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# BREAKDOWN POINTS OF AFFINE EQUIVARIANT ESTIMATORS OF MULTIVARIATE LOCATION AND COVARIANCE MATRICES

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Finite-sample replacement breakdown points are derived for different types of estimators of multivariate location and covariance matrices. The role of various equivariance properties is illustrated. The breakdown point is related to a measure of performance based on large deviations probabilities. Finally, we show that one-step reweighting preserves the breakdown point.

**1. Introduction.** Several notions of robustness have been considered for estimators of a multivariate location parameter $\boldsymbol{\mu} \in \mathbb{R}^p$. One of these concepts is the breakdown point, a global measure of robustness suggested by Hodges (1967) and Hampel (1968). A simple and appealing finite-sample version of this concept was given by Donoho and Huber (1983). Roughly, this finite-sample replacement breakdown point measures the minimum fraction of outliers that will spoil the estimate completely. Estimators with zero breakdown point can therefore not be robust. Recently, He, Jurečková, Koenker and Portnoy (1988) established a relation between the replacement breakdown point and certain measures of performance based on large deviations. Their results show that the breakdown point is not just an attractive robustness concept, but that it also has a stochastic motivation.

A natural condition for multivariate estimators is equivariance under affine transformations. To combine affine equivariance with a high breakdown point is not trivial. Donoho (1982) discusses several affine equivariant multivariate methods, showing that their breakdown point goes down to 0 as the dimension $p$ increases. Stahel (1981) and Donoho (1982) independently introduced an affine estimator of multivariate location and covariance with a high breakdown point in any dimension. Another estimator with this combination of properties was the minimum volume ellipsoid estimator [Rousseeuw (1985)].

But what is the best possible value of the breakdown point? For covariance estimators the maximal breakdown point was derived by Davies (1987). In our paper we are mainly concerned with upper bounds for pure location estimators satisfying various equivariance properties. Section 2 discusses these types of equivariance and gives an upper bound on the breakdown point. In order to

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investigate to what extent this bound is sharp, Sections 2 and 3 study several examples including the $L_1$ estimator, Oja's (1983) generalized median and smooth $S$-estimators in the sense of Rousseeuw and Yohai (1984).

Section 4 extends the results of He, Jurečková, Koenker and Portnoy (1988) to multivariate location estimators. Estimators with maximal breakdown point satisfy a minimax property: They maximize least favorable tail performance over the class of algebraically tailed distributions.

To combine high breakdown point with high asymptotic efficiency, it is often suggested to start with a high breakdown estimate and then to take a one-step improvement which preserves the breakdown point and obtains a better efficiency. Section 5 shows that the breakdown point is preserved if one does a one-step reweighting by computing the usual weighted mean and covariance matrix, where the weights are based on the Mahalanobis distances with respect to the initial estimates.

**2. Maximal breakdown point of equivariant estimators.** Let $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be a collection of $n$ points in $\mathbb{R}^p$ and denote by $\mathbf{t}_n(\mathbf{X}) \in \mathbb{R}^p$ a location estimate based on $\mathbf{X}$. We say that $\mathbf{t}_n$ is *translation equivariant* if $\mathbf{t}_n(\mathbf{X} + \mathbf{v}) = \mathbf{t}_n(\mathbf{X}) + \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^p$, where $\mathbf{X} + \mathbf{v} = \{\mathbf{x}_1 + \mathbf{v}, \dots, \mathbf{x}_n + \mathbf{v}\}$. When $\mathbf{t}_n$ is equivariant not only under shifts of $\mathbf{X}$ but also under affine transformations, then $\mathbf{t}_n$ is called *affine equivariant*, i.e.,

$$\mathbf{t}_n(\mathbf{AX} + \mathbf{v}) = \mathbf{A}\mathbf{t}_n(\mathbf{X}) + \mathbf{v} \tag{2.1}$$

for all nonsingular $p \times p$ matrices $\mathbf{A}$ and $\mathbf{v} \in \mathbb{R}^n$, where $\mathbf{AX} + \mathbf{v} = \{\mathbf{Ax}_1 + \mathbf{v}, \dots, \mathbf{Ax}_n + \mathbf{v}\}$. Although this condition is quite natural, it turns out that some well-known estimators of multivariate location fail to satisfy it. The condition can be relaxed by requiring (2.1) only for orthogonal matrices, and it is then referred to as *orthogonal equivariance* or rigid motion equivariance. At the end of this section we shall consider a translation equivariant estimator which is not orthogonal equivariant, and also an orthogonal equivariant estimator which is not affine equivariant. A covariance estimate $\mathbf{C}_n(\mathbf{X}) \in \mathrm{PDS}(p)$, the class of all positive definite symmetric $p \times p$ matrices, is said to be affine equivariant if $\mathbf{C}_n(\mathbf{AX} + \mathbf{v}) = \mathbf{AC}_n(\mathbf{X})\mathbf{A}^{\mathrm{T}}$ for all $\mathbf{v} \in \mathbb{R}^p$ and nonsingular $\mathbf{A}$, where $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$.

We measure the robustness of $\mathbf{t}_n$ and $\mathbf{C}_n$ by means of the finite-sample replacement breakdown point [Donoho and Huber (1983)]. The breakdown point of a location estimator $\mathbf{t}_n$ at a collection $\mathbf{X}$ is defined as the smallest fraction $m/n$ of outliers that can take the estimate over all bounds:

$$\varepsilon^*(\mathbf{t}_n, \mathbf{X}) = \min_{1 \le m \le n} \left\{ \frac{m}{n} : \sup_{\mathbf{Y}_m} \| \mathbf{t}_n(\mathbf{X}) - \mathbf{t}_n(\mathbf{Y}_m) \| = \infty \right\}, \tag{2.2}$$

where the supremum is taken over all possible corrupted collections $\mathbf{Y}_m$ that are obtained from $\mathbf{X}$ by replacing $m$ points of $\mathbf{X}$ by arbitrary values. Although $\varepsilon^*(\mathbf{t}_n, \mathbf{X})$ appears to depend on $\mathbf{X}$, for almost $\mathbf{t}_n$ this will not be the case. However, location estimators $\mathbf{t}_n$ with $\varepsilon^*(\mathbf{t}_n, \mathbf{X})$ depending on $\mathbf{X}$ do exist [see, for instance, Huber (1984)]. The breakdown point of a covariance estimator $\mathbf{C}_n$

at a collection $\mathbf{X}$ is defined as the smallest fraction $m/n$ of outliers that can either take the largest eigenvalue $\lambda_1(\mathbf{C}_n)$ over all bounds, or take the smallest eigenvalue $\lambda_p(\mathbf{C}_n)$ arbitrarily close to 0:

$$\varepsilon^*(\mathbf{C}_n, \mathbf{X}) = \min_{1 \le m \le n} \left\{ \frac{m}{n} : \sup_{\mathbf{Y}_m} D(\mathbf{C}_n(\mathbf{X}), \mathbf{C}_n(\mathbf{Y}_m)) = \infty \right\},$$

where the supremum is taken over the same corrupted collections $\mathbf{Y}_m$ as in (2.2), and where $D(\mathbf{A}, \mathbf{B}) = \max\{|\lambda_1(\mathbf{A}) - \lambda_1(\mathbf{B})|, |\lambda_p(\mathbf{A})^{-1} - \lambda_p(\mathbf{B})^{-1}|\}$, with $\lambda_1(\mathbf{A}) \ge \cdots \ge \lambda_p(\mathbf{A})$ being the ordered eigenvalues of the matrix $\mathbf{A}$.

Donoho and Huber (1983) also considered other finite-sample versions, such as addition breakdown. We personally prefer replacing observations to adding observations because replacement contamination is simple, realistic and generally applicable. Indeed, from an intuitive point of view, outliers are not some faulty observations that are added at the end of the sample, but they treacherously hide themselves by replacing some of the data points that should have been observed. Moreover, as we will see in Section 4, the replacement breakdown point also has a stochastic interpretation.

First we show that the breakdown point of any affine equivariant estimator is itself invariant under affine transformations.

