The Annals of Statistics
1992, Vol. 20, No. 3, 1454–1467

ADMISSIBLE ESTIMATORS OF VARIANCE COMPONENTS OBTAINED VIA SUBMODELS

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Simultaneous estimation of the vector of variance components under unbalanced mixed models w.r.t. the ordinary quadratic loss function is considered. A new method of constructing invariant quadratic admissible estimators, both with and without the condition of unbiasedness, is presented. Using this method, admissible estimators for the factorial models with imbalance at the last stage and the unbalanced $(p-1)$-way nested factors design are constructed.

1. Introduction. In recent years the problem of estimation of the variance components in mixed linear models has been intensively investigated. An excellent reference is a recent monograph by Rao and Kleffe (1988). This paper is devoted to the problem of constructing admissible estimators of variance components (with and without the condition of unbiasedness) under the ordinary quadratic loss function. Most relevant are the papers by Olsen, Seely and Birkes (1976) and by LaMotte (1982). Theorem 3.14 of LaMotte, which gives necessary and sufficient conditions for admissibility in the general linear model, is of great theoretical importance, but unfortunately is too complex to provide an explicit characterization of admissible estimators of variance components in mixed models. To apply them requires, in most cases, solving a system of linear equations with the number of unknowns proportional to the square of the number of observations. For more details the reader may refer to the work of Kleffe and Seifert (1986) and Klonecki and Zontek (1989a). An explicit complete characterization of admissible estimators of variance components is now available only for some special mixed models—models with two variance components [Gnot and Kleffe (1983)] and models with commuting covariance matrices [Klonecki and Zontek (1987)]. Also explicit formulae of locally best (unbiased and biased) estimators can be derived for a broad class of mixed models [see Rao and Kleffe (1988)].

In this paper we present a new method of constructing admissible estimators of variance components that is applicable to many important applications for unbalanced models. Its novelty lies in the idea of using known admissible estimators under some simple models to construct admissible estimators under some other, more complex models. Although it does not give all admissible estimators, it allows assessment of admissibility of a large class of estimators, in particular of some unbiased estimators based on the so-called cell means

Received May 1990; revised October 1991.
AMS 1980 *subject classifications*. Primary 62F10; secondary 62J10.
Key words and phrases. Unbalanced mixed linear models, invariant quadratic estimators, admissible estimation of variance components, submodels.

statistics [Yates (1934); Rankin (1974); Searle (1971); Tan and Tabatabai (1988) and Hocking (1988)]. For details see Klonecki and Zontek (1989b). An important feature of the method is that it allows construction of admissible (biased) nonnegative estimators when a nonnegative estimator is available in the specified submodel. Another attractive feature of the method is its adaptability to computer algorithms.

The paper is organized as follows. Section 2 recalls definitions and establishes notation used throughout the paper. In Section 3, the definition of a submodel is presented. It also contains the main result of this article, a relationship between an admissible estimator under a specified submodel and the corresponding admissible estimator under the primal model. In Section 4, a necessary and sufficient condition for an admissible estimator to be nonnegative is establish as well as some formulae one can apply to construct admissible estimators in some special cases. Section 5 is devoted to estimation with the condition of unbiasedness. It shows that a formula relating admissible unbiased estimators in submodels to corresponding admissible unbiased estimators in the primal models also can be established. In Section 6 admissible estimators are provided for the model with two variance components: for some factorial models with imbalance in the last stage and the unbalanced $(p-1)$-way nested random factors design. Also given are comments on how to construct nonnegative admissible estimators.

2. Preliminaries and notation. A representation of the variance component model with p variance components is

$$X = D\beta + \sum_{i=1}^{p} J_i \varepsilon_i. \tag{2.1}$$

The design matrix D and the matrices $J_1, \dots, J_p$ are known and $\varepsilon_1, \dots, \varepsilon_p$ are mutually uncorrelated random vectors such that for all $i = 1, \dots, p$,

$$E\varepsilon_i = 0,$$
$$\operatorname{cov} \varepsilon_i = \sigma_i I_{r_i},$$

where I_r denotes the $r \times r$ unit matrix. The regression vector β and the variance components $\sigma_1 \geq 0, \dots, \sigma_p \geq 0$ are unknown.

With this notation $D\beta$ is a representation of the mean vector of X and $\sum_{i=1}^{p} \sigma_i J_i J_i'$ is the covariance matrix of X. This is usually written as

$$X \sim \left(D\beta, \sum_{i=1}^{p} \sigma_i J_i J_i' \right). \tag{2.2}$$

We are interested in estimation of the vector of the variance components $\sigma = (\sigma_1, \dots, \sigma_p)'$ when $p \geq 2$ and when β is treated as a nuisance parameter. Attention is confined to estimators of the form $\hat{\sigma} = (X'L_1X, \dots, X'L_pX)'$, where $L_i = NL_iN$, while N is the orthogonal projection onto the intersection of the null space of D' and the space spanned by all columns of $J_1, \dots, J_p$. These estimators are invariant to the fixed effects design matrix D. A maximal invariant statistic is

$$NX \sim \left(0, \sum_{i=1}^{p} \sigma_i M_i\right), \tag{2.3}$$

where $M_i = NJ_iJ_i'N$. For convenience we shall call this model an invariant version of model (2.1). Estimators of σ belonging to the considered class are called invariant quadratic (IQE) and those that are also unbiased for σ are called invariant quadratic unbiased (IQUE).

