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Mereology on Topological and Convergence Spaces

Daniel R. Patten

Abstract We show that a standard axiomatization of mereology is equivalent to the condition that a topological space is discrete, and consequently, any model of general extensional mereology is indistinguishable from a model of set theory. We generalize these results to the Cartesian closed category of convergence spaces.

1 Introduction

In 1981, Clarke [6] proposed an axiomatization of mereology—the study of the relationship between *part* and *whole*—with one primitive *connection*: two regions are connected if they intersect. Some ten years later Randell, Cui, and Cohn [15] weakened this definition of connection by requiring only that the topological closures of the two regions intersect. Although Casati and Varzi [5] discuss several approaches to mereology, their "favored strategy" uses this definition of connection; in turn, they define parthood in terms of connection: one thing is a part of another if and only if anything connected to the former is also connected to the latter. In contrast, Guarino, Carrara, and Giaretta [8] used this definition of connection but chose, for reasons unclear, not to define parthood in terms of connection—an uneconomical choice since it increases the number of primitives in the theory.

Although those cited above have examined thoroughly the relational logic implied by the mereotopological axioms, they have neglected the topological consequences of these same axioms. In this paper, we remedy this oversight in two ways. First, we determine the topological structures determined by the mereotopological axioms of [15]; in particular, we show that the definition of Randell, Cui, and Cohn is equivalent to the condition that a topological space is discrete. Second, we generalize this result to **CONV**, the Cartesian closed category of convergence spaces, of which **TOP**, the category of topological spaces, is a full subcategory.[1]

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2 Mereology on Topological Spaces

Conventional treatment of mereology begins by defining the binary relations connection, as in [15], and parthood, with the condition that parthood is a partial order. We take an ostensibly different, but in fact equivalent, approach: rather than explicitly require parthood to be a partial order, we define connection to be an extensional relation.[2]

Definition 2.1 Let $(X, \mathcal{T})$ be a topological space. Let $|$ be an extensional binary relation on $2^X - \{\emptyset\}$ such that $A|B$ if and only if the topological closures of A and B intersect; formally, we write

$$(\forall A)(\forall B)\big(A|B \Leftrightarrow \mathrm{cl}(A) \cap \mathrm{cl}(B) \neq \emptyset\big). \tag{1}$$

Two subsets A and B of X are *connected* if and only if $A|B$. We refer to $|$ as the *connection* relation.

Proposition 2.2 *The connection relation is reflexive and symmetric.*

Proof Suppose that A and B are nonempty subsets of a topological space X.

(Reflexivity) Since $\mathrm{cl}(A) \cap \mathrm{cl}(A) = \mathrm{cl}(A) \neq \emptyset$, it follows that $A|A$.

(Symmetry) If $A|B$, then $\mathrm{cl}(B) \cap \mathrm{cl}(A) = \mathrm{cl}(A) \cap \mathrm{cl}(B) \neq \emptyset$, and thus $B|A$. □

Connection is not an equivalence relation: it is not transitive. To see this, consider the following counterexample. Suppose that X has the discrete topology and that $\emptyset \neq A \subset B \subseteq X$. Since every set of a discrete space is closed, it follows that $\mathrm{cl}(B) \cap \mathrm{cl}(X - A) = B \cap (X - A) \neq \emptyset$; thus $B|(X - A)$. Likewise $A|B$ because $\mathrm{cl}(A) \cap \mathrm{cl}(B) = A \cap B \neq \emptyset$. But it is not the case that $A|(X - A)$; otherwise $A \cap (X - A) = \mathrm{cl}(A) \cap \mathrm{cl}(X - A) \neq \emptyset$, which is absurd.

Definition 2.3 Let $(X, \mathcal{T})$ be a topological space. Let $\leq$ be a relation on $2^X - \{\emptyset\}$ such that

$$(\forall A)(\forall B)\big(A \leq B \Leftrightarrow (\forall C)(C|A \rightarrow C|B)\big). \tag{2}$$

A subset A of X is a *part* of a subset B of X if and only if $A \leq B$. We refer to $\leq$ as the *parthood* relation.