LEMMA 2.1. *Let $\mathbf{X}$ be a collection of $n$ points on $\mathbb{R}^p$, and let $\mathbf{t}_n(\mathbf{X}) \in \mathbb{R}^p$ and $\mathbf{C}_n(\mathbf{X}) \in \mathrm{PDS}(p)$ be location and covariance estimates based on $\mathbf{X}$.*

(i) *When $\mathbf{t}_n$ is translation equivariant, then for any $\mathbf{v} \in \mathbb{R}^p$ it holds that $\varepsilon^*(\mathbf{t}_n, \mathbf{X} + \mathbf{v}) = \varepsilon^*(\mathbf{t}_n, \mathbf{X})$.*

(ii) *When $\mathbf{t}_n$ is affine (orthogonal) equivariant, then for any $\mathbf{v} \in \mathbb{R}^p$ and for any nonsingular (orthogonal) $p \times p$ matrix $\mathbf{A}$ it holds that $\varepsilon^*(\mathbf{t}_n, \mathbf{A}\mathbf{X} + \mathbf{v}) = \varepsilon^*(\mathbf{t}_n, \mathbf{X})$.*

(iii) *When $\mathbf{C}_n$ is affine equivariant, then for any $\mathbf{v} \in \mathbb{R}^p$ and for any nonsingular $p \times p$ matrix $\mathbf{A}$ it holds that $\varepsilon^*(\mathbf{C}_n, \mathbf{A}\mathbf{X} + \mathbf{v}) = \varepsilon^*(\mathbf{C}_n, \mathbf{X})$.*

PROOF. Let $\mathbf{A}$ be a nonsingular $p \times p$ matrix and $\mathbf{v} \in \mathbb{R}^p$. Denote by $\mathbf{Y}_m$ a corrupted collection that differs from $\mathbf{X}$ in at most $m$ points, so that $\mathbf{A}\mathbf{Y}_m + \mathbf{v}$ differs from $\mathbf{A}\mathbf{X} + \mathbf{v}$ in at most $m$ points. When $\mathbf{t}_n$ is affine equivariant we have that $\|\mathbf{t}_n(\mathbf{A}\mathbf{X} + \mathbf{v}) - \mathbf{t}_n(\mathbf{A}\mathbf{Y}_m + \mathbf{v})\| = \|\mathbf{A}[\mathbf{t}_n(\mathbf{X}) - \mathbf{t}_n(\mathbf{Y}_m)]\|$. In that case, together with the fact that for symmetric $p \times p$ matrices $\mathbf{M}$ one has

$$\lambda_p(\mathbf{M}) = \inf_{\mathbf{y}} \frac{\mathbf{y}^{\mathrm{T}}\mathbf{M}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} \quad \text{and} \quad \lambda_1(\mathbf{M}) = \sup_{\mathbf{y}} \frac{\mathbf{y}^{\mathrm{T}}\mathbf{M}\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}}, \tag{2.3}$$

we obtain

$$\lambda_p(\mathbf{A}^{\mathrm{T}}\mathbf{A}) \le \frac{\|\mathbf{t}_n(\mathbf{A}\mathbf{X} + \mathbf{v}) - \mathbf{t}_n(\mathbf{A}\mathbf{Y}_m + \mathbf{v})\|^2}{\|\mathbf{t}_n(\mathbf{X}) - \mathbf{t}_n(\mathbf{Y}_m)\|^2} \le \lambda_1(\mathbf{A}^{\mathrm{T}}\mathbf{A}).$$

This means that $\sup_{\mathbf{Y}_m}\|\mathbf{t}_n(\mathbf{X}) - \mathbf{t}_n(\mathbf{Y}_m)\|$, taken over all possible $\mathbf{Y}_m$, is finite or infinite at the same time as $\sup_{\mathbf{Z}_m}\|\mathbf{t}_n(\mathbf{A}\mathbf{X} + \mathbf{v}) - \mathbf{t}_n(\mathbf{Z}_m)\|$, taken over all corrupted collections $\mathbf{Z}_m$ that differ from $\mathbf{AX} + \mathbf{v}$ in at most $m$ points. This proves (ii) for the case that $\mathbf{t}_n$ is affine equivariant. Clearly, if $\mathbf{A}$ is orthogonal the argument above can be repeated for orthogonal equivariant $\mathbf{t}_n$, and if we take $\mathbf{A} = \mathbf{I}$ the argument can be repeated for translation equivariant $\mathbf{t}_n$. This leaves us with proving (iii).

In that case, (2.3) and affine equivariance of $\mathbf{C}_n$ imply that for any collection $\mathbf{S}$ of $n$ points

$$(2.4)\qquad \lambda_1(\mathbf{C}_n(\mathbf{S}))\lambda_p(\mathbf{AA}^{\mathrm{T}}) \le \lambda_1(\mathbf{C}_n(\mathbf{AS} + \mathbf{v})) \le \lambda_1(\mathbf{C}_n(\mathbf{S}))\lambda_1(\mathbf{AA}^{\mathrm{T}}).$$

Apply (2.4) to $\mathbf{S} = \mathbf{X}$ and $\mathbf{S} = \mathbf{Y}_m$. If we write $\alpha = [\lambda_1(\mathbf{AA}^{\mathrm{T}}) - \lambda_p(\mathbf{AA}^{\mathrm{T}})]\lambda_1(\mathbf{C}_n(\mathbf{X}))$ and if we suppress the notation $\mathbf{C}_n$ for a moment, we find that

$$\lambda_1(\mathbf{AX} + \mathbf{v}) - \lambda_1(\mathbf{AY}_m + \mathbf{v}) \le \lambda_p(\mathbf{AA}^{\mathrm{T}})[\lambda_1(\mathbf{X}) - \lambda_1(\mathbf{Y}_m)] + \alpha,$$

$$\lambda_1(\mathbf{AX} + \mathbf{v}) - \lambda_1(\mathbf{AY}_m + \mathbf{v}) \ge \lambda_1(\mathbf{AA}^{\mathrm{T}})[\lambda_1(\mathbf{X}) - \lambda_1(\mathbf{Y}_m)] - \alpha.$$

Inequalities that relate $\lambda_p(\mathbf{AX} + \mathbf{v}) - \lambda_p(\mathbf{AY}_m + \mathbf{v})$ to $\lambda_p(\mathbf{X}) - \lambda_p(\mathbf{Y}_m)$ can be obtained similarly. As in the first part of the proof it follows that $\sup_{\mathbf{Y}_m} D(\mathbf{C}_n(\mathbf{X}), \mathbf{C}_n(\mathbf{Y}_m))$ and $\sup_{\mathbf{Z}_m} D(\mathbf{C}_n(\mathbf{AX} + \mathbf{v}), \mathbf{C}_n(\mathbf{Z}_m))$, taken over all corrupted collections $\mathbf{Z}_m$ that differ from $\mathbf{AX} + \mathbf{v}$ in at most $m$ points, are finite or infinite at the same time, which proves (iii). □

It is natural to ask for the maximal breakdown point of an estimator satisfying one of the equivariance properties mentioned above. The next theorem gives the upper bound for translation equivariant location estimators.

THEOREM 2.1. *Let* $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ *be a collection of* $n$ *points* $\mathbb{R}^p$. *When* $\mathbf{t}_n$ *is translation equivariant, then* $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) \le \lfloor (n+1)/2 \rfloor / n$, *where* $\lfloor u \rfloor$ *denotes the nearest integer less than or equal to* $u$.

PROOF. Because $\mathbf{t}_n$ is translation equivariant, according to Lemma 2.1 we may assume that $\mathbf{t}_n(\mathbf{X}) = \mathbf{0}$. Suppose that $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) > \lfloor (n+1)/2 \rfloor / n$. This means that there would exist a constant $k$ such that

$$(2.5)\qquad \|\mathbf{t}_n(\mathbf{Y})\| \le k < \infty$$

for all corrupted collections $\mathbf{Y}$ obtained by replacing $\lfloor (n+1)/2 \rfloor$ points of $\mathbf{X}$. Denote by $q = n - \lfloor (n+1)/2 \rfloor$ the number of points of $\mathbf{X}$ that are not replaced. Since $2q \le n$, for any $\mathbf{v} \in \mathbb{R}^p$ we can always construct a collection $\mathbf{Y}_{\mathbf{v}}$ containing $\mathbf{x}_1, \dots, \mathbf{x}_q, \mathbf{x}_1 + \mathbf{v}, \dots, \mathbf{x}_q + \mathbf{v}$ and also a corresponding collection $\mathbf{Z}_{\mathbf{v}} = \mathbf{Y}_{\mathbf{v}} - \mathbf{v}$ containing $\mathbf{x}_1 - \mathbf{v}, \dots, \mathbf{x}_q - \mathbf{v}, \mathbf{x}_1, \dots, \mathbf{x}_q$. Both collections contain at least $q$ points of $\mathbf{X}$ so according to (2.5) we must have $\|\mathbf{t}_n(\mathbf{Y}_{\mathbf{v}})\| \le k$ as well as $\|\mathbf{t}_n(\mathbf{Y}_{\mathbf{v}}) - \mathbf{v}\| = \|\mathbf{t}_n(\mathbf{Z}_{\mathbf{v}})\| \le k$, using that $\mathbf{t}_n$ is translation equivariant. Clearly, for large $\mathbf{v} \in \mathbb{R}^p$ these two inequalities cannot both be true. □

As the class of affine (orthogonal) equivariant estimators is contained in the class of translation equivariant estimators, the upper bound $\lfloor (n+1)/2 \rfloor / n$

obviously also holds for this smaller class. It then becomes of interest whether there exist estimators with these equivariance properties that attain this upper bound. The following two examples show that the upper bound $\lfloor (n+1)/2 \rfloor /n$ is sharp for translation and orthogonal equivariant estimators.