To compute the quadratic risk of an IQE estimator one needs the fourth moments of NX. Here it is assumed that they are as under normality, although the normality itself is not needed. The problem of admissible quadratic estimation of variance components under normality among all invariant estimators (not only quadratic) with respect to the ordinary quadratic risk function was investigated by Farrell, Klonecki and Zontek (1989).

Throughout the paper we shall use the following notation. For any linear operator V: $\mathscr{V}_1 \to \mathscr{V}_2$, where $\mathscr{V}_1$ and $\mathscr{V}_2$ stand for finite-dimensional vector spaces with inner products (to be denoted by $\langle \cdot, \cdot \rangle$), the image, the null space, the adjoint operator and the Moore–Penrose generalized inverse are denoted by $\mathrm{Im}(V)$, $\mathscr{N}(V)$, $V^{\#}$ and V^{+}, respectively. The symbol $\mathscr{M}_{r\times t}$ stands for the space of all linear transformations mapping $\mathscr{R}^t$ into $\mathscr{R}^r$. We identify $\mathscr{M}_{r\times t}$ with the space of all $r \times t$ matrices. If $V \in \mathscr{M}_{r\times t}$, we write V' instead of $V^{\#}$ and $\mathscr{M}_t$ instead of $\mathscr{M}_{t\times t}$. The space of all self-adjoint operators (symmetric matrices) in $\mathscr{M}_t$ is denoted by $\mathscr{S}_t$. For any linear operator V in $\mathscr{M}_{r\times t}$ we denote by $V \otimes V$ the linear operator which maps $\mathscr{S}_t$ into $\mathscr{S}_r$ and is defined for each C in $\mathscr{S}_t$ by $(V \otimes V)C = VCV'$. For any given variance components model (2.1), we denote by $\mathscr{L}$ the set of all linear operators which map $\mathscr{R}^p$ into $\mathrm{Im}(N \otimes N)$.

Let $\mathscr{W}_t$ stand for the space of all linear operators mapping $\mathscr{S}_t$ into itself. As is customary, if $W \in \mathscr{W}_n$, $C \in \mathscr{S}_n$ and $L \in \mathscr{L}$, then WC denotes the value of W at C, while WL denotes the superposition of W and L. Also, if $a \in \mathscr{R}^t$ and $A \in \mathscr{S}_n$, then Aa' represents the linear operator mapping $\mathscr{R}^t$ into $\mathscr{S}_n$ defined for each $b \in \mathscr{R}^t$ by $(Aa')b = (a'b)A$. Clearly, if $t = 1$ and $a = 1$, then for every matrix A in $\mathscr{S}_n$ symbol $A^{\#}$ denotes the linear operator which maps $\mathscr{S}_n$ into $\mathscr{R}$ and is defined by $A^{\#}C = \mathrm{tr}(AC)$ for every matrix C in $\mathscr{S}_n$. We say that operator B is associated with symmetric matrices $B_1, \dots, B_p$, if $B = \sum_{i=1}^{p} B_i e_i'$, where e_i is the ith column of I_p. Moreover, symbol P_r, $r = 1, 2, \dots$, stands for the orthogonal projection onto the null space of $\mathbf{1}_r'$, where $\mathbf{1}_r$ is the r vector of ones. The nonnegative orthant of $\mathscr{R}^r$ is denoted by $\mathscr{R}_+^r$. Other notation will be introduced as needed.

The operators of $\mathscr{L}$ provide a convenient shorthand method for writing IQE estimators of σ. They can be written as $L^{\#}Y$, where $L \in \mathscr{L}$, while $Y = NXX'N$, exposing the fact that IQE estimators can be treated as linear estimators with respect to the random matrix Y. Under the adopted assumption regarding the moments of NX,

$$EY = M\sigma,$$
$$\mathrm{Cov}\, Y = 2M\sigma \otimes M\sigma,$$

where M stands for the operator in $\mathscr{L}$ which is associated with matrices $M_1, \ldots, M_p$. This will be written schematically as

$$Y \sim (M\sigma, 2M\sigma \otimes M\sigma).$$

When $\operatorname{Im}(M)$ is a p-dimensional subspace, which is henceforth adopted as a constant assumption, σ is invariantly estimable.