Proposition 2.4 *The parthood relation is a partial order.*

Proof Suppose that A, B, and C are nonempty subsets of a topological space X.

(Reflexivity) Since $(\forall C)(C|A \rightarrow C|A)$, it follows that $A \leq A$.

(Transitivity) Let $A \leq B$ and $B \leq C$. If $D|A$, then by definition of parthood $D|B$, and so $D|C$. Therefore $A \leq C$.

(Antisymmetry) Let $A \leq B$ and $B \leq A$. By definition of parthood, it follows that $(\forall C)(C|A \leftrightarrow C|B)$. Thus, by hypothesis, we have $A = B$. □

Notice that connection need not be extensional for parthood to be reflexive and transitive; connection, however, must be extensional for parthood to be antisymmetric. To see this, suppose that the parthood relation is antisymmetric, but do not require that connection to be extensional. Let A and B be subsets of a topological space X. Suppose that $(\forall C)(C|A \leftrightarrow C|B)$. By definition of parthood, it follows that $A \leq B$ and $B \leq A$. Thus, by hypothesis, we have $A = B$. Therefore, if we require parthood to be a partial order, then we must require connection to be extensional.

While it is conceivable that additional topological constraints could guarantee parthood to be antisymmetric, no separation axiom will suffice. To see this, suppose that connection is defined as above but without the requirement that it be extensional. Consider the standard topology on $\mathbb{R}$. Let $A = \{1/n : n \in \mathbb{Z}^+\}$, and let $B = A \cup \{0\}$. Then $\mathrm{cl}(A) = B = \mathrm{cl}(B)$, and so both $A \leq B$ and $B \leq A$. But $A \neq B$. Since $\mathbb{R}$ is a T_6-space, we conclude that no separation axiom implies antisymmetry of parthood.

Definition 2.5 A *mereology* is a triple $(X, \mathcal{T}, \leq)$ in which $(X, \mathcal{T})$ is a topological space and $\leq$ is a parthood relation.[3] When no reasonable confusion is likely, we refer to $(X, \mathcal{T}, \leq)$ by X.

We now ask two questions. Given a mereology $(X, \mathcal{T}, \leq)$, what is the structure of $\mathcal{T}$? Conversely, which topological spaces are candidates for mereologies? To motivate an answer, we study a simple example.

Example 2.6 Consider $X = \{a, b, c\}$, and let $(X, \mathcal{T}, \leq)$ be a mereology. Which of the nine nonhomeomorphic topologies on X can $\mathcal{T}$ be?

If $\mathcal{T} = \{\emptyset, \{a\}, X\}$, then $\mathrm{cl}(\{b\}) = \{b, c\} = \mathrm{cl}(\{c\})$, and so for any $A \subseteq X$ both $A|\{b\}$ and $A|\{c\}$; thus $\{b\} \leq \{c\}$ and $\{c\} \leq \{b\}$, which implies that $\{b\} = \{c\}$, which is absurd.

Of the eight remaining topologies, we can apply minor variations of this reductio ad absurdum to all but the discrete topology, which is, as is easily verified, a mereology.

Not only can we generalize the argument given in Example 2.6 to show that every mereology is discrete, but we can also show that every discrete topological space is a mereology.

Theorem 2.7 *A topological space is a mereology if and only if it is discrete.*

Proof Let X be a topological space. It suffices to show that the connection relation is extensional if and only if X is discrete.