COORDINATEWISE MEDIAN. A simple way to obtain a multivariate translation equivariant location estimator with high breakdown point is to take a one-dimensional translation equivariant location estimator with high breakdown point and construct its multivariate analogue coordinatewise. Define $\mathbf{t}_n(\mathbf{X}) = (t_{n1}(\mathbf{X}) \cdots t_{np}(\mathbf{X}))^{\mathrm{T}}$ coordinatewise by $t_{nj}(\mathbf{X}) = \mathrm{median}_{1 \le i \le n} x_{ij}$ for $j = 1, \ldots, p$, where $\mathbf{x}_i = (x_{i1} \cdots x_{ip})^{\mathrm{T}}$ for $i = 1, \ldots, n$. Clearly, the breakdown point $\lfloor (n+1)/2 \rfloor /n$ of the univariate median is preserved. Note that $\mathbf{t}_n$ is translation equivariant but not orthogonal equivariant.

There are several other ways of generalizing the one-dimensional median to higher dimensions. One of the oldest generalized medians is the following example of an orthogonal equivariant estimator.

$L_1$ ESTIMATOR. Define the $L_1$ estimate as the vector $\mathbf{t}_n$ that minimizes $\sum_{i=1}^n \|\mathbf{x}_i - \mathbf{t}\|$. Because the euclidean norm is invariant under orthogonal transformations it follows that the $L_1$ estimator is orthogonal equivariant. However, it is not affine equivariant. The breakdown point is independent of the dimension $p$ and $\mathbf{X}$, and equals that of the univariate median.

THEOREM 2.2. *Let $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a collection of n points in $\mathbb{R}^p$. Then the $L_1$ estimator has breakdown point $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) = \lfloor (n+1)/2 \rfloor /n$.*

PROOF. Since $\mathbf{t}_n$ is translation equivariant, according to Lemma 2.1 we may assume $\mathbf{t}_n(\mathbf{X}) = \mathbf{0}$. Put $M = \max_{1 \le i \le n} \|\mathbf{x}_i\|$, and let $B(\mathbf{0}, 2M)$ be the sphere with center $\mathbf{0}$ and radius $2M$. Denote by $\mathbf{Y}_m = \{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ a corrupted collection obtained by replacing at most $m = \lfloor (n-1)/2 \rfloor$ points of $\mathbf{X}$ and let $\mathbf{t}_n(\mathbf{Y}_m)$ minimize $\sum_{i=1}^n \|\mathbf{y}_i - \mathbf{t}\|$.

We show that $\sup_{\mathbf{Y}_m} \|\mathbf{t}_n(\mathbf{Y}_m)\|$, taken over all possible $\mathbf{Y}_m$, is finite. Denote by $d = \inf_{\mathbf{v} \in B(\mathbf{0}, 2M)} \|\mathbf{t}_n(\mathbf{Y}_m) - \mathbf{v}\|$ the distance between $\mathbf{t}_n(\mathbf{Y}_m)$ and $B(\mathbf{0}, 2M)$, so that $\|\mathbf{t}_n(\mathbf{Y}_m)\| \le d + 2M$. Then for each of the $\lfloor (n-1)/2 \rfloor$ replaced $\mathbf{y}_j$'s, it holds that

$$\|\mathbf{y}_j - \mathbf{t}_n(\mathbf{Y}_m)\| \ge \|\mathbf{y}_j\| - \|\mathbf{t}_n(\mathbf{Y}_m)\| \ge \|y_j\| - (d + 2M). \tag{2.6}$$

Suppose that the distance between $\mathbf{t}_n(\mathbf{Y}_m)$ and $B(\mathbf{0}, 2M)$ is large, i.e., $d > 2M\lfloor (n-1)/2 \rfloor$. Since $\mathbf{X} \subset B(\mathbf{0}, M)$, for each of the $n - \lfloor (n-1)/2 \rfloor$ original $\mathbf{x}_k$'s in $\mathbf{Y}_m$ we would then have that

$$\|\mathbf{x}_k - \mathbf{t}_n(\mathbf{Y}_m)\| \ge M + d \ge \|\mathbf{x}_k\| + d. \tag{2.7}$$

From (2.6) and (2.7) it would follow that

$$\sum_{i=1}^{n} \|\mathbf{y}_i - \mathbf{t}_n(\mathbf{Y}_m)\| \geq \sum_{i=1}^{n} \|\mathbf{y}_i\| + \left(n - \left\lfloor \frac{n-1}{2} \right\rfloor\right) d - \left\lfloor \frac{n-1}{2} \right\rfloor (d + 2M)$$
$$\geq \sum_{i=1}^{n} \|\mathbf{y}_i\| + d - 2M \left\lfloor \frac{n-1}{2} \right\rfloor > \sum_{i=1}^{n} \|\mathbf{y}_i\|.$$

This is a contradiction with the fact that $\mathbf{t}_n(\mathbf{Y}_m)$ minimizes $\sum_{i=1}^{n} \|\mathbf{y}_i - \mathbf{t}\|$. Therefore $d \leq 2M\lfloor (n-1)/2 \rfloor$, hence $\sup_{\mathbf{Y}_m} \|\mathbf{t}_n(\mathbf{Y}_m)\| \leq d + 2M \leq 2M\lfloor (n+1)/2 \rfloor$. We conclude that $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) \geq \lfloor (n+1)/2 \rfloor / n$. The other inequality is obtained directly from Theorem 2.1. □

**3. Affine equivariance and breakdown point.** Is the upper bound $\lfloor (n+1)/2 \rfloor / n$ also sharp for affine equivariant estimators? Davies (1987) showed that for covariance estimators this is no longer the case. When the collection $\mathbf{X}$ is in *general position*, i.e., no $p + 1$ points are contained in some hyperplane of dimension smaller than $p$, and if $n \geq p + 1$, the breakdown point of any affine equivariant covariance estimator $\mathbf{C}_n$ is at most $\lfloor (n - p + 1)/2 \rfloor / n$. Although the result is stated for pairs $(\mathbf{t}_n, \mathbf{C}_n)$, it is only shown that the covariance part might break down if one replaces $\lfloor (n - p + 1)/2 \rfloor$ points or more, regardless of what happens with the location part. This means that the upper bound $\lfloor (n - p + 1)/2 \rfloor / n$ does not have to apply to affine equivariant *location* estimators, especially not to those that are defined without a corresponding covariance part. Such an estimator is Oja's (1983) affine equivariant multivariate median.

OJA'S ESTIMATOR. Consider the volumes $\Delta(\mathbf{t}, \mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_p})$ of all simplexes formed by $\mathbf{t} \in \mathbb{R}^p$ and all possible subcollections $\mathbf{x}_{i}, \ldots, \mathbf{x}_{i_p}$ from $\mathbf{X}$. Oja's multivariate median is the vector $\mathbf{t}_n$ in $\mathbb{R}^p$ that minimizes

$$\sum_{\{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_p}\} \subset \mathbf{X}} \Delta\left(\mathbf{t}, \mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_p}\right).$$

This location estimator is affine equivariant and is defined without any covariance part. In the simple case of four points in $\mathbb{R}^2$ (so $n \geq p + 1$ is satisfied) it is not difficult to see that when one point is replaced, Oja's solution will always stay within the convex hull of the remaining three original points. Hence, even if $\mathbf{X}$ is in general position, $\lfloor (n - p + 1)/2 \rfloor / n$ is not generally valid as an upper bound for the breakdown point of affine equivariant location estimators.

The example of Oja's estimator seems to suggest that affine equivariant location estimators may have a breakdown point greater than $\lfloor (n - p + 1)/2 \rfloor / n$. This may be due to the fact that location estimators only break down if we can make them infinitely large by replacing points of $\mathbf{X}$, whereas covariance estimators also break down if we can make them infinitely "small." Therefore we may have to replace more points in order to let a location estimator break down.

In any case, the upper bound $\lfloor (n+1)/2 \rfloor / n$ of Theorem 2.1 still holds, and we want to know how close we can get to this bound. The first example of an affine equivariant multivariate estimator with a high breakdown point was the Stahel–Donoho estimator. Donoho (1982) showed that it is affine equivariant and computed the addition breakdown point. By a slight adjustment of his proof one can show that if the collection $\mathbf{X}$ is in general position, the *replacement* breakdown point equals $(\lfloor (n+1)/2 \rfloor - p)/n$, which is smaller than the upper bound $\lfloor (n-p+1)/2 \rfloor / n$ for affine equivariant covariance estimators. We give two examples of estimates with a breakdown point that is equal to this upper bound.