An IQE estimator is said to be admissible if it is admissible among the class of all IQE estimators; it is said to be an admissible unbiased estimator if it is an IQUE estimator and admissible among the class of all IQUE estimators. When an admissible (unbiased) estimator of σ is available, an admissible (unbiased) estimator of any parametric·function of σ can be obtained by applying a well-known lemma of Shinozaki (1975).

3. Main results.

3.1. *Submodels.* The concept of a submodel involving $p - 1$ variance components of model (2.1) involving p variance components is crucial for the considerations of this paper. For simplicity of notation the discarded variance component is fixed throughout the paper as σ_p.

DEFINITION 3.1. If S is any matrix such that

$$\operatorname{Im}(S) = \operatorname{Im}\left(\sum_{i=1}^{p-1} M_i\right), \tag{3.1}$$

then each variance component model with $p - 1$ variance components whose invariant version reduces to

$$X^* \sim \left(0, \sum_{i=1}^{p-1} \sigma_i M_i^*\right), \tag{3.2}$$

where $M_i^* = (S \otimes S)^+ M_i$, $i = 1, \ldots, p - 1$, is called a submodel specified by matrix S of model (2.1).

Since, in view of this definition, (3.2) is a maximal invariant statistic of all submodels specified by matrix S, every IQE estimator of variance components under any submodel resulting from matrix S is a linear function of $Y^* = X^*(X^*)'$ only. It should also be noted that the maximal invariant statistics (3.2) specified by different matrices S satisfying (3.1) must not be identical. Also we would like to point out that matrix $N^* = S^+S$ plays for each submodel specified by matrix S the same role as matrix N in the primal model (2.1). Examples illustrating the concept of a submodel are given in Section 6.

We now show that one can construct an admissible estimator of $\sigma = (\sigma_1, \ldots, \sigma_{p-1}, \sigma_p)'$ under model (2.1) when there is available an admissible estimator of $\sigma^* = (\sigma_1, \ldots, \sigma_{p-1})'$ under its submodels with variance components $\sigma_1, \ldots, \sigma_{p-1}$ specified by an appropriate matrix S.

3.2. *Construction of an admissible estimator via submodels.* To establish the theorem on which the construction is based we need the following lemma.

Let $\hat{\sigma} = L^{\#}Y$ be any (invariant) estimator of σ under model (2.1) and let S be a matrix fulfilling condition (3.1). Then $\hat{\sigma}^* = [(S \otimes S)'LT']^{\#}Y^*$, where $T = (I_{p-1}|0) \in \mathscr{M}_{(p-1)\times p}$, represents an estimator of $\sigma^* = (\sigma_1, \dots, \sigma_{p-1})'$ under model (3.2). We now show that the quadratic risk $r(\hat{\sigma}, \sigma) = E(\hat{\sigma} - \sigma)'(\hat{\sigma} - \sigma)$ of estimator $\hat{\sigma}$ at point $\sigma = T'\sigma^*$ under model (2.1) is not smaller than the quadratic risk $r(\hat{\sigma}^*, \sigma^*)$ of estimator $\hat{\sigma}^*$ at point σ^* under model (3.2).

LEMMA 3.1. *For any vector σ^* in $\mathscr{R}_+^{p-1}$,*

$$r(\hat{\sigma}, T'\sigma^*) \ge r(\hat{\sigma}^*, \sigma^*) \tag{3.3}$$

holds with equality for all $\sigma^* \in \mathscr{R}_+^{p-1}$ if and only if

$$S'(Le_p)S = 0.$$

PROOF. First observe that $r(\hat{\sigma}, \sigma)$ can be written as

$$r(\hat{\sigma}, \sigma) = 2\operatorname{tr} L^{\#}(M\sigma \otimes M\sigma)L + \sigma'\big(M^{\#}L - I_p\big)\big(L^{\#}M - I_p\big)\sigma.$$

Writing L as $L = LT'T + Le_pe_p'$, the first term on the right-hand side becomes

$$2\operatorname{tr} TL^{\#}(M\sigma \otimes M\sigma)LT' + 2e_p'L^{\#}(M\sigma \otimes M\sigma)Le_p$$

and when $\sigma = T'\sigma^*$, the second one takes the form of

$$\sigma'\big(M^{\#}L - I_p\big)T'T\big(L^{\#}M - I_p\big)'\sigma + \big(\sigma'M^{\#}Le_p\big)^2.$$

Using the fact that $M\sigma = (S \otimes S)M^*\sigma^*$, where M^* is the operator associated with matrices $M_1^*, \dots, M_{p-1}^*$, we obtain

$$\begin{aligned}
r(\hat{\sigma}, T'\sigma^*)&\\
&= \operatorname{tr}\big\{2TL^{\#}(S \otimes S)[(M^*\sigma^*) \otimes (M^*\sigma^*)](S \otimes S)'LT'\big\}\\
&\quad + (\sigma^*)'\big[(M^*)^{\#}(S \otimes S)'LT' - I_{p-1}\big]\big[TL^{\#}(S \otimes S)M^* - I_{p-1}\big]\sigma^*\\
&\quad + 2e_p'L^{\#}(S \otimes S)[(M^*\sigma^*) \otimes (M^*\sigma^*)](S \otimes S)'Le_p\\
&\quad + \big[(\sigma^*)'(M^*)^{\#}(S \otimes S)'Le_p\big]^2\\
&= r(\hat{\sigma}^*, \sigma^*) + E\big[\langle S'Le_pS, Y^*\rangle^2\big].
\end{aligned}$$