(Necessity) Let X have the discrete topology. Suppose that A and B are subsets of X and that $C|A$ if and only if $C|B$ for every subset C of X. If $x \in A$, then $\{x\} \subseteq A$. Since X is discrete, it follows that $\mathrm{cl}(\{x\}) \cap \mathrm{cl}(A) = \{x\} \cap A \neq \emptyset$; thus $\{x\}|A$, and so by hypothesis $\{x\}|B$, which implies that $\{x\} \cap B = \mathrm{cl}(\{x\}) \cap \mathrm{cl}(B) \neq \emptyset$, from which we infer that $x \in B$. An exactly similar argument shows that $x \in A$ whenever $x \in B$. Therefore $A = B$.

(Sufficiency) Let the connection relation be extensional. Without loss of generality, suppose that A is a nonempty subset of X. Since $\mathrm{cl}(\mathrm{cl}(A)) \cap \mathrm{cl}(A) = \mathrm{cl}(A) \neq \emptyset$, it follows that $\mathrm{cl}(A)|A$ and $A|\mathrm{cl}(A)$, and so by hypothesis $\mathrm{cl}(A) = A$. Thus every subset of X is closed; that is, as a topological space X is discrete. □

In view of Theorem 2.7, we can simplify Definition 2.1 by replacing (1) with

$$(\forall A)(\forall B)(A|B \Leftrightarrow A \cap B \neq \emptyset). \tag{3}$$

As a consequence, we obtain a corollary to Theorem 2.7.

Corollary 2.8 *If A and B are nonempty subsets of a mereology X, then $A \leq B$ if and only if $A \subseteq B$.*

Proof Let A and B be nonempty subsets of a topological space X.

(Necessity) Suppose that $A \subseteq B$. If $C|A$ for some subset C of X, then $C \cap B \supseteq C \cap A \neq \emptyset$. Thus $C|B$, and therefore $A \leq B$.

(Sufficiency) Suppose that $A \leq B$. If $x \in A$, then $\{x\} \subseteq A$, and so $\{x\} \cap A = \emptyset$, which implies that $\{x\}|A$; thus $\{x\}|B$, from which it follows that $\{x\} \cap B \neq \emptyset$, and so $x \in B$. Therefore, we conclude that $A \subseteq B$. □

Corollary 2.8 shows that the parthood relation is equivalent to the subset relation and thereby effectively demonstrates the equivalence of mereology and set theory. We can obtain a stronger result than this, namely, the equivalence of general extensional mereology and set theory. To proceed, we present several definitions and intermediate results.

Definition 2.9 Let X be a mereology. Let $<$ be a binary relation on $2^X - \{\emptyset\}$ such that

$$(\forall A)(\forall B)\big(A < B \Leftrightarrow (A \leq B \wedge A \neq B)\big). \tag{4}$$

A subset A of X is a *proper part* of a subset B of X if and only if $A < B$. We refer to $<$ as the *proper parthood* relation.

Proposition 2.10 *The proper parthood relation is asymmetric and transitive.*

Proof Let A, B, and C be subsets of a mereology X.

(Asymmetry) To the contrary, suppose that $A < B$ and $B < A$. By definition of proper parthood, it follows that $A \leq B$, $B \leq A$, and $A \neq B$. By antisymmetry of parthood, however, we see that $A = B$, in contradiction to the former result. Therefore, we conclude that $B \nless A$.

(Transitivity) Suppose that $A < B$ and $B < C$. By definition of proper parthood, it follows that $A \leq B$, $B \leq C$, $A \neq B$, and $B \neq C$. By transitivity of parthood, we see that $A \leq C$. If $A = C$, then $B \leq A$, and so antisymmetry of parthood implies that $A = B$, in contradiction to the result that $A \neq B$. Thus $A \neq C$, and therefore $A < C$. □

Definition 2.11 Let X be a mereology. Let $\odot$ be a binary relation on $2^X - \{\emptyset\}$ such that

$$(\forall A)(\forall B)\big(A \odot B \Leftrightarrow (\exists C)(C \leq A \wedge C \leq B)\big). \tag{5}$$

Two subsets A and B of X *overlap* if and only if $A \odot B$. We refer to $\odot$ as the *overlap* relation.