Rousseeuw (1985) introduced the minimum volume ellipsoid (MVE) estimator, and showed it to be affine equivariant with breakdown point $(\lfloor n/2 \rfloor - p + 1)/n$. Also this breakdown point is smaller than the covariance upper bound $\lfloor (n-p+1)/2 \rfloor / n$. We will adjust the MVE estimator such that it does attain this upper bound.

Minimum volume ellipsoid estimator. Let $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ have $n \geq p + 1$ points. Find $\mathbf{t}_n \in \mathbb{R}^p$ and $\mathbf{C}_n \in \mathrm{PDS}(p)$ to minimize the determinant of $\mathbf{C}$ subject to

$$\#\{i: (\mathbf{x}_i - \mathbf{t})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{x}_i - \mathbf{t}) \leq c^2\} \geq \left\lfloor \frac{n+p+1}{2} \right\rfloor. \tag{3.1}$$

So, $\mathbf{t}_n$ and $\mathbf{C}_n$ determine the center and the covariance structure of the minimum volume ellipsoid covering at least $\lfloor (n+p+1)/2 \rfloor$ points. When every subcollection of $\lfloor (n+p+1)/2 \rfloor$ points of $\mathbf{X}$ contains at least $p+1$ points in general position, there exists at least one solution $(\mathbf{t}_n, \mathbf{C}_n)$ in $\mathbb{R}^p \times \mathrm{PDS}(p)$. Even if some $\lfloor (n+p+1)/2 \rfloor$ points lie on a lower-dimensional hyperplane $H$, then one can still define $\mathbf{t}_n \in \mathbb{R}^p$ as the center of the minimum volume ellipsoid inside $H$ covering at least $\lfloor (n+p+1)/2 \rfloor$ points.

The number $c$ is a fixed chosen constant and has no influence on the value of $\mathbf{t}_n$, which is taken as the MVE estimate of location. However, the choice of $c$ determines the magnitude of $\mathbf{C}_n$, which can be taken as the MVE estimate of covariance. The value of $c$ can be chosen in agreement with an assumed underlying distribution in order to obtain a consistent covariance estimate. For instance, if one assumes $X_1, \dots, X_n$ to be a sample from an elliptical distribution $P_{\boldsymbol{\mu}, \boldsymbol{\Sigma}}$ with density $(\det \boldsymbol{\Sigma})^{-1/2} f[\{(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\}^{1/2}]$, then a natural choice for $c$ would be the value for which $P_{\boldsymbol{\mu}, \boldsymbol{\Sigma}}\{(X - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (X - \boldsymbol{\mu}) \leq c^2\} = \int_{\|\mathbf{x}\| \leq c} f(\|\mathbf{x}\|)\, d\mathbf{x}$ equals $\frac{1}{2}$. In case one assumes $X_1, \dots, X_n$ to be a normal sample, $c^2$ will be $\chi^2_{0.50}[p]$. An algorithm to compute $\mathbf{t}_n$ and $\mathbf{C}_n$ is described in Rousseeuw and Leroy (1987), page 259.

Before we derive the breakdown point of the MVE estimates, we first prove the following property for ellipsoids:

$$E(\mathbf{t}, \mathbf{C}) = \{\mathbf{x}: (\mathbf{x} - \mathbf{t})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{x} - \mathbf{t}) \leq 1\}, \tag{3.2}$$

where $\mathbf{t} \in \mathbb{R}^p$ and $\mathbf{C} \in \mathrm{PDS}(p)$.

LEMMA 3.1. *Consider* $\mathbf{v}_1, \dots, \mathbf{v}_{p+1} \in \mathbb{R}^p$ *that span a nonempty simplex. Let* $E(\mathbf{t}, \mathbf{C})$ *be an ellipsoid as in* (3.2), *which contains* $\mathbf{v}_1, \dots, \mathbf{v}_{p+1}$. *Then for every* $V > 0$ *there exists a constant* $M > 0$, *which only depends on* $\mathbf{v}_1, \dots, \mathbf{v}_{p+1}$, *such that if* $\|t\| > M$ *it follows that the volume of* $E(\mathbf{t}, \mathbf{C})$ *is larger than* $V$.

PROOF. Denote by $0 < \lambda_p \leq \cdots \leq \lambda_1 < \infty$ the eigenvalues of $\mathbf{C}$. The volume of $E(\mathbf{t}, \mathbf{C})$ equals $\alpha_p \sqrt{\lambda_1 \cdots \lambda_p}$, where $\alpha_p = \pi^{p/2} / \Gamma(p/2 + 1)$, and the axes of $E(\mathbf{t}, \mathbf{C})$ have lengths $\sqrt{\lambda_j}$, $j = 1, \dots, p$.

Because $E(\mathbf{t}, \mathbf{C})$ contains the nonempty simplex spanned by $\mathbf{v}_1, \dots, \mathbf{v}_{p+1}$, there exists a constant $\beta > 0$, which only depends on $\mathbf{v}_1, \dots, \mathbf{v}_{p+1}$, such that all axes are longer than $\beta$, i.e., for all $j = 1, \dots, p$,

$$\lambda_j > \beta^2. \tag{3.3}$$

Without loss of generality we may assume that $\mathbf{0} \in E(\mathbf{t}, \mathbf{C})$. According to (2.3), for every $\mathbf{v} \in E(\mathbf{t}, \mathbf{C})$ we have that $\|\mathbf{v} - \mathbf{t}\|^2 \leq (\mathbf{v} - \mathbf{t})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{v} - \mathbf{t}) \lambda_1 \leq \lambda_1$. In particular, this holds for $\mathbf{v} = \mathbf{0}$, so $\|\mathbf{t}\|^2 \leq \lambda_1$. This means that if we take $M = V / (\alpha_p \beta^{p-1})$, then from (3.3) it follows that the volume of $E(\mathbf{t}, \mathbf{C})$ equals $\alpha_p \sqrt{\lambda_1 \cdots \lambda_p} \geq \alpha_p \beta^{p-1} > V$. □

THEOREM 3.1. *Let* $\mathbf{X}$ *be a collection of* $n \geq p + 1$ *points in* $\mathbb{R}^p$ *in general position, and let* $\mathbf{t}_n$ *and* $\mathbf{C}_n$ *be the* MVE *estimates of location and covariance. If* $p = 1$, *then* $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) = \lfloor (n+1)/2 \rfloor / n$ *and* $\varepsilon^*(\mathbf{C}_n, \mathbf{X}) = \lfloor n/2 \rfloor / n$. *When* $p \geq 2$, *then* $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) = \varepsilon^*(\mathbf{C}_n, \mathbf{X}) = \lfloor (n - p + 1)/2 \rfloor / n$.

PROOF. We extend the proof of Proposition 3.1 in Rousseeuw (1985). Without loss of generality we may assume that $c$ equals 1 in (3.1). When $p = 1$, $\mathbf{t}_n$ is the midpoint of the shortest interval covering at least $\lfloor n/2 \rfloor + 1$ points, and $\mathbf{C}_n$ is proportional to the length of this interval. Even if this interval would have length 0, $\mathbf{t}_n$ is always defined. It is not difficult to see that one needs to replace at least $\lfloor (n+1)/2 \rfloor$ points to make $\|\mathbf{t}_n\|$ infinitely large. By placing $(n/2)$ points in one of the remaining $n - \lfloor (n-1)/2 \rfloor$ points, $\mathbf{C}_n$ can be made $\mathbf{0}$.

For $p \geq 2$, we first show that $\varepsilon^*(\mathbf{t}_n, \mathbf{X})$ and $\varepsilon^*(\mathbf{C}_n, \mathbf{X})$ are at least $\lfloor (n - p + 1)/2 \rfloor / n$. Replace at most $m = \lfloor (n - p + 1)/2 \rfloor - 1$ points of $\mathbf{X}$. Because every subcollection of $\lfloor (n + p + 1)/2 \rfloor$ points of the corrupted collection $\mathbf{Y}_m$ contains at least $\lfloor (n + p + 1)/2 \rfloor - (\lfloor (n - p + 1)/2 \rfloor - 1) = p + 1$ points $\mathbf{x}_i$ of the original collection $\mathbf{X}$ in general position, there exists at least one solution $(\mathbf{t}_n(\mathbf{Y}_m), \mathbf{C}_n(\mathbf{Y}_m))$ in $\mathbb{R}^p \times \mathrm{PDS}(p)$. Denote by $E_m = E(\mathbf{t}_n(\mathbf{Y}_m), \mathbf{C}_n(\mathbf{Y}_m))$ the minimum volume ellipsoid of type (3.2) covering at least $\lfloor (n + p + 1)/2 \rfloor$ points of $\mathbf{Y}_m$.