In view of this formula the first assertion of the lemma is evident and the second one is obtained from the fact that $Y^* \sim (M^*\sigma^*, 2M^*\sigma^* \otimes M^*\sigma^*)$ and that $\operatorname{Im}(S') = \operatorname{Im}(M^*\mathbf{1}_{p-1})$. $\square$

Theorem 3.1 provides a formula for an admissible estimator of σ under the primal model in terms of an admissible estimator of σ^* under a submodel specified by a matrix S. As before, X^* stands for the invariant version of the submodels specified by matrix S and $Y^* = X^*(X^*)'$. Moreover, $B_0 = (N - SS^+)M_p^+$ and $q = \mathrm{tr}(B_0 M_p)$. We also assume that M_p and SS^+ commute. This condition is met trivially when J_p is the $n \times n$ unit matrix.

THEOREM 3.1. *If $(L^*)^{\#}Y^*$ is an admissible estimator of σ^* under a submodel specified by matrix S of model* (2.1), *then for any vector $\alpha \in \mathscr{R}_+^p$, with its last coordinate equal to* 1, *expression $L^{\#}Y$, where*

$$L = L_0 + \frac{1}{2+q} B_0\left[\left(I_p - L_0^{\#}M\right)\alpha\right]', \tag{3.4}$$

while $L_0 = (S^+ \otimes S^+)'L^*T$, represents an admissible estimator of σ under model (2.1).

PROOF. To begin, notice that if L is given by (3.4), then using the readily proven formulae $S'B_0S = 0$ and $(S \otimes S)'LT' = L^*$ we get from Lemma 3.1 that

$$r(L^{\#}Y, T'\sigma^*) = r\left((L^*)^{\#}Y^*, \sigma^*\right).$$

Now suppose to the contrary that $L_1^{\#}Y$ is better than $L^{\#}Y$. Since by Lemma 3.1,

$$\begin{aligned} r\left((L^*)^{\#}Y^*, \sigma^*\right) &= r(L^{\#}Y, T'\sigma^*) \geq r\left(L_1^{\#}Y, T'\sigma^*\right) \\ &\geq r\left(\left[(S \otimes S)'L_1T'\right]^{\#}Y^*, \sigma^*\right), \end{aligned}$$

$[(S \otimes S)'L_1T']^{\#}Y^*$ must be as good as $(L^*)^{\#}Y^*$. But since $(L^*)^{\#}Y^*$ is admissible for σ^* by assumption, we get that $(S \otimes S)'L_1T' = L^*$ and, again using Lemma 3.1, that $(S \otimes S)'L_1e_p = 0$. This in turn entails that

$$\begin{aligned} (SS^+ \otimes SS^+)L_1 &= (S^+ \otimes S^+)'(S \otimes S)'L_1(T'T + e_pe_p') \\ &= (S^+ \otimes S^+)'L^*T = L_0. \end{aligned}$$

Hence $L_1 \in \mathscr{A} = L_0 + \{\Pi Z\colon Z \in \mathscr{L}\}$, where $\Pi = N \otimes N - SS^+ \otimes SS^+$. Also observe that $L \in \mathscr{A}$. If $\mathrm{Im}(S) = \mathrm{Im}(N)$, then $\Pi = 0$ and $\mathscr{A}$ consists of exactly one element. This contradicts the assumption that $(L_1^*)^{\#}Y$ is better than $L^{\#}Y$ so that in this case the proof is terminated.

Otherwise it is sufficient to show that for any α, as specified in the theorem, $L^{\#}Y$ is the unique locally best estimator at point $\sigma = \alpha$ among the class $\{\tilde{L}^{\#}Y\colon \tilde{L} \in \mathscr{A}\}$, because this contradicts the assumption that $L_1^{\#}Y$ is better than $L^{\#}Y$. The former follows from the fact [see LaMotte (1982)] that $\Pi(2W + w)L = \Pi M\alpha\alpha'$, where $W = M\alpha \otimes M\alpha$ and $w = M\alpha\alpha'M^{\#}$, which readily can be verified by noting that under the assumption that SS^+ and M_p commute the following formulae hold: $\Pi W(S^+ \otimes S^+)' = 0$, $\Pi(2W + w)B_0 = (2 + q)B_0^+$ and $\Pi M\alpha\alpha' = B_0^+\alpha'$. $\square$

4. Corollaries. We shall now deduce from Theorem 3.1 a number of results that may be useful for constructing nonnegative admissible estimators of variance components.