While reflexive and symmetric, the overlap relation is not transitive. In view of Theorem 2.7, it follows that two sets overlap if and only if they intersect.

Definition 2.12 (see [5]) Let A and B be subsets of a mereology X. The *weak supplementation principle* asserts that if A is a proper part of B, then there is a part of B not overlapping A. Formally, we write

$$(\forall A)(\forall B)\big(A < B \rightarrow (\exists C)(C \leq B \wedge \neg(C \odot A))\big). \tag{6}$$

The *strong supplementation principle* asserts that unless B is a part of A there is a part of B not overlapping A. Formally, we write

$$(\forall A)(\forall B)\big(\neg(B \leq A) \rightarrow (\exists C)(C \leq B \wedge \neg(C \odot A))\big). \tag{7}$$

Proposition 2.13 *The strong supplementation principle implies the weak supplementation principle.*

Proof Suppose that the strong supplementation principle holds in a mereology X. Let A and B be subsets of X such that $A < B$. By definition of proper parthood, it follows that $A \leq B$ and $A \neq B$. By the antisymmetry of parthood $\neg(B \leq A)$. By hypothesis, therefore, we conclude that there exists a subset C of X such that $C \leq B$ and $\neg(C \odot A)$. □

Observe that the proof of Proposition 2.13 does not require connection to be extensional. If, on the other hand, connection is extensional, then the strong supplementation principle will hold in every mereology.

Proposition 2.14 *The strong supplementation principle holds in every mereology.*

Proof Let X be a mereology, and suppose that $\neg(B \leq A)$ for some nonempty subsets A and B of X. By Corollary 2.8, it follows that $\neg(\mathrm{cl}(B) \subseteq \mathrm{cl}(A))$, which implies that there exists an $x \in X$ such that $x \in \mathrm{cl}(B)$ and $x \notin \mathrm{cl}(A)$. By the former conjunct we have $\mathrm{cl}(\{x\}) \subseteq \mathrm{cl}(B)$, and so by Corollary 2.8 again we infer that $\{x\} \leq B$; by the latter conjunct and Theorem 2.7, we have $\mathrm{cl}(\{x\}) \cap \mathrm{cl}(A) = \{x\} \cap \mathrm{cl}(A) = \emptyset$, and so $\neg(\{x\}|A)$, which implies that $\neg(\{x\} \odot A)$. □

Definition 2.15 (see [5]) The *general sum* of all sets satisfying a predicate φ is the set

$$\Sigma\varphi = \iota A \forall B \big(A \odot B \leftrightarrow \exists C(\varphi C \wedge C \odot B)\big). \tag{8}$$

Proposition 2.16 *Let X be a mereology. If $\Sigma\varphi$ exists, then it is the supremum of $\Phi = \{A \subseteq X \mid \varphi A\}$.*

Proof First, we establish that $U = \Sigma\varphi$ is an upper bound of Φ. By (8), we have

$$U \odot C \leftrightarrow \exists B(\varphi B \wedge B \odot C). \tag{9}$$

Suppose that φS holds. An instance of the contrapositive of the strong supplementation principle is

$$(\forall A)(A \leq S \to A \odot U) \to S \leq U. \tag{10}$$

If $A \leq S$, then $A \odot S$, and so by (9) it follows that $A \odot U$; thus by (10) we infer that $S \leq U$.

Now we show that U is the least upper bound of Φ. Let V be any other upper bound of Φ. An instance of the strong supplementation principle is

$$\neg(U \leq V) \to (\exists A)(A \leq U \wedge \neg A \odot V). \tag{11}$$

Thus, if $\neg(U \leq V)$, then there exists an A such that $A \leq U$ and $\neg(A \odot V)$. The former conjunct implies that $A \odot U$, and so by (9) there exists B such that φB holds and $B \odot A$. Since $B \odot A$, there exists E such that $E \leq B$ and $E \leq A$; because V is a least upper bound of Φ, it follows that $B \leq V$, and so $E \leq V$ and $E \leq A$, which implies that $A \odot V$, in contradiction to $\neg(A \odot V)$. Therefore, we conclude that $U \leq V$. □

Finally, we define a general extensional mereology (sometimes referred to as **GEM** in the literature). If connection is extensional, then any mereology is a general extensional mereology.