Let $V$ denote the volume of the smallest sphere with center $\mathbf{0}$ containing all points of $\mathbf{X}$. The corrupted collection $\mathbf{Y}_m$ still contains at least $n - (\lfloor (n - p + 1)/2 \rfloor - 1) \geq \lfloor (n + p + 1)/2 \rfloor$ points of $\mathbf{X}$. The smallest sphere with center $\mathbf{0}$ containing these $\lfloor (n+p+1)/2 \rfloor$ points of $\mathbf{X}$ must then have a volume less than $V$. At the same time this sphere is also an ellipsoid containing at least $\lfloor (n+p+1)/2 \rfloor$ points of $\mathbf{Y}_m$. Therefore $E_m$, being the smallest ellipsoid of this kind, must also have a volume less than $V$. On the other hand the ellipsoid $E_m$ covers some subcollection of $\lfloor (n+p+1)/2 \rfloor$ points of $\mathbf{Y}_m$. As we have seen above, such a subcollection must contain $p+1$ points $\mathbf{x}_i$ of the original collection $\mathbf{X}$ in general position. Since these $p+1$ $\mathbf{x}_i$'s span a nonempty simplex, it follows from Lemma 3.1 that there exists a constant $M > 0$, which only depends on $\mathbf{X}$, such that $\|\mathbf{t}_n(\mathbf{Y}_m)\| > M$ would force the volume of $E_m$ to be greater than $V$. As we have just seen that this cannot be the case we conclude that

$$\|\mathbf{t}_n(\mathbf{Y}_m)\| \le M. \tag{3.4}$$

These considerations about $E_m$ also show the covariance estimator does not break down either. Similar to (3.3), the fact that $E_m$ contains $p+1$ original $\mathbf{x}_i$ in general position implies that there exists a constant $\beta$, which only depends on $\mathbf{X}$, such that

$$\lambda_j(\mathbf{C}_n(\mathbf{Y}_m)) > \beta^2 > 0 \tag{3.5}$$

for $j = 1, \dots, p$. Since the volume of $E_m$, which is proportional to the product of the eigenvalues, is always less than $V$, there must also exist a constant $0 < \alpha < \infty$, which only depends on $\mathbf{X}$, such that $\lambda_1(\mathbf{C}_n(\mathbf{Y}_m)) \le \alpha$. Together with (3.4) and (3.5) this proves that both $\varepsilon^*(\mathbf{t}_n, \mathbf{X})$ and $\varepsilon^*(\mathbf{C}_n, \mathbf{X})$ are at least $\lfloor (n-p+1)/2 \rfloor / n$.

For the affine equivariant covariance estimate $\mathbf{C}_n$ the value $\lfloor (n-p+1)/2 \rfloor)/n$ is also an upper bound, therefore $\varepsilon^*(\mathbf{C}_n, \mathbf{X}) = \lfloor (n-p+1)/2 \rfloor)/n$. For $\varepsilon^*(\mathbf{t}_n, \mathbf{X})$ the other inequality is obtained as follows. Take any $p$ points of $\mathbf{X}$ and consider the $(p-1)$-dimensional hyperplane $H$ they determine. Replace $m = \lfloor (n-p+1)/2 \rfloor$ other points of $\mathbf{X}$ by points on $H$. Then $H$ contains $\lfloor (n-p+1)/2 \rfloor + p = \lfloor (n+p+1)/2 \rfloor$ points of the corrupted collection $\mathbf{Y}_m$. The minimum volume ellipsoid covering these $\lfloor (n+p+1)/2 \rfloor$ points has a zero volume. Because $\mathbf{X}$ is in general position we can construct $\mathbf{Y}_m$ such that no other lower-dimensional hyperplane contains $\lfloor (n+p+1)/2 \rfloor$ points of $\mathbf{Y}_m$, therefore $\mathbf{t}_n(\mathbf{Y}_m)$ must lie on $H$. By sending the contaminated points on $H$ to $\infty$, one of the axes of $E_m$ becomes infinitely large, and so the center $\mathbf{t}_n(\mathbf{Y}_m)$ of $E_m$ becomes infinitely large. This proves $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) \le \lfloor (n-p+1)/2 \rfloor / n$. □

The MVE location estimator suffers from the same poor rate of convergence as the least median of squares (LMS) regression estimator [Rousseeuw (1984)]. In order to obtain $\sqrt{n}$-consistency, Rousseeuw and Yohai (1984) considered smoothed versions of the LMS estimator. These $S$ estimators generalize easily to multivariate location and covariance, in which case they become smoothed versions of the MVE estimator.

*S* ESTIMATORS. Let $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ have $n \geq p + 1$ points. Find $\mathbf{t}_n \in \mathbb{R}^p$ and $\mathbf{C}_n \in \mathrm{PDS}(p)$ to minimize the determinant of $\mathbf{C}$ subject to

$$\frac{1}{n} \sum_{i=1}^{n} \rho\Big[\big\{(\mathbf{x}_i - \mathbf{t})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{x}_i - \mathbf{t})\big\}^{1/2}\Big] \leq b. \tag{3.6}$$

Note that one obtains the MVE estimates when $nb = n - \lfloor (n + p + 1)/2 \rfloor$ and $\rho(\cdot) = 1 - \mathbf{1}_{[-c, c]}(\cdot)$. Rousseeuw and Yohai (1984), aiming at both asymptotic normality and a high breakdown point, assumed the following conditions on $\rho$:

(R1) $\rho$ is symmetric, twice continuously differentiable, and $\rho(0) = 0$.

(R2) there exists a constant $c > 0$ such that $\rho$ is strictly increasing on $[0, c]$ and constant on $[c, \infty)$.

A typical example of such a $\rho$ function is the biweight function $\rho_{B,c}(u)$, which is $u^2/2 - u^4/(2c^2) + u^6/(6c^4)$ on $[-c, c]$ and $c^2/6$ outside $[-c, c]$.

Let $r = b/\sup \rho$ and denote by $\lceil u \rceil$ the nearest integer greater than or equal to $u$. When every subcollection of $\lceil n - nr \rceil$ points of $\mathbf{X}$ contains at least $p + 1$ points in general position, there exists at least one solution $(\mathbf{t}_n, \mathbf{C}_n)$ in $\mathbb{R}^p \times \mathrm{PDS}(p)$. The constant $0 < b < \sup \rho$ can be chosen in agreement with an assumed underlying distribution. If one assumes $X_1, \dots, X_n$ to be a sample from an elliptical distribution $P_{\boldsymbol{\mu}, \boldsymbol{\Sigma}}$ a natural choice for $b$ is

$$\mathbb{E}\rho\Big[\big\{(X_1 - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(X_1 - \boldsymbol{\mu})\big\}^{1/2}\Big] = \int \rho(\|\mathbf{x}\|) f(\|\mathbf{x}\|)\, d\mathbf{x}.$$

The choice of the (tuning) constant $c$ then determines the value of $b$.

Properties of *S* estimators have been investigated by Davies (1987) and Lopuhaä (1989). *S* estimators defined by $\rho$ functions satisfying (R1) and (R2) have exactly the same asymptotic behavior as multivariate *M* estimators defined with the same $\rho$ function [Lopuhaä (1989)]. However, in contrast with *M* estimators, *S* estimators have a high breakdown point in any dimension $p$. In order to encompass *S* estimators defined by smooth $\rho$ such as $\rho_{B,c}$, we complement the breakdown result of Davies (1987), who considers functions $\rho$ that are equal to 0 in a neighborhood of the origin.

THEOREM 3.2. *Let* $\mathbf{X}$ *be a set of* $n \geq p + 1$ *points in* $\mathbb{R}^p$ *in general position. Write* $r = b/\sup \rho$. *If* $r \leq (n - p)/(2n)$ *then S estimates defined by a function* $\rho$ *that satisfies* (R1) *and* (R2) *have breakdown point* $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) = \varepsilon^*(\mathbf{C}_n, \mathbf{X}) = \lceil nr \rceil / n$.

PROOF. The proof is similar to that of Theorem 3.1. As we can always rescale the function $\rho$ we may assume that $c = 1$ and that $\sup \rho = 1$, so that $b$ in (3.6) equals $r$. We first show that $\varepsilon^*(\mathbf{t}_n, \mathbf{X})$ and $\varepsilon^*(\mathbf{C}_n, \mathbf{X})$ are at least $\lceil nr \rceil / n$. Replace at most $m = \lceil nr \rceil - 1$ points of $\mathbf{X}$. Because $r \leq (n - p)/(2n)$, every subcollection of $\lceil n - nr \rceil$ points of the corrupted collection $\mathbf{Y}_m$ contains at least $\lceil n - nr \rceil - (\lceil nr \rceil - 1) \geq p + 1$ points $\mathbf{x}_i$ of the original collection $\mathbf{X}$ in general position. So there exists at least one solution $(\mathbf{t}_n(\mathbf{Y}_m),\ \mathbf{C}_n(\mathbf{Y}_m))$ in $\mathbb{R}^p \times \mathrm{PDS}(p)$. Denote by $E_m = E(\mathbf{t}_n(\mathbf{Y}_m), \mathbf{C}_n(\mathbf{Y}_m))$ the smallest ellipsoid of type (3.2) that satisfies (3.6).

Since $nr - \lceil nr \rceil + 1$ is always strictly positive and $\rho$ is continuous, we can find a smallest sphere with center $\mathbf{0}$ and radius, say $R$, such that $\sum_{i=1}^{n} \rho(\|\mathbf{x}_i\|/R) = nr - \lceil nr \rceil + 1$. Denote by $V$ the volume of this sphere. The collection $\mathbf{Y}_m$ contains $n - m$ points of $\mathbf{X}$, say $\mathbf{x}_1, \dots, \mathbf{x}_{n-m}$. The smallest sphere with center $\mathbf{0}$ and radius $M$ such that for these points $\sum_{i=1}^{n-m} \rho(\|\mathbf{x}_i\|/M) = nr - \lceil nr \rceil + 1$ must then have a volume less than $V$. At the same time this sphere is an ellipsoid for which

$$\sum_{\mathbf{y}_i \in \mathbf{y}_m} \rho(\|\mathbf{y}_i\|/M) \le \sum_{i=1}^{n-m} \rho(\|\mathbf{x}_i\|/M) + \lceil nr \rceil - 1 = nr.$$

Therefore $E_m$, being the smallest ellipsoid of this kind, must also have a volume less than $V$. On the other hand, it follows from constraint (3.6) that $E_m$ must cover some subcollection of $\lceil n - nr \rceil$ points of $\mathbf{Y}_m$. As we have seen above, such a subcollection must contain $p + 1$ points $\mathbf{x}_i$ of the original collection $\mathbf{X}$ in general position. At this point, we invoke Lemma 3.1 and use exactly the same argument as is in the first part of the proof of Theorem 3.1 to conclude that $\varepsilon^*(\mathbf{t}_n, \mathbf{X})$ and $\varepsilon^*(\mathbf{C}_n, \mathbf{X})$ are at least $\lceil nr \rceil / n$.

The other inequalities are obtained as follows. Replace $m = \lceil nr \rceil$ points of $\mathbf{X}$. Without loss of generality denote the corrupted collection by $\mathbf{Y}_m = \{\mathbf{y}_1, \dots, \mathbf{y}_n\}$, with $\mathbf{y}_j = \mathbf{x}_j$ for $j = \lceil nr \rceil + 1, \dots, n$. Let $E(\mathbf{t}, \mathbf{C})$ be any ellipsoid of type (3.2) that satisfies

$$\sum_{i=1}^{n} \rho\left[\left\{(\mathbf{y}_i - \mathbf{t})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{y}_i - \mathbf{t})\right\}^{1/2}\right] = nr \tag{3.7}$$

and suppose that all replaced points $\mathbf{y}_1, \dots, \mathbf{y}_{\lceil nr \rceil}$ are outside $E(\mathbf{t}, \mathbf{C})$. Then $\sum_{i=1}^{n} \rho[\{(\mathbf{y}_i - \mathbf{t})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{y}_i - \mathbf{t})\}^{1/2}]$ would equal

$$\sum_{j=\lceil nr \rceil + 1}^{n} \rho\left[\left\{(\mathbf{x}_j - \mathbf{t})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{x}_j - \mathbf{t})\right\}^{1/2}\right] + \lceil nr \rceil. \tag{3.8}$$

When $nr \in \mathbb{N}$, it follows from $r \le (n - p)/(2n)$ that $n - \lceil nr \rceil \ge p + nr \ge p + 1$. In that case the summation in (3.8) runs over at least $p + 1$ points in general position. Since $\rho$ is strictly increasing it then follows that this sum must be strictly positive. When $nr \notin \mathbb{N}$, then $\lceil nr \rceil > nr$. Either way, we would find that (3.8) is strictly greater than $nr$. This is a contradiction with the fact that $E(\mathbf{t}, \mathbf{C})$ satisfies (3.7), so we conclude that at least one replacement, say $\mathbf{y}_1$, must be inside $E(\mathbf{t}, \mathbf{C})$.

Similarly, suppose that all $n - \lceil nr \rceil$ original points $\mathbf{x}_{\lceil nr \rceil + 1}, \dots, \mathbf{x}_n$ are outside $E(\mathbf{t}, \mathbf{C})$. In that case we would find that

$$\sum_{i=1}^{n} \rho\left[\left\{(\mathbf{y}_i - \mathbf{t})^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{y}_i - \mathbf{t})\right\}^{1/2}\right] \ge n - \lceil nr \rceil. \tag{3.9}$$

However, from $r \le (n - p)/(2n)$ it follows that $\lceil n - nr \rceil - \lceil nr \rceil \ge p$. Because always $n - \lceil n - nr \rceil > nr - 1$, we would find that the right-hand side of (3.9) is strictly greater than $nr$. As $E(\mathbf{t}, \mathbf{C})$ satisfies (3.7) this cannot be the case, so apart from $\mathbf{y}_1$ the ellipsoid $E(\mathbf{t}, \mathbf{C})$ must also contain at least one original $\mathbf{x}_i$.

By sending $\mathbf{y}_1$ to $\infty$ we can make one of the axes of $E(\mathbf{t}, \mathbf{C})$ infinitely large. This means that for every $\mathbf{t}$ and $\mathbf{C}$ that satisfy (3.7), we can make both $\|\mathbf{t}\|$ and the largest eigenvalue $\lambda_1(\mathbf{C})$ infinitely large. Since $\mathbf{t}_n(\mathbf{Y}_m)$ and $\mathbf{C}_n(\mathbf{Y}_m)$ must satisfy (3.7) both estimates break down. □

REMARK 3.2. The breakdown point in Theorem 3.2 is at its highest when $r = (n-p)/(2n)$. In that case the $S$ estimates have breakdown point $\lceil (n-p)/2 \rceil / n = \lfloor (n-p+1)/2 \rfloor)/n$.

**4. Breakdown and large deviations.** The replacement breakdown point as defined in Section 2 is not only a simple and appealing robustness concept. Recently, He, Jurečková, Koenker and Portnoy (1988) showed that it also has a stochastic interpretation. We extend their result to multivariate location estimators. For any $\mathbf{x}_i$ in a collection $\mathbf{X}$ write $\mathbf{x}_i = (x_{i1} \cdots x_{ip})^{\mathrm{T}}$, and instead of $\mathbf{t}_n(\mathbf{X})$ write $\mathbf{t}_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$, which is then invariant under permutations of the $\mathbf{x}_i$. We say that a location estimate $\mathbf{t}_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is *nondecreasing in* $\mathbf{x}_i$ if for any $\mathbf{x}_i = (x_{i1} \cdots x_{ip})^{\mathrm{T}}$ and $\tilde{\mathbf{x}} = (\tilde{x}_{i1} \cdots \tilde{x}_{ip})^{\mathrm{T}}$ with $\tilde{x}_{ij} \geq x_{ij}$ it holds that

$$t_{nj}(\mathbf{x}_1, \dots, \tilde{\mathbf{x}}_i, \dots, \mathbf{x}_n) \geq t_{nj}(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_n)$$

for all $j = 1, \dots, p$.

In this section we consider $\mathbf{x}_1, \dots, \mathbf{x}_n$ as a sample $X_1, \dots, X_n$ from a spherically symmetric distribution $P_{\boldsymbol{\mu}}$ with a density of the form $f(\|\mathbf{x} - \boldsymbol{\mu}\|)$, $f(y) > 0$. We say that $P_{\boldsymbol{\mu}}$ is *algebraically* tailed, if for some $m > 0$,

$$-\log P_{\boldsymbol{\mu}}(\|X_1 - \mu\| > a) \sim m \log a \quad \text{as } a \to \infty. \tag{4.1}$$

Examples are the multivariate Cauchy distribution and the multivariate student distribution. We say that $P_{\boldsymbol{\mu}}$ is *exponentially* tailed, if for some $b > 0$ and $r > 0$,

$$-\log P_{\boldsymbol{\mu}}(\|X_1 - \mu\| > a) \sim ba^r \quad \text{as } a \to \infty.$$

The multivariate normal is an example of such a distribution.

Jurečková (1981) considered

$$B(a, \mathbf{t}_n) = \frac{-\log P_{\boldsymbol{\mu}}(\|\hat{\mathbf{t}}_n - \boldsymbol{\mu}\| > a)}{-\log P_{\boldsymbol{\mu}}(\|X_1 - \boldsymbol{\mu}\| > a)}$$

as a measure of performance for $\hat{\mathbf{t}}_n = \mathbf{t}_n(X_1, \dots, X_n)$, and showed that in the case $p = 1$, under certain conditions on $\mathbf{t}_n$, it holds that

$$1 \leq \liminf_{a \to \infty} B(a, \mathbf{t}_n) \leq \limsup_{a \to \infty} B(a, \mathbf{t}_n) \leq n.$$

For exponentially tailed distributions, the sample mean $\overline{X}_n$ performed optimally with $B(a, \overline{X}_n)$ tending to $n$, while for algebraically tailed distributions the lack of robustness of $\overline{X}_n$ was expressed by $B(a, \overline{X}_n)$ tending to 1. In the multivariate setting one has something similar: When $X_1, \dots, X_n$ are normally distributed with mean zero then $\bar{X}_n =_{\mathscr{D}} (1/\sqrt{n})X_1$ so that $B(a, \bar{X}_n)$ tends to $n$, and when $X_1, \dots, X_n$ have a multivariate Cauchy distribution then $\bar{X}_n =_{\mathscr{D}} X_1$ so that $B(a, \bar{X}_n) = 1$.

He, Jurečkova, Koenker and Portnoy (1988) related $B(a, \mathbf{t}_n)$ to the finite-sample replacement breakdown point $\varepsilon^* = m^*/n$ of univariate $\mathbf{t}_n$. The following theorem extends their result to multivariate location estimators in case $P_{\boldsymbol{\mu}}$ is algebraically tailed.

THEOREM 4.1. *Let $X_1, \dots, X_n$ be a sample from a spherically symmetric distribution $P_{\boldsymbol{\mu}}$ with density $f(\|\mathbf{x} - \boldsymbol{\mu}\|) > 0$, and suppose that $P_{\boldsymbol{\mu}}$ is algebraically tailed. Let $\mathbf{t}_n(X_1, \dots, X_n)$ be a translation equivariant estimate, which is nondecreasing in every argument $X_i$, $i = 1, \dots, n$. Suppose that at any collection $\mathbf{X}$ the breakdown point $\varepsilon^*(\mathbf{t}_n, \mathbf{X}) = m^*/n$ is independent of $\mathbf{X}$. Then*

$$m^* \le \liminf_{a \to \infty} B(a, \mathbf{t}_n) \le \limsup_{a \to \infty} B(a, \mathbf{t}_n) \le n - m^* + 1.$$

PROOF. Since $\mathbf{t}_n$ is translation equivariant we can restrict ourselves to the case $\boldsymbol{\mu} = \mathbf{0}$, and write $P$ for $P_{\mathbf{0}}$. We first show that for some constant $A > 0$,

$$(4.2) \qquad P\big(\|\hat{\mathbf{t}}_n\| > a\big) \ge (2p)^{-n+m^*-1} P\big(\|X_1\| > \sqrt{p}\,(a + A)\big)^{n-m^*+1}.$$

For $i = 1, \dots, n - m^* + 1$ we have that $\{\|X_i\|^2 > a\} \subset \bigcup_{j=1}^{p} \{X_{ij}^2 > a/p\}$. Because $X_i$ is spherically distributed around $\mathbf{0}$, it holds for every $j = 1, \dots, p$ that $P(X_{ij}^2 > a/p) = P(X_{i1}^2 > a/p)$. It follows that

$$(4.3) \quad P\big(\|X_1\|^2 > a\big)^{n-m^*+1} \le p^{n-m^*+1} P\big(X_{11}^2 > a/p, \dots, X_{n-m^*+1,1}^2 > a/p\big).$$

The probability on the right-hand side of (4.3) equals

$$(4.4) \qquad \sum_{k=0}^{n-m^*+1} \sum_{\substack{\{i_1, \dots, i_k\} \subset \\ \{1, \dots, n-m^*+1\}}} \left[ \prod_{l=1}^{k} P\big(X_{i_l 1} > \sqrt{a/p}\big) \times \prod_{l=k+1}^{n-m^*+1} P\big(X_{i_l 1} < -\sqrt{a/p}\big) \right].$$

Since every $X_i$ is spherically distributed around $\mathbf{0}$, each $P(X_{i_l 1} < -\sqrt{a/p})$ equals $P(X_{i_l 1} > \sqrt{a/p})$. Together with (4.3) and (4.4), using that $\sum_{k=0}^{n-m^*+1} \binom{n-m^*+1}{k}$ equals $2^{n-m^*+1}$, we obtain

$$(4.5) \qquad \begin{aligned} &P\big(\|X_1\|^2 > a\big)^{n-m^*+1} \\ &\quad \le (2p)^{n-m^*+1} P\big(X_{11} > \sqrt{a/p}, \dots, X_{n-m^*+1,1} > \sqrt{a/p}\big). \end{aligned}$$

Next define the vector $\tilde{X} = (\tilde{X}_1 \cdots \tilde{X}_p)^{\mathrm{T}}$ by $\tilde{X}_j = \min\{X_{1j}, \dots, X_{n-m^*+1,j}\}$ for $j = 1, \dots, p$. Then for every $i = 1, \dots, n - m^* + 1$, the vector $X_i - \tilde{X}$ has only nonnegative coordinates. Since $\mathbf{t}_n$ is translation equivariant and is nondecreasing in every argument, we find for its first coordinate that

$$
\begin{aligned}
t_{n1}(X_1,\ldots,X_n) &= t_{n1}(X_1 - \tilde{X},\ldots,X_{n-m^*+1} - \tilde{X},\ldots,X_n - \tilde{X}) + \tilde{X}_1 \\
&\geq t_{n1}(\mathbf{0},\ldots,\mathbf{0},X_{n-m^*+1} - \tilde{X},\ldots,X_n - \tilde{X}) + \tilde{X}_1.
\end{aligned} \tag{4.6}
$$

The collection $\mathbf{Y}_{m^*} = \{\mathbf{0},\ldots,\mathbf{0},X_{n-m^*+2} - \tilde{X},\ldots,X_n - \tilde{X}\}$ has at least $n - m^* + 1$ points of the collection $\mathbf{X}_0 = \{\mathbf{0},\ldots,\mathbf{0}\}$. Because $\mathbf{t}_n$ has breakdown point $m^*/n$ at $\mathbf{X}_0$ there exists a constant $A > 0$, which no longer depends on $X_1,\ldots,X_n$, such that $\|\mathbf{t}_n(\mathbf{Y}_{m^*})\| \leq A$. Now if

$$X_{11} > \sqrt{a/p}\,,\ldots,X_{n-m^*+1,1} > \sqrt{a/p}\,,$$

then obviously $\tilde{X}_1 > \sqrt{a/p}$, so that it follows from (4.6) that $t_{n1}(X_1,\ldots,X_n) \geq \sqrt{a/p} - A$ and therefore $\|\mathbf{t}_n(X_1,\ldots,X_n)\| \geq \sqrt{a/p} - A$. Together with (4.5) this implies that

$$P\big(\|X_1\|^2 > a\big)^{n-m^*+1} \leq (2p)^{n-m^*+1}P\big(\|\hat{\mathbf{t}}_n\| > \sqrt{a/p} - A\big),$$

which proves (4.2). Taking logarithms in (4.2) gives

$$
\begin{aligned}
B(a,\mathbf{t}_n) &\leq (n - m^* + 1)\,\frac{-\log P\big(\|X_1\| > \sqrt{p}\,(a + A)\big)}{-\log P(\|X_1\| > a)} \\
&\quad + \frac{(n - m^* + 1)\log(2p)}{-\log P(\|X_1\| > a)}.
\end{aligned} \tag{4.7}
$$

With (4.1) we obtain $\limsup_{a\to\infty} B(a,\mathbf{t}_n) \leq n - m^* + 1$.

The lower bound on $\limsup_{a\to\infty} B(a,\mathbf{t}_n)$ is obtained similarly. We have $\{\|\hat{\mathbf{t}}_n\| > a\} \subset \bigcup_{j=1}^{p}\{|\hat{t}_{nj}| > a/\sqrt{p}\}$. Therefore, we find that

$$P\big(\|\hat{\mathbf{t}}_n\| > a\big) \leq \sum_{j=1}^{p} P\big(\hat{t}_{nj} > a/\sqrt{p}\big) + \sum_{j=1}^{p} P\big(\hat{t}_{n1} < -a/\sqrt{p}\big). \tag{4.8}$$

Suppose $\hat{t}_{nj} > a/\sqrt{p}$. Then define the vector $\tilde{X}' = (\tilde{X}_1' \cdots \tilde{X}_p')^{\mathrm{T}}$ as follows. Define $\tilde{X}_j' = X_{(n-m^*+1)j}$ as the $(n - m^* + 1)$th order statistic of the $n$ coordinates $X_{1j},\ldots,X_{nj}$. Next, consider the $n - m^* + 1$ vectors that correspond with the $n - m^* + 1$ smallest first coordinates among $X_{1j},\ldots,X_{nj}$. We may assume these vectors are $X_1,\ldots,X_{n-m^*+1}$. For $k \neq j$ define $\tilde{X}_k' = \max\{X_{1k},\ldots,X_{n-m^*+1,k}\}$. Then for every $i = 1,\ldots,n - m^* + 1$ the vector $X_i - \tilde{X}'$ has only nonpositive coordinates. Similar to (4.6), use that $\mathbf{t}_n$ is translation equivariant and nondecreasing in every argument, and obtain

$$t_{nj}(X_1,\ldots,X_n) \leq t_{nj}\big(\mathbf{0},\ldots,\mathbf{0},X_{n-m^*+2} - \tilde{X}',\ldots,X_n - \tilde{X}'\big) + \tilde{X}_j'. \tag{4.9}$$

By a similar reasoning as before, there exists a constant $A > 0$ which no longer depends on $X_1,\ldots,X_n$, such that $\|\mathbf{t}_n(\mathbf{Y}_{m^*})\| \leq A$, where $\mathbf{Y}_{m^*} = \{\mathbf{0},\ldots,\mathbf{0},X_{n-m^*+2} - \tilde{X}',\ldots,X_n - \tilde{X}'\}$. This means that if $\hat{t}_{nj} > a/\sqrt{p}$ it follows from (4.9) that $\tilde{X}_j' > a/\sqrt{p} - A$. By definition of $\tilde{X}_j'$ this means that

$\|X_i\| > a/\sqrt{p} - A$ for at least $m^*$ of the $X_1, \dots, X_n$. A similar argument holds when $\hat{t}_{nj} < -a/\sqrt{p}$. Therefore, with (4.8) we obtain

$$P(\|\hat{\mathbf{t}}_n\| > a) \le 2p\binom{n}{m^*}P(\|X_1\| > a/\sqrt{p} - A)^{m^*}.$$

Taking logarithms gives an inequality similar to (4.7), to which we apply (4.1) and obtain $\liminf_{a\to\infty} B(a, t_n) \ge m^*$. □

REMARK 4.1. When $P_\mu$ is the multivariate normal distribution, $\limsup_{a\to\infty} B(a, \mathbf{t}_n) \le n - m^* + 1$ is still valid. To see this, use that if $X_1 = (X_{11} \cdots X_{1p})^{\mathrm{T}}$ has a $N(\mathbf{0}, \mathbf{I})$ distribution it holds that

$$-\log P(\|X_1\| > a) \sim -\log P(|X_{11}| > a) \sim \tfrac{1}{2}a^2 \quad \text{as } a \to \infty. \tag{4.10}$$

Similar to (4.2) one can obtain $P(\|\hat{\mathbf{t}}_n\| > a) \ge 2^{-n+m^*-1}P(|X_{11}| > a + A)^{n-m^*+1}$. Taking logarithms gives

$$B(a, \mathbf{t}_n) \le (n - m^* + 1)\frac{-\log P(|X_{11}| > a + A)}{-\log P(\|X_1\| > a)} + \frac{(n - m^* + 1)\log 2}{-\log P(\|X_1\| > a)}$$

and if we apply (4.10) we obtain $\limsup_{a\to\infty} B(a, \mathbf{t}_n) \le n - m^* + 1$.

Remark 4.1 and Theorem 4.1 indicate that estimators with a high breakdown point necessarily sacrifice performance at light tailed distributions. However, Theorem 4.1 implies that estimators with maximal breakdown point satisfy a minimax property: They maximize least favorable tail performance over the class of algebraically tailed distributions. For such $\mathbf{t}_n$ with maximal breakdown point $\lfloor (n+1)/2 \rfloor / n$, such as the coordinatewise median of Section 2, Theorem 4.1 (for $n$ odd) boils down to

$$\lim_{a\to\infty} B(a, \mathbf{t}_n) = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

**5. Breakdown point of one-step reweighted estimators.** A high breakdown point is often counterbalanced by a low asymptotic efficiency. A possible way to avoid this is to use robust estimates as a diagnostic tool to select the "good" observations from a (corrupted) collection [see, for instance, Rousseeuw and van Zomeren (1990)]. Once the "good" observations have been identified, classical methods could be applied to obtain final estimates of location and covariance. If the breakdown point of the initial robust estimates is preserved, we may be able to combine high breakdown point with high efficiency. In this section we will show that for the usual weighted sample mean and sample covariance the breakdown point of the initial estimates is preserved. Asymptotic properties of these estimates are still under investigation.

Assume throughout this section that the collection $\mathbf{X} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is in general position. Let $\mathbf{t}_{0,n}(\mathbf{X}) \in \mathbb{R}^p$ and $\mathbf{C}_{0,n}(\mathbf{X}) \in \mathrm{PDS}(p)$ denote initial (robust) estimates of location and covariance (scatter) based on $\mathbf{X}$. Consider the (robust) ellipsoid

$$E(\mathbf{t}_{0,n}, \mathbf{C}_{0,n}, c_0) = \{\mathbf{x}: (\mathbf{x} - \mathbf{t}_{0,n})^{\mathrm{T}} \mathbf{C}_{0,n}^{-1} (\mathbf{x} - \mathbf{t}_{0,n}) \le c_0^2\},$$

where $c_0$ is a constant that only determines its magnitude. We think of $\mathbf{t}_{0,n}$, $\mathbf{C}_{0,n}$ and $c_0$ as MVE estimates with $c = c_0$ in (3.6), or $S$ estimates defined with $b/\sup \rho \le (n-p)/(2n)$ and with $c = c_0$ in (R2). However, any combination of $\mathbf{t}_{0,n}$, $\mathbf{C}_{0,n}$ and $c_0$ will do as long as they have the property that for any collection $\mathbf{X}$

$$\#\{i: \mathbf{x}_i \in E(\mathbf{t}_{0,n}(\mathbf{X}), \mathbf{C}_{0,n}(\mathbf{X}), c_0)\} \ge \left\lfloor \frac{n+p+1}{2} \right\rfloor. \tag{5.1}$$

This is to prevent $E(\mathbf{t}_{0,n}, \mathbf{C}_{0,n}, c_0)$ from implosion after replacing $m \le \lfloor (n-p+1)/2 \rfloor - 1$ points of $\mathbf{X}$.

For $i = 1, \dots, n$ compute Mahalanobis distances

$$d_0(\mathbf{x}_i) = \left\{(\mathbf{x}_i - \mathbf{t}_{0,n}(\mathbf{X}))^{\mathrm{T}} \mathbf{C}_{0,n}(\mathbf{X})^{-1} (\mathbf{x}_i - \mathbf{t}_{0,n}(\mathbf{X}))\right\}^{1/2}.$$

Identify observations with relatively small $d_0(\mathbf{x}_i)$ as "good" observations, and identify observations with relatively large $d_0(\mathbf{x}_i)$ as outliers. Next compute the weighted sample mean and sample covariance, by assigning smaller weights to outlying observations. Let $w: [0, \infty) \to [0, \infty)$ be a function satisfying:

(W1) $w(\cdot)$ is nonincreasing and bounded.

(W2) $w(y) > 0$ for $y \in [0, c_0]$, and there exists a constant $c_1 \ge c_0$ such that $w(y) = 0$ for $y \in (c_1, \infty)$.

Define weighted estimates by

$$\mathbf{t}_{1,n}(\mathbf{X}) = \frac{\sum_{i=1}^n w(d_0(\mathbf{x}_i))\mathbf{x}_i}{\sum_{i=1}^n w(d_0(\mathbf{x}_i))},$$

$$\mathbf{C}_{1,n}(\mathbf{X}) = \frac{\sum_{i=1}^n w(d_0(\mathbf{x}_i))(\mathbf{x}_i - \mathbf{t}_{1,n}(\mathbf{X}))(\mathbf{x}_i - \mathbf{t}_{1,n}(\mathbf{X}))^{\mathrm{T}}}{\sum_{i=1}^n w(d_0(\mathbf{x}_i))}.$$

A typical choice for $w(\cdot)$ would be the function $\mathbf{1}_{[0,c_1]}(\cdot)$, in which case $\mathbf{t}_{1,n}$ and $\mathbf{C}_{1,n}$ are simply the sample mean and sample covariance of the "good" observations.

It is not difficult to see that the affine equivariance of $\mathbf{t}_{0,n}$ and $\mathbf{C}_{0,n}$ carries over to $\mathbf{t}_{1,n}$ and $\mathbf{C}_{1,n}$. We will show that also the breakdown point of the initial estimates is preserved. We need the following property of eigenvalues.

LEMMA 5.1. *For symmetric $p \times p$ matrices $\mathbf{A}$ and $\mathbf{B}$ it holds that $\lambda_1(\mathbf{A} + \mathbf{B}) \le \lambda_1(\mathbf{A}) + \lambda_1(\mathbf{B})$, and $\lambda_p(\mathbf{A} + \mathbf{B}) \ge \lambda_p(\mathbf{A}) + \lambda_p(\mathbf{B})$.*

PROOF. Apply (2.3) and use standard properties of infima and suprema. