow from the proofs of [7, Propositions 1.3.4, 1.3.5]. Since $H(t) = G(t) + F(t)$ for all $t \geq 0$, and

$$\Big\|\int_t^\infty R(s)g(t-s)\,ds\Big\| \leq \|g\|_\infty \int_t^\infty \big\|R(s)\big\|\,ds \to 0, \quad t \to +\infty, \tag{5.8}$$

it suffices to show that the function $t \to L(t) := \int_0^t R(t-s)q(s)\,ds$, $t \geq 0$ is in class $W_0^1([0,\infty) : X)$ (resp., $e - W_0^1([0,\infty) : X)$), provided that $q \in W_0^1([0,\infty) : X)$ (resp., $q \in e - W_0^1([0,\infty) : X)$) satisfies (5.4) (resp., (5.5)). Clearly, for every $x \geq 0$ and $l > 0$, we have by an elementary argumentation involving Fubini's theorem:

$$\begin{aligned}
\frac{1}{l}\int_{x+t}^{x+t+l} \big\|q(s)\big\|\,ds &\leq \frac{1}{l}\int_{x+t}^{x+t+l}\Big[\int_0^s \big\|R(s-r)\big\|\big\|q(r)\big\|\,dr\Big]\,ds \\
&\leq \int_0^{x+t}\Big[\frac{1}{l}\int_{x+t}^{x+t+l}\big\|R(s-r)\big\|\,ds\Big]\big\|q(r)\big\|\,dr \\
&\quad + \int_{x+t}^{x+t+l}\Big[\frac{1}{l}\int_r^{x+t+l}\big\|R(s-r)\big\|\,ds\Big]\big\|q(r)\big\|\,dr.
\end{aligned}$$

For the estimation of the second addend, we can use the inequality

$$\begin{aligned}
&\int_{x+t}^{x+t+l}\Big[\frac{1}{l}\int_r^{x+t+l}\big\|R(s-r)\big\|\,ds\Big]\big\|q(r)\big\|\,dr \\
&\quad\leq \int_{x+t}^{x+t+l}\Big[\int_0^\infty \big\|R(v)\big\|\,dv\Big]\frac{1}{l}\big\|q(r)\big\|\,dr \\
&\quad\leq \Big[\int_0^\infty \big\|R(v)\big\|\,dv\Big]\cdot\frac{1}{l}\int_{x+t}^{x+t+l}\big\|q(r)\big\|\,dr, \quad x \geq 0, l > 0;
\end{aligned}$$

therefore, since $q \in W_0^1([0,\infty) : X)$ (resp., $q \in e - W_0^1([0,\infty) : X)$), we have that the second addend is in the same class, as well, by the uniform integrability of $\|R(\cdot)\|$. For the first addend, it suffices to observe that condition (5.4) (resp., (5.5)) holds for $q(\cdot)$.

Example 5.4. It is very simple to prove that (5.5) holds provided that $(R(t))_{t\geq 0} \subseteq L(X,Y)$ is exponentially decaying, as well as that there exists a finite constant $M \geq 1$ such that

$$\int_0^t e^{\omega(t-s)}\big\|q(s)\big\|\,ds \leq M, \quad t \geq 0,$$

where $\omega < 0$ denotes the exponential growth bound of $(R(t))_{t\geq 0}$.

Example 5.5. Assume that there exist two numbers $a \in (0,1)$ and $b \in (1,\infty)$ satisfying $\|R(t)\| \leq Mt^{-a}$, $t \in (0,1)$ and $\|R(t)\| \leq Mt^{-b}$, $t \geq 1$, so that Proposition (5.3)(ii)–(iii) can be applied; here, $M \geq 1$ is a finite constant. Since for any $l \geq 1$ we have

$$\int_0^{x+t} \Bigl[\frac{1}{l}\int_{x+t-r}^{x+t-r+l} \|R(v)\|\,dv\Bigr]\|q(r)\|\,dr$$
$$\leq \frac{M}{l(1-a)}\int_0^{x+t}\bigl\{(x+t-r+1)^{1-a}-(x+t-r)^{1-a}\bigr\}\|q(r)\|\,dr$$
$$+\frac{M}{l(b-1)}\int_0^{x+t}\bigl\{(x+t-r+1)^{1-b}-(x+t-r+l)^{1-b}\bigr\}\|q(r)\|\,dr$$
$$\leq \frac{M}{l(1-a)}\int_0^{x+t}(x+t-r)^{-a}\|q(r)\|\,dr$$
$$+\frac{M(l-1)}{l(b-1)}\int_0^{x+t}(x+t-r+1)^{-b}\|q(r)\|\,dr,$$

condition (5.5) holds if the following conditions are satisfied.