Definition 2.17 (see [5]) A *general extensional mereology* is a mereology X in which the weak supplementation principle holds and $\Sigma\varphi$ exists for every relation φ on X.

Theorem 2.18 *Every mereology is a general extensional mereology.*

Proof Let X be a mereology. By Theorem 2.7, it follows that X is a discrete topological space. Since the supremum of $\{A \subseteq X \mid \varphi A\}$ is $\bigcup_{\varphi A} A$, which always exists, it suffices to show that $\bigcup_{\varphi A} A$ satisfies Definition 2.15.

If $\bigcup_{\varphi A} A$ and B overlap, then there exists $C \subseteq X$ that is a subset of both $\bigcup_{\varphi A} A$ and B. Now C intersects some A for which φA holds. Since $A \cap C$ is a subset of both A and B, it follows that $A \cap C$ is a part of both A and B, and so A and B overlap.

Conversely, suppose that $B \subseteq X$ and B overlaps with some $C \subseteq X$ such that φC. Since B intersects C, it follows that B intersects $\bigcup_{\varphi A} A$, and so B and $\bigcup_{\varphi A} A$ overlap. □

3 Mereology on Convergence Spaces

The results of the previous section show that general extensional mereology reduces to set theory. To prevent this reduction, we might relax the definition of connection; it is not obvious, however, in which way we should do this, for we cannot remove extensionality from connection without simultaneously removing antisymmetry from parthood. Alternatively, we observe that **TOP**, the category of topological spaces, is a full subcategory of **CONV**, the category of convergence spaces; subsequently, we can amend the definition of mereology to be over convergence spaces rather than topological spaces. Since topological spaces are "coarser" structures than convergence spaces, the "finer" structure of the latter might prevent reduction of general extensional mereology to set theory, not unlike the way that increasing microscopic resolution allows the observation of formerly indistinguishable features.

We present, for the reader's convenience, those concepts from the theory of convergence spaces used throughout our discussion. For wider coverage of convergence spaces we refer the reader to [1]. To begin, we consider filters, which albeit large, abstract objects, are easily defined.[4]

Definition 3.1 Let X be a set. A nonempty collection $\mathcal{F}$ of subsets of X is a *filter* if and only if

1. $\emptyset \notin \mathcal{F}$,
2. $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$, and
3. $A \in \mathcal{F}$ and $B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

If A is a subset of X, then the filter $\{B | A \subseteq B \subseteq X\}$ denoted by $[A]$ is the *principal filter generated by* A. For the *point filter* $\{x\}$ we write $[x]$ to abbreviate $[\{x\}]$. We denote the set of all filters on X by $\Phi(X)$. When $\mathcal{F} \supseteq \mathcal{G}$, we say that $\mathcal{F}$ is *finer* than $\mathcal{G}$ and $\mathcal{G}$ is *coarser* than $\mathcal{F}$.

Some authors[5] allow filters to contain the empty set; in that case, they usually distinguish between improper and proper filters, that is, those filters that contain the empty set and those that do not.

Every filter is contained in some ultrafilter; in fact, each filter is the intersection of those ultrafilters that contain it. Free filters, which cannot be exhibited since their existence depends on the axiom of choice, do not contain any finite sets.[6]

Whereas a topology is a unary relation that defines the open subsets of a set, a convergence structure is a binary relation between a set and its filters that determines which filters "converge" to which points.

Definition 3.2 Let X be a set. A *convergence structure* is a relation $\downarrow$ between $\Phi(X)$ and X such that for every $x \in X$ and $\mathcal{F}, \mathcal{G} \in \Phi(X)$,

1. $[x] \downarrow x$,
2. $\mathcal{F} \downarrow x$ and $\mathcal{F} \subseteq \mathcal{G}$ implies $\mathcal{G} \downarrow x$, and
3. $\mathcal{F} \downarrow x$ and $\mathcal{G} \downarrow x$ implies $\mathcal{F} \cap \mathcal{G} \downarrow x$.

A *convergence space* is a pair $(X, \downarrow)$ in which X is a set and $\downarrow$ is a convergence structure. When no reasonable confusion is likely, we refer to $(X, \downarrow)$ by X. We read $\mathcal{F} \downarrow x$ as "$\mathcal{F}$ converges to x."

Observe that the finite intersection property of convergence spaces implies that $\mathcal{F} \cap [x]$ converges to x whenever $\mathcal{F}$ converges to x. Thus, in view of Definition 3.1, the filter $\mathcal{F}$ must contain some set that contains x. For concision, we say that $\mathcal{F}$ is "located" at x.

There are several extant notions of convergence spaces. For example, [3] and [9] do not require convergence structures to satisfy the finite intersection property—such a space is called a *generalized convergence space*, whereas our definition, which coincides with that of [1], is called a *limit space*.[7] Although the two formulations of convergence space coincide when they are pseudotopolgoical, there are some distinctions: for example, finite limit spaces are pretopological, but finite generalized convergence spaces need not be pretopological.[8]

Definition 3.3 Let X be a convergence space. For every $x \in X$, the *neighborhood filter of* x is the set

$$\mathcal{N}_x = \bigcap \left\{ \mathcal{F} \in \Phi(X) \;\middle|\; \mathcal{F} \downarrow x \right\}.$$

Every element of N_x is a *neighborhood of* x. A set is *open* if and only if it is a neighborhood of each of its members.

Definition 3.4 Let X be a convergence space. The *closure* of a subset A of X is the set

$$\mathrm{cl}(A) = \left\{ x \in X \;\middle|\; (\exists \mathcal{F})(\mathcal{F} \downarrow x \wedge A \in \mathcal{F}) \right\}.$$

A set A is *closed* if and only if $\mathrm{cl}(A) = A$.

Definition 3.5 Let $(X, \mathcal{T})$ be a topological space. For every $x \in X$, define $\mathcal{U}_x$, the set of all topological neighborhoods of x. The *topological convergence structure* on X is defined by

$$\mathcal{F} \downarrow x \text{ if and only if } \mathcal{F} \supseteq \mathcal{U}_x.$$

A convergence space is *topological* if and only if it has the topological convergence structure.

Example 3.6 Definitions 3.3 and 3.4 coincide with the usual topological notions when the convergence space is itself topological. Although closure, as defined in Definition 3.4, is extensive and preserves both binary unions and the empty set, it does not satisfy all of the Kuratowski closure axioms; in particular, closure is not an idempotent operation.[9] To see this, define a convergence structure[10] on $X = \{0, 1, 2\}$ by the equivalences

$$\mathcal{F} \downarrow 0 \Leftrightarrow \mathcal{F} \supseteq [\{0, 1\}],$$
$$\mathcal{F} \downarrow 1 \Leftrightarrow \mathcal{F} \supseteq [\{1, 2\}],$$
$$\mathcal{F} \downarrow 2 \Leftrightarrow \mathcal{F} \supseteq [\{0, 2\}].$$

The only nonempty closed set of X is X itself; equivalently, the only nonempty open set of X is X itself. In particular $\mathrm{cl}(\mathrm{cl}(\{0\})) = \mathrm{cl}(\{0,2\}) = X$. Thus, the closure operator of convergence spaces, called the Katětov closure operator, does not satisfy the Kuratowski closure axioms.[11]

Example 3.7 Since **TOP** is a full subcategory of **CONV**, every topological space is a convergence space; the converse, however, does not hold. To see this, suppose that $(Y, \downarrow)$ is a topological convergence space on $Y = \{0, 1, 2\}$; that is, some topology $\mathcal{T}$ induces the convergence structure $\downarrow$. It follows that $\mathcal{U}_0 = [\{0, 1\}]$ and $\mathcal{U}_1 = [\{1, 2\}]$. Thus $\{0, 1\}$ and $\{1, 2\}$ must both be open in $(Y, \mathcal{T})$, and so $\{1\}$ is open in $(Y, \mathcal{T})$, which implies that $\{1\}$ is a neighborhood of 1 in $(Y, \mathcal{T})$, from which we infer that $\{1\} \in \mathcal{U}_1$; but $\{1\} \notin [\{1, 2\}]$. Therefore Y is not a topological convergence space.[12]

Since we wish to generalize Theorem 2.7, we need a correct formulation of a discrete convergence space. Propositions 3.9 and 3.10 show that Definition 3.8 is, indeed, such a formulation.

Definition 3.8 A convergence space X is *discrete* if and only if $\mathcal{A}$ is finer than $[x]$ whenever $\mathcal{A}$ converges to x in X.

Proposition 3.9 *Every subset of a discrete convergence space is closed.*

Proof Let X be a discrete convergence space. If $x \in \mathrm{cl}(\emptyset)$, then $\emptyset \in [x]$ since $[x] \downarrow x$. This, however, contradicts the condition that $\emptyset$ does not belong to any filter; thus $x \notin \mathrm{cl}(\emptyset)$ or, equivalently, $\mathrm{cl}(\emptyset) = \emptyset$. Suppose that A is a nonempty subset of X. If $x \in A$, then $\{x\} \subseteq A$, and so $A \in [x]$. Since $[x] \downarrow x$, it follows that $x \in \mathrm{cl}(A)$; thus $A \subseteq [A]$. Conversely, if $x \in \mathrm{cl}(A)$, then there exists a filter $\mathcal{F}$ converging to x that contains A. By hypothesis $[x] \subseteq \mathcal{F}$, which implies that $A \cap x \neq \emptyset$, and so $x \in A$; thus $\mathrm{cl}(A) \subseteq A$. □

Proposition 3.10 *A convergence space is discrete if and only if it is a discrete topological space.*

Proof (Necessity) Let $(X, \mathcal{T})$ be a discrete topological space. We must show that $(X, \mathcal{T})$ induces a discrete convergence structure on X. For each $x \in X$, define $\mathcal{N}_x$ to be the collection of all neighborhoods of x. Define a convergence structure $(X, \downarrow)$ on X by $\mathcal{F} \downarrow x$ if and only if $\mathcal{F} \supseteq \mathcal{N}_x$. Since $(X, \mathcal{T})$ is discrete, every subset of X is open in $\mathcal{T}$; in particular $\{x\}$ is open in $\mathcal{T}$, and so $\{x\} \in \mathcal{N}_x$. Thus, if $\mathcal{F}$ converges to x, then $\mathcal{F}$ contains $\{x\}$, which implies that $\mathcal{F}$ is the point filter $[x]$; therefore, $(X, \downarrow)$ is discrete.

(Sufficiency) Let $(X, \downarrow)$ be a discrete convergence space. We must show that $(X, \downarrow)$ is topological for the discrete topology $\mathcal{T}$ on X. By hypothesis $\mathcal{F} \downarrow x$ if and only if $[x] \subseteq \mathcal{F}$. Since $\mathcal{U}_x = [x]$ for every $x \in X$, it follows that $\mathcal{F} \downarrow x$ if and only if $\mathcal{F} \supseteq \mathcal{U}_x$. Therefore $(X, \downarrow)$ is topological and, in particular, discrete. □

Having assembled the necessary verbal apparatus, we can now state the main result of the paper.

Theorem 3.11 *A convergence space is a mereology if and only if it is discrete.*

Proof Let X be a convergence space. It suffices to show that the connection relation is extensional if and only if X is discrete. The argument is exactly similar to the proof of Theorem 2.7. □

If we generalize the definition of connection from **TOP** to **CONV**, then all results of the previous section hold mutatis mutandis.[13]

4 Conclusions

We have investigated mereology[14] on topological and convergence spaces. Theorem 2.7 establishes that every mereology, as defined by Definition 2.5, is a discrete topological space; likewise, Theorem 3.11 shows that every mereology defined on a convergence space must be a discrete convergence space. The latter result allows us to generalize Theorem 2.18 to convergence spaces: thus, every mereology defined on a convergence space is a general extensional mereology, that is, a mereology in which the weak supplementation principle holds and the general sum exists for each predicate of the mereology.

A topological space is simply a subcollection of the power set of the underlying set; a discrete topological space, then, is the entire power set. Thus, our work demonstrates that general extensional mereology is indistinguishable from set theory—not a surprising result when one considers that general extensional mereology was developed in pursuit of an alternative to set theory. We attribute this reduction (of general extensional mereology to set theory) to the extensionality imposed on connection (or, equivalently and more conventionally, the antisymmetry imposed on parthood); the results of Section 3 show that the reduction is not a topological consequence since it also occurs in **CONV** of which **TOP** is a full subcategory.

One possible objection to our work is that it assumes the elements of a mereology are sets. While we concede this point, we also answer that the assumption arises not from any peculiarity of our work but from the mereotopological axioms in [15]. In particular, the axioms require that the topological closures of two connected regions intersect; but the only objects that have topological closures are indeed sets.

Notes

1. See Mac Lane [11] on category theory, and in particular, Blair et al. [3] on the categories **CONV** and **TOP**. See Kelley [10] or Munkres [13] on topology. Although we provide some of the basic definitions of convergence spaces, we refer the reader to Beattie and Butzmann [1] for a fuller treatment of convergence spaces.

2. A relation R on a set S is extensional if and only if $a = b$ whenever $Rca \leftrightarrow Rcb$ for every a, b, and $c \in S$.

3. In view of Definition 2.1 and Proposition 2.4, the parthood relation must be a partial order. Moreover, the topological space $(X, \mathcal{T})$ completely determines $(X, \mathcal{T}, \leq)$.

4. For further discussion on filters, see Bourbaki [4]; for a historical treatment of their development, see Mashaal [12].

5. For example, Heckmann [9].

6. See [4] for further discussion.

7. Our definition also coincides with that of Binz [2]. See Preuß [14] for classification of convergence spaces.

8. A pseudotopological (or Choquet) space is a convergence space in which a filter $\mathcal{F}$ converges to x if and only if every ultrafilter finer than $\mathcal{F}$ also converges to x; a pretopological space is convergence space in which the neighborhood filter of x converges to x.

9. For details on the Kuratowski closure axioms, see [10].

10. We can view this particular space as a reflexive digraph. See [3] for more on this valuable, yet largely unexplored, perspective.

11. For properties of the Katětov closure operator, see Dikranjan [7].

12. A simpler, though less pedagogically fruitive, argument is that since the Katětov closure operator is not idempotent on X, the convergence space X cannot be topological.

13. There is at least one other approach to this generalization of connection from **TOP** to **CONV**. Instead of using limit spaces, we could have used general convergence spaces and, in place of Theorem 3.11, prove that a convergence space is a mereology if and only if it is postdiscrete and pretopological (see [3]).

14. In particular, we investigated mereology with a connection relation, often referred to as *mereotopology*. Since we also extended the definition of connection to convergence spaces, we were reluctant to use that term, while hoping that context could resolve any ambiguities.

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Department of Electrical Engineering and Computer Science
Syracuse University
Syracuse, New York 13244–4100
USA
drpatten@syr.edu