4.1. *An alternative formula for* (3.4):

COROLLARY 4.1. *If*

$$\left[\omega^*\left(2I_{p-1} + K^*\omega^*\right)^{-1}\right]'(B^*)^{\#}Y^*, \tag{4.1}$$

where $K^* = (M^*)^{\#}B^*$, *is an admissible estimator of* σ^* *under a submodel specified by matrix* S *of model* (2.1), *then for any vector* $\alpha^* \in \mathscr{R}_+^{p-1}$, *expression*

$$\left[\omega\left(2I_p + K\omega\right)^{-1}\right]' B^{\#}Y, \tag{4.2}$$

where

$$\omega = \begin{pmatrix} \omega^* & 0 \\ (\alpha^*)' & 1 \end{pmatrix}, \qquad K = M^{\#}B = \begin{pmatrix} K^* & 0 \\ k' & q \end{pmatrix},$$

$k = TB^{\#}M_p$, *while* $B = (S^+ \otimes S^+)'B^*T + B_0 e_p'$, *represents an admissible estimator of* σ *under model* (2.1).

PROOF. Substituting $L^* = B^*\omega^*(2I_{p-1} + K^*\omega^*)^{-1}$ and $\alpha = ((\alpha^*)', 1)'$ into the right side of (3.4), we find it equal to $B\omega(2I_p + K\omega)^{-1}$. Noting that $k = (\mathrm{tr}(M_p B_1), \dots, \mathrm{tr}(M_p B_{p-1}))'$ and applying Theorem 3.1 gives the result at once. □

4.2. *A necessary and sufficient condition for estimator* (3.4) *to be nonnegative*. The condition is given in Corollary 4.2. Here $a \geq b$, where $a = (a_1, \dots, a_s)'$ and $b = (b_1, \dots, b_s)'$, means that $a_i \geq b_i$.

COROLLARY 4.2. *Estimator* (3.4) *is nonnegative if and only if estimator* $(L^*)^{\#}Y^*$ *is nonnegative and*

$$\left[I_{p-1} - (L^*)^{\#}M^*\right]T\alpha \geq m,$$

where $m = TL_0^{\#}M_p$.

PROOF. It follows straightforwardly from (3.4) by noting that

$$TL_0^{\#}M = \left((L^*)^{\#}M^* \middle| m\right)$$

and that $L_0^{\#}Y$ and $\mathrm{tr}(B_0 Y)$ are uncorrelated. □

From Corollary 4.2 we conclude that in order that (4.2) be nonnegative, it is necessary and sufficient that (4.1) be nonnegative and that

$$2\left[(2I_{p-1} + K^*\omega^*)^{-1}\right]'\alpha^* \geq \left[\omega^*(2I_{p-1} + K^*\omega^*)^{-1}\right]'k.$$

Evidently, $\alpha^* = (\omega^*)'k/2 \geq 0$ fulfills this condition.

4.3. *Admissible estimators for a special class of variance components models.* The method of constructing admissible estimators via submodels presented here is especially attractive for those models (2.1) which have the structure

$$X = H\tilde{X} + \varepsilon_p, \tag{4.3}$$

where H is a matrix of full rank,

$$\tilde{X} = D^*\beta + \sum_{i=1}^{p-1} J_i^*\varepsilon_i, \tag{4.4}$$

while D^* and $J_1^*, \dots, J_{p-1}^*$ are matrices of constants such that the null space $\mathscr{N}[(D^*)']$ is contained in $\mathrm{Im}(J_1^*| \cdots |J_{p-1}^*)$. In this case $N = I - HD^*(HD^*)^+$. The relevant matrix for model $\tilde{X}$ is given by $N^* = I - D^*(D^*)^+$.

Corollary 4.3. *If matrix* $S = NH$ *fulfills condition* (3.1), *then* (4.4) *is a submodel of* (4.3) *and the operator* B *defined in Corollary* 4.1 *takes the form*

$$B = (H^+ \otimes H^+)' B^* T + (I_n - HH^+)e_p'. \tag{4.5}$$

Proof. The first assertion follows from the easily proven fact that $S^+ = N^*H^+$ and that H^+H is a unit matrix. The second part is obtained by noting that $SS^+ = HH^+ - DD^+$. □

5. Admissible unbiased estimators. It is interesting that for some of the presented results on estimation of variance components one can establish analogous results for estimation with the condition of unbiasedness. Before we state them, it may be in order to make the following comments. If $L^\# Y$ is an admissible unbiased estimator of σ, then $\mathrm{Im}(L)$ is a p dimensional subspace contained in the smallest quadratic subspace containing $\mathrm{Im}(M)$ and if B is any operator in $\mathscr{L}$ such that $\mathrm{Im}(B) = \mathrm{Im}(L)$, then $B^\# M$ is invertible and $(B^\# M)^{-1}B^\# Y$ is the same as $L^\# Y$. The point that needs to be stressed here is that every operator B in $\mathscr{L}$ such that $\dim \mathrm{Im}(B) = p$ and $\mathrm{Im}(B) \cap \mathscr{N}(M^\#) = \{0\}$ uniquely determines an unbiased estimator of σ. The problem of constructing admissible unbiased estimators of σ is here addressed by indicating operators B in $\mathscr{L}$ that lead to admissible unbiased estimators. It is also worthwhile to notice that if $L^\# Y$ is an admissible (biased) estimator of σ and if $\dim \mathrm{Im}(L) = p$, then $(L^\# M)^{-1}L^\# Y$ is an admissible unbiased estimator of σ. The invertibility of $L^\# M$ follows from Theorem 2.3 of LaMotte (1980).

As before set $L_0 = (S^+ \otimes S^+)' L^* T$, $B_0 = (N - SS^+) M_p^+$, $q = \operatorname{tr}(B_0 M_p)$ and assume that SS^+ and M_p commute. Now also assume that $SS^+ M_p \neq M_p$, in which case $q \neq 0$.

THEOREM 5.1. *If $(L^*)^{\#} Y^*$ is an admissible unbiased estimator of σ^* under a submodel specified by matrix S of model* (2.1), *then $L^{\#} Y$ with*

$$L = L_0 + \frac{1}{q} B_0 \left[\left(I_p - L_0^{\#} M \right) e_p \right]',$$

is an admissible unbiased estimator of σ under model (2.1).

PROOF. To establish the assertion it suffices to check that $L^{\#} Y$ is an unbiased estimator of σ and admissible among the class of IQUE estimators. Since the former is evident and the latter can be established along the same lines as Theorem 3.1, we omit the proof. □

The notation used below is as in Corollary 4.1.

COROLLARY 5.1. *If $[(K^*)^{-1}]'(B^*)^{\#} Y^*$ is an admissible unbiased estimator of σ^* under a submodel of* (2.1) *resulting from matrix S, then K is invertible and $(K^{-1})' B^{\#} Y$ is an admissible unbiased estimator of σ under the primal model.*

6. Applications. We shall now illustrate applications of the obtained results to constructing admissible estimators of variance components for three special variance component models.

6.1. *Model with two variance components.* For $p = 2$, equation (2.1) becomes $X = D\beta + J_1 \varepsilon_1 + J_2 \varepsilon_2$. Assume here also that $J_2 = I_n$, in which event $N = I_n - DD^+$, and that $\operatorname{Im}(M_1) \neq \operatorname{Im}(M_2)$. Since matrix $S = NJ_1$ fulfills condition (3.1) and since $S^+ = C^+ J_1' N$, where $C = J_1' N J_1$, we notice that $X^* \sim (0, \sigma_1 CC^+)$ is a submodel of the model with two variance components considered here. Admissible estimators of $\sigma = (\sigma_1, \sigma_2)'$ can now be obtained via Corollary 4.1 provided we can compute explicitly the Moore–Penrose g-iverse C^+.

It can fairly easily be shown that the operator associated with matrices

$$B_1 = NJ_1 (C^+)^2 J_1' N,$$
$$B_2 = N - NJ_1 C^+ J_1' N,$$

can be substituted for B in (4.2). With this choice of B,

$$K = \begin{pmatrix} \operatorname{tr}(CC^+) & 0 \\ \operatorname{tr} B_1 & \operatorname{tr} B_2 \end{pmatrix}.$$

For model (2.1) with $p = 2$ variance components the class of all estimators of form (4.2) together with the class of all unique Bayes IQE estimators constitutes the minimal complete class first obtained by Gnot and Kleffe (1983).

6.2. *Factorial models.* Although the developments below apply to all unbalanced factorial models having structure (4.3), we shall deal, for reasons of simplicity, only with two two-way classification random models with imbalance at the last stage, the model with interaction and the model without interaction.

(i) *Model with interaction.* In the notation of (2.1) this model is specified by the following matrices: $D = \mathbf{1}_n$, $J_1 = J_3(I_a \otimes \mathbf{1}_b)$, $J_2 = J_3(\mathbf{1}_a \otimes I_b)$, $J_3 = \operatorname{diag}(\mathbf{1}_{n_{11}}, \dots, \mathbf{1}_{n_{ab}})$ and $J_4 = I_n$, where $\otimes$ denotes the Kronecker product, while $n = \sum_{i=1}^a \sum_{j=1}^b n_{ij}$, $n_{ij} \geq 1$. This model has the structure (4.3) postulated in Corollary 4.3:

$$X = H\tilde{X} + \varepsilon_4,$$

where $H = J_3$, while

$$\tilde{X} = \mathbf{1}_{ab}\beta + (I_a \otimes \mathbf{1}_b)\varepsilon_1 + (\mathbf{1}_a \otimes I_b)\varepsilon_2 + \varepsilon_3.$$

Notice that $J_3'\mathbf{1}_n$ is the vector of cell sizes, whereas $\tilde{X}$ is a balanced additive model with one observation per cell and that in this case $N = P_n = M_4$. Also notice that the matrix $S = NH$ fulfills condition (3.1) and that $\tilde{X}$ belongs to the class of submodels specified by this matrix. For this model there exists a best unbiased estimator and also there is available a complete characterization of admissible estimators. As shown by Zontek and Klonecki (1990) estimator (4.1) is admissible under model $\tilde{X}$ when B^* stands for the linear operator associated with the matrices

$$B_1^* = \frac{1}{b^2}M_1^*, \qquad B_2^* = \frac{1}{a^2}M_2^* \quad \text{and} \quad B_3^* = N^* - \frac{1}{b}M_1^* - \frac{1}{a}M_2^*$$

and when

$$\omega^* = \begin{pmatrix} 1 & 0 & 0 \\ \omega_{21} & 1 & 0 \\ \omega_{31} & \omega_{32} & 1 \end{pmatrix}, \qquad \omega_{ij} \geq 0.$$

To derive the corresponding admissible biased estimators given in Corollary 4.1 and the admissible unbiased estimator given in Corollary 5.1 one needs to calculate K^*, k and B_0, which are

$$K^* = (M^*)^{\#}B^* = \begin{pmatrix} a-1 & 0 & 0 \\ 0 & b-1 & 0 \\ (a-1)/b & (b-1)/a & (a-1)(b-1) \end{pmatrix},$$

$$k = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{1}{n_{ij}}(a-1, b-1, (a-1)(b-1))'$$

and

$$B_0 = I_n - J_3 J_3^+.$$

It is a matter of elementary calculation to show that the resulting admissible estimator (4.2) is nonnegative when

$$\omega_{21} \geq 0, \qquad \omega_{32} \geq \frac{b-1}{2a},$$

$$\omega_{31} \geq \frac{a-1}{2b} + \frac{b-1}{a(b+1)}(1 + a\omega_{32})\omega_{21}$$

and when $\alpha^* = (\omega^*)'k/2$. The first three conditions ensure nonnegativeness of the estimator of σ^* under model $\tilde{X}$, whereas the last one ensures the nonnegativeness of the estimator of σ under model X. Since $J_3^+ = \operatorname{diag}((1/n_{11})\mathbf{1}'_{n_{11}}, \dots, (1/n_{ab})\mathbf{1}'_{n_{ab}})$, expression $J_{p-1}^* X$ represents the vector of cell means. Consequently, admissible estimators obtained via Theorems 4.1 and 5.1 belong to the class of the so-called cell means estimators. The unbiased estimator is that arising from the sum of squares suggested by Yates (1934).

(ii) *Model without interaction.* The equation of the two-way classification model without interaction can also be written as

$$X = H\tilde{\tilde{X}} + \varepsilon_3, \tag{6.1}$$

where $H = \operatorname{diag}(\mathbf{1}_{n_{11}}, \dots, \mathbf{1}_{n_{ab}})$, while

$$\tilde{\tilde{X}} = \mathbf{1}_{ab}\beta + (I_a \otimes \mathbf{1}_b)\varepsilon_1 + (\mathbf{1}_a \otimes I_b)\varepsilon_2.$$

Now, however, the matrix NH does not satisfy condition (3.1). A natural choice of a matrix that fulfills it is $S = P_n H(I_a \otimes \mathbf{1}_b | \mathbf{1}_a \otimes I_b)$, where $n = \operatorname{tr} H'H$. Its Moore–Penrose g-inverse [see Zontek (1989)] is $S^+ = N^* W^{-1} S'$, where $N^* = \operatorname{diag}(P_a, P_b)$, while $W = S'S + \operatorname{diag}(\mathbf{1}_a\mathbf{1}'_a, \mathbf{1}_b\mathbf{1}'_b)$.

A submodel specified by this matrix is $X^* = N^*(\varepsilon'_1, \varepsilon'_2)'$, for which a complete characterization of admissible estimators is readily obtained.

The operator B appearing in Corollaries 4.1 and 5.1 becomes

$$B = (S \otimes S)(W \otimes W)^{-1} M^* T + (P_n - SW^{-1}S')e'_3, \tag{6.2}$$

where $T = (I_2 | 0_2)$ and M^* is the operator associated with matrices

$$M_1^* = \operatorname{diag}(P_a, 0), \quad M_2^* = \operatorname{diag}(0, P_b).$$

Model X is a particular case of the crossed classification model without interaction. For such models with empty cells admissible estimators have been obtained by Zontek (1989).

It is also worthwhile to point out that, in this event, the admissible unbiased estimator of σ determined by (6.2) may differ from the relevant cell means unbiased estimator derived from the sum of squares given by Westfall and Bremer [(1991), formula (3-6)]. That corresponds to the operator

$$B = (H^+ \otimes H^+)' QT + (P_n - SW^{-1}S')e'_3,$$

where Q is associated with matrices

$$Q_1 = P_{ab}(I_a \otimes \mathbf{1}_b\mathbf{1}'_b)P_{ab}, \qquad Q_2 = P_{ab}(\mathbf{1}_b\mathbf{1}'_b \otimes I_b)P_{ab}.$$

The admissibility of this cell means the estimator cannot be established by the method developed in this paper. As a matter of fact, for all the special cases of model (6.1) that we investigated, numerical calculations seem to suggest that it is inadmissible among the class of IQUE estimators.

The above considerations carry over to all factorial models with imbalance at the last stage. If the highest order interaction term is included in the model, then there exists a matrix S such that the specified random model (3.2) is balanced and the unbiased admissible estimator obtained via Theorem 5.1 coincides with the cell means estimator as defined by Westfall and Bremer (1991).

6.3. *The random, unbalanced* ($p-1$)*-way nested classification model.* We begin by showing that this model can be presented as a superposition of a chain of nested submodels.

Let $m_0 = 1 < m_1 < \cdots < m_p = m_{p+1}$ and for $i = 0, 1, \dots, p$ let H_i be a block diagonal $m_{i+1} \times m_i$ matrix defined by

$$H_i = \operatorname{diag}\left(\mathbf{1}_{n_{i1}}, \dots, \mathbf{1}_{n_{im_i}}\right),$$

where $n_{ij} \geq 1$. With this notation $m_{i+1} = \sum_{j=1}^{m_i} n_{ij}$, $H_0 = \mathbf{1}_{m_1}$ and $H_p = I_{m_p}$.

Following LaMotte (1972), we write the random, unbalanced ($p-1$)-way nested classification model as

$$X^{(p)} = H_{p-1}X^{(p-1)} + \varepsilon_p, \tag{6.3}$$

where

$$X^{(p-1)} = H_{p-2}X^{(p-2)} + \varepsilon_{p-1},$$

$$\vdots$$

$$X^{(1)} = H_0\beta + \varepsilon_1,$$

while $\beta \in \mathscr{R}$ and, as previously, $\varepsilon_1, \dots, \varepsilon_p$ stand for uncorrelated random vectors such that $E\varepsilon_i = 0$ and $\operatorname{Cov}\varepsilon_i = \sigma_i I_{m_i}$, $i = 1, \dots, p$.

Let $S_i = P_{m_{i+1}}H_i$, $i = 1, \dots, p-1$. From Corollary 4.3 it follows that model $X^{(i)}$, $i = 1, \dots, p-1$, is a submodel, specified by matrix S_i, of model $X^{(i+1)}$. The first model $X^{(1)}$ in this chain of submodels involves only one variance component, namely, σ_1, and expression $[1/(m_1+1)](X^{(1)})'P_{m_1}X^{(1)}$, being of form (4.1), is its admissible estimator. Using Corollaries 4.1 and 4.3 repeatedly to the successive submodels we obtain an admissible estimator of σ under the primal model. It will be of the form (4.2) with ω being a lower triangular $p \times p$ matrix with nonnegative entries and with unit diagonals and B being the linear map associated with matrices

$$B_i = \left(H_i^+ \cdots H_p^+\right)'\left(I_{m_i} - H_{i-1}H_{i-1}^+\right)H_i^+ \cdots H_p^+, \qquad i = 1, \dots, p.$$

It also should be evident that ω can be chosen so as to yield nonnegative admissible estimators.

If in model (6.3) the first $1 \leq t < p$ vectors $\varepsilon_1, \dots, \varepsilon_t$ are treated as fixed effects, then the linear operator associated with matrices $B_{t+1}, \dots, B_p$ becomes a relevant operator B for the construction of the admissible estimators under the resulting mixed nested model with $p - t$ variance components. Therefore, for such mixed models, admissible nonnegative estimators also easily can be constructed.

The problem of constructing admissible estimators for nested models was also considered in an earlier work by Klonecki and Zontek (1989a), where some alternative admissible estimators are presented.

7. Final comments. The method developed in the paper provides a class of attractive alternatives to the MINQE(I) and MINQE(U, I) as well to other estimators available in the literature. In fact, it includes nonnegative estimators and, moreover, the estimators belonging to this class have generally a better risk performance than the unbiased estimators [see Zontek and Klonecki (1990)]. Selection of an estimator from this class can be based on its risk, which is easily calculated. Our numerous calculations seem to indicate that the risk function of an admissible (biased) estimator $L^{\#}Y$ is flatter with increasingly larger dimensions of $\mathrm{Im}(L)$. The problem of which estimators to recommend in particular cases requires further studies.

Acknowledgments. We would like to thank an Associate Editor and two referees for helpful criticisms of earlier versions of this paper and for bringing the reference of LaMotte (1972) to our attention. Also, we would like to acknowledge valuable suggestions and encouragement from Arthur Cohen, the former Editor.

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