(i) The mapping $t \mapsto \int_0^t (t-r)^{-a}\|q(r)\|\,dr$, $t>0$ is bounded as $t \to +\infty$.
(ii) We have $\lim_{t\to+\infty}\int_0^t (t+1-r)^{-b}\|q(r)\|\,dr = 0$.

These conditions hold for a substantially large class of functions $q(\cdot)$.

Example 5.6. Assume now that $(R(t))_{t>0} \subseteq L(X,Y)$ is strongly continuous and satisfies the estimate $\|R(t)\| \leq Me^{-ct}t^{\beta-1}$, $t>0$ for some finite constants $c,\beta,M>0$. Dividing the integral $\int_{x+t-r}^{x+t-r+l}$ into two parts $\int_{x+t-r}^{x+t-r+1}$ and $\int_{x+t-r+1}^{x+t-r+l}$, for $l \geq 1$, and estimating the integrand $e^{-cv}v^{\beta-1}$ on $[x+t-r, x+t-r+1]$ by $v^{\beta-1}$ (on $[x+t-r+1, x+t-r+l]$ by e^{-cv}), as was done in the previous example, it can be easily verified that (5.5) holds if the following conditions are satisfied.

(i) The mapping $t \mapsto \int_0^t (t-r)^{\beta-1}\|q(r)\|\,dr$, $t>0$ is bounded as $t \to +\infty$.
(ii) The mapping $t \mapsto \int_0^t e^{-c(t-r)}\|q(r)\|\,dr$, $t>0$ is bounded as $t \to +\infty$.

We invite the interested reader to find some sufficient conditions ensuring the asymptotically (Stepanov) almost periodicity of finite convolution products provided that the function $q \in W_0^1([0,\infty):X)$ (resp., $q \in e-W_0^1([0,\infty):X)$) satisfies some other conditions from (5.4) (resp., (5.5)). It would take too long to examine the cases in which the function $g(\cdot)+q(\cdot)$ belongs to the classes $AAPW^p([0,\infty):X)$, $e-AAPW^p([0,\infty):X)$, $AAPSW^p([0,\infty):X)$, or $e-AAPSW^p([0,\infty):X)$.

6. Abstract Volterra integro-differential equations: Weyl-almost periodicity and asymptotical Weyl-almost periodicity of solutions

We start the work in this section by stating the following proposition (a similar result can be stated for degenerate operator families subgenerated by a pair of closed linear operators; see [26] for more details).

Proposition 6.1. *Suppose that* $abs(|a|) < \infty$*, that* $abs(k) < \infty$*, that* $1 \leq p < \infty$*, and that* $\mathcal{A}$ *is a subgenerator of a mild, strongly Laplace transformable,* (a,k)*-regularized* C_2*-uniqueness family* $(R_2(t))_{t\geq 0}$*. Denote by* D *the set consisting of all eigenvectors* x *of operator* $\mathcal{A}$ *corresponding to eigenvalues* $\lambda \in \mathbb{C}$ *of operator* $\mathcal{A}$ *for which the mapping*

$$f_{\lambda,x}(t) := \mathcal{L}^{-1}\Big(\frac{\tilde{k}(z)}{1-\lambda\tilde{a}(z)}\Big)(t)C_2x, \quad t \geq 0$$

is (equi)-Weyl-p*-almost periodic. Then the mapping* $t \mapsto R_2(t)x$*,* $t \geq 0$ *is (equi)-Weyl-*p*-almost periodic for all* $x \in \operatorname{span}(D)$*.*

Let $f : [0,\infty) \to X$ be Weyl-p-almost periodic. As in the case of Stepanov almost periodic functions, the Bohr–Fourier coefficients

$$P_r(f) = \lim_{t\to\infty}\frac{1}{t}\int_{\alpha}^{t+\alpha} e^{-irs}f(s)\,ds$$

exist for all $r \in \mathbb{R}$, independently of $\alpha \in \mathbb{R}$, and the assumption $P_r(f) = 0$ for all $r \in \mathbb{R}$ implies that $f(t) = 0$ for almost everywhere $t \in \mathbb{R}$. In particular, $f(\cdot)$ satisfies (P1) and the argumentation contained in the proof of [22, Theorem 4.5] shows that the following result holds true.

Theorem 6.2. *Let* $\mathcal{A}$ *be the integral generator of a Weyl-*p*-almost periodic* (a,k)*-regularized* C*-resolvent family* $(R(t))_{t\geq 0}$ *for some* $p \in [1,\infty)$*, let* $\overline{R(C)} = \overline{D(\mathcal{A})} = X$*, and let* $k(0) \neq 0$*. Denote*

$$\mathcal{R} := \big\{r \in \mathbb{R} : \tilde{a}(ir) \text{ exists}\big\}.$$

Suppose that $k(t)$ *and* $|a|(t)$ *satisfy* (P1), $\lim_{\Re z\to\infty}\tilde{a}(z) = 0$ *as well as that*

$$P_r^k = \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{-irs}k(s)\,ds = 0, \quad r \in \mathcal{R}.$$

Then we have

(Q): $P_r^R x \in \mathcal{A}[\tilde{a}(ir)P_r^R x]$, $r \in \mathcal{R}$, $x \in X$ *and the mapping*

$$R(t)P_r^R x = \mathcal{L}^{-1}\Big(\frac{\tilde{k}(z)\tilde{a}(ir)}{\tilde{a}(ir)-\tilde{a}(z)}\Big)(t)CP_r^R x, \quad t \geq 0, x \in X,$$

is Weyl-p*-almost periodic for all* $r \in \mathcal{R}$ *and* $x \in X$*.*

Suppose, in addition, that

$$R(t)P_r^R x = k(t)CP_r^R x, \quad t \geq 0, r \in \mathbb{R}\setminus\mathcal{R}, x \in X.$$

Then the set D *consisting of all eigenvectors of operator* $\mathcal{A}$ *corresponding to eigenvalues* $\lambda \in \{0\} \cup \{\tilde{a}(ir)^{-1} : r \in \mathcal{R}, \tilde{a}(ir) \neq 0\}$ *of operator* $\mathcal{A}$ *is total in* X*.*

Suppose now that $\alpha \in (0,2)\setminus\{1\}$ and that $r \in \mathbb{R}\setminus\{0\}$. Then the function $t \mapsto E_\alpha((ir)^\alpha t^\alpha)$, $t \geq 0$ is bounded and uniformly continuous so that its Weyl-p-almost periodicity for some $p \in [1,\infty)$ would imply its almost periodicity (see Theorem 4.2), which is a contradiction (see [23]). Keeping this observation in mind, as well as the fact that any two Weyl-p-almost periodic functions having the same

Bohr–Fourier coefficients need to be identical almost everywhere (see [11], [12]), as in [23] we can deduce the following results.

Theorem 6.3. *Let $C \in L(X)$ be injective, let $1 \leq p < \infty$, let A be a closed single-valued linear operator, and let $\overline{R(C)} = X$. Suppose that $\alpha \in (0,2) \setminus \{1\}$ and that A generates a Weyl-p-almost periodic (g_α, C)-resolvent family $(R(t))_{t\geq 0}$. Then $A = 0 \in L(X)$ and $R(t) = C$, $t \geq 0$.*

Proposition 6.4. *Suppose that $(S(t))_{t\geq 0}$ is a bounded C-regularized semigroup with the integral generator $\mathcal{A}$. If $x \in X$ satisfies that the mapping $t \mapsto S(t)x$, $t \geq 0$ is Weyl-p-almost periodic for some $p \in [1,\infty)$, then the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is almost periodic.*

Theorem 6.5. *Suppose that $1 \leq p < \infty$ and that $(S(t))_{t\geq 0}$ is a C-regularized semigroup with the integral generator $\mathcal{A}$. Then the following holds.*

(i) *Let $x \in X$ satisfy that the mapping $t \mapsto S(t)x$, $t \geq 0$ is W^p-bounded. Then the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is bounded.*

Suppose that the mapping $t \mapsto S(t)x$, $t \geq 0$ is W^p-bounded for all $x \in X$. Then we have the following.

(ii) *The mapping $t \mapsto S(t)C^2x$, $t \geq 0$, is bounded and uniformly continuous for all $x \in X$, and there exists a finite constant $M \geq 0$ such that $\|S(t)C\| \leq M$, $t \geq 0$. Therefore, if $x \in X$ satisfies that the mapping $t \mapsto S(t)C^2x$, $t \geq 0$, is Weyl-p-almost periodic, then it is almost periodic.*

(iii) *If $R(C)$ is dense in X, then the mapping $t \mapsto S(t)Cx$, $t \geq 0$ is bounded and uniformly continuous for all $x \in X$. Therefore, if $x \in X$ satisfies that the mapping $t \mapsto S(t)Cx$, $t \geq 0$, is Weyl-p-almost periodic, then it is almost periodic.*

Due primarily to the fact that it is very difficult to satisfactorily introduce some Fréchet topologies on the spaces of (asymptotically, equi)-Weyl-almost periodic functions, we will not consider here (asymptotically, equi)-Weyl-almost periodic functions depending on two parameters: composition theorems and semilinear differential Cauchy inclusions. We close the section with the observation that composition theorems for Weyl-pseudo-almost automorphic functions have been investigated by Abbas [2].

7. Examples and applications

The main aim of this section is to present some illustrative examples and applications of our abstract results established in the previous sections. The main assumption will be that $\mathcal{A}$ is a multivalued linear operator on a Banach space X satisfying the condition examined by Favini and Yagi [17, (P), p. 47]:

(P) There exist finite constants $c, M > 0$ and $\beta \in (0,1]$ such that

$$\Psi := \Psi_c := \big\{\lambda \in \mathbb{C} : \Re\lambda \geq -c\big(|\Im\lambda| + 1\big)\big\} \subseteq \rho(\mathcal{A})$$

and

$$\big\|R(\lambda : \mathcal{A})\big\| \leq M\big(1 + |\lambda|\big)^{-\beta}, \quad \lambda \in \Psi.$$

Suppose now that condition (P) holds and $\beta > \theta$. Then the degenerate strongly continuous semigroup $(T(t))_{t>0} \subseteq L(X)$ generated by $\mathcal{A}$ satisfies estimate $\|T(t)\| \leq M_0 e^{-ct} t^{\beta-1}$, $t > 0$ for some finite constant $M_0 > 0$ (see [23]). Define

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu} \int_0^\infty s^\nu \Phi_\gamma(s) T(st^\gamma)x\, ds, \quad t > 0, x \in X,$$
$$S_\gamma(t) := T_{\gamma,0}(t), \quad t > 0; \qquad S_\gamma(0) := I,$$

and, following Wang, Chen, and Xiao [38],

$$P_\gamma(t) := \gamma T_{\gamma,1}(t)/t^\gamma, \quad t > 0. \tag{7.1}$$

We start our work by inquiring into the existence and uniqueness of the following abstract Cauchy inclusion of first order

$$u'(t) \in \mathcal{A}u(t) + f(t), \quad t \in \mathbb{R}. \tag{7.2}$$

Following Zaidman [39], by a mild solution of (7.2) we mean the X-valued continuous function $u(\cdot)$ given by

$$t \mapsto u(t) := \int_{-\infty}^t T(t-s) f(s)\, ds, \quad t \in \mathbb{R}.$$

In our concrete situation, it is clear that any of two parts of Proposition 5.1 can be applied. If $f : \mathbb{R} \to Y$ is bounded and Weyl-almost periodic, then there exists a unique bounded and (equi)-Weyl-almost periodic solution of (7.2); on the other hand, if $f : \mathbb{R} \to Y$ is equi-Weyl-almost periodic, then there exists a unique bounded and equi-Weyl-almost periodic solution of (7.2).

We continue our exposition in this section by investigating the existence and uniqueness of fractional relaxation inclusions with Weyl–Liouville derivatives of order $\gamma \in (0, 1]$, thus continuing the work of Mu, Zhoa, and Peng [30] and the author [24] (for fractional differential equations with delay, the reader may consult Abbas [1] and the references cited therein).

7.1. Fractional relaxation inclusions with Weyl–Liouville derivatives. Throughout this section, we assume that $\gamma \in (0, 1]$. Following the method applied in the proof of [30, Lemma 6], we say that a continuous function $u : \mathbb{R} \to X$ is a *mild solution* of fractional relaxation inclusion

$$D_{t,+}^\gamma u(t) \in \mathcal{A}u(t) + f(t), \quad t \in \mathbb{R} \tag{7.3}$$

if and only if

$$u(t) = \int_{-\infty}^t (t-s)^{\gamma-1} P_\gamma(t-s) f(s)\, ds, \quad t \in \mathbb{R}.$$

Set

$$R_\gamma(t) := t^{\gamma-1} P_\gamma(t), \quad t > 0.$$

Then we know by [27] that

$$\|R_\gamma(t)\| \leq M_1 t^{\gamma\beta-1}, \quad t \in (0, 1] \qquad \text{and} \qquad \|R_\gamma(t)\| \leq M_2 t^{-1-\gamma}, \quad t \geq 1,$$

as well as the fact that Proposition 5.1 can be applied again, producing the same final results as in the case of consideration of abstract differential inclusion (7.3). If $f : \mathbb{R} \to Y$ is bounded and Weyl-almost periodic, then there exists a unique bounded and (equi)-Weyl-almost periodic solution of (7.3); on the other hand, if $f : \mathbb{R} \to Y$ is equi-Weyl-almost periodic, then there exists a unique bounded and equi-Weyl-almost periodic solution of (7.3).

7.2. Fractional relaxation inclusions with Caputo derivatives. Suppose that $\gamma \in (0,1)$ and that $\mathcal{A}$ is a multivalued linear operator on a Banach space X. Of concern is the fractional relaxation inclusion

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), \quad t > 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^{\gamma}$ denotes the Caputo fractional derivative of order γ, $x_0 \in X$, and $f : [0,\infty) \to X$ is asymptotically (equi)-Weyl almost periodic. By a mild solution of $(\mathrm{DFP})_{f,\gamma}$, we mean any function $u \in C([0,\infty) : X)$ satisfying that

$$u(t) = S_{\gamma}(t)x_0 + \int_0^t R_{\gamma}(t-s)f(s)\,ds, \quad t \geq 0. \tag{7.4}$$

Let $x_0 \in X$ be a point of continuity of $(T(t))_{t>0}$ (see [17] for more details). Then x_0 is also a point of continuity of subordinated fractional resolvent family $(S_{\gamma}(t))_{t>0}$, and since $\|S_{\gamma}(t)\| \leq M_1 t^{\gamma(\beta-1)}$, $t > 0$ for some finite constant $M_1 > 0$, the mapping $t \mapsto S_{\gamma}(t)x_0$, $t \geq 0$ is continuous and tending to zero as t tends to $+\infty$. Therefore, Proposition 5.3 is receptive to applications, giving some sufficient conditions for the existence of a unique asymptotically Weyl-almost periodic solution of abstract Cauchy inclusion $(\mathrm{DFP})_{f,\gamma}$.

Summa summarum, we can apply Propositions 5.1 and 5.3 in the study of existence and uniqueness of Weyl-almost periodic solutions of the well-known Poisson heat equation (see [17])

$$\begin{cases} \frac{\partial}{\partial t}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), & t \in \mathbb{R}, x \in \Omega; \\ v(t,x) = 0, & (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

and asymptotically Weyl-almost periodic solutions of the Poisson heat equation

$$\begin{cases} \frac{\partial}{\partial t}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), & t \geq 0, x \in \Omega; \\ v(t,x) = 0, & (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

in the space $X := L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ almost everywhere $x \in \Omega$, $m \in L^{\infty}(\Omega)$ and $1 < p < \infty$, as well as their fractional analogues associated with the use of Weyl–Liouville or Caputo derivatives. Let us recall that the multivalued linear operator $\mathcal{A} := AB^{-1}$, where $A := \Delta - b$ acts on X with the Dirichlet boundary conditions, and B is the multiplication operator by the function $m(x)$, satisfies the condition (P) with $\beta = 1/p$ and some finite constants $c, M > 0$; recall also that the condition [17, (3.42)] on $m(x)$ ensures that one gets the better exponent β in (P), provided that $p > 2$.

We close the article with the observation that our results seem to be new even for strongly continuous semigroups of operators that are degenerate or nondegenerate in time, as well as for nondegenerate fractional resolvent families (then the mild solution of the corresponding abstract Cauchy inclusion (7.4) is obtained by replacing $S_\gamma(\cdot)$ and $R_\gamma(\cdot)$ in this equation with $T(\cdot)$; the use of C-regularized semigroups is also possible). Our results also seem to be new for semigroups and fractional resolvent families of linear operators generated by almost sectorial linear operators (see [26], [32], [37], [38]), so that Proposition 5.3 is applicable in the analysis of existence and uniqueness of asymptotically Weyl-almost periodic solutions of the following fractional equation with higher order differential operators in the Hölder space $X = C^\alpha(\overline{\Omega})$ (see [37] for more details):

$$\begin{cases} \mathbf{D}_t^\gamma u(t,x) = -\sum_{|\beta|\leq 2m} a_\beta(t,x) D^\beta u(t,x) - \sigma u(t,x) + f(t,x), & t \geq 0, x \in \Omega; \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases}$$

Some other applications in the analysis of asymptotically Weyl-almost periodic solutions of abstract Volterra integro-differential equations and inclusions can be given, for example, to equations considered by Agarwal, de Andrade, and Cuevas [3], de Andrade and Lizama [14], and Ponce and Warma [33] (see also [23], [26]).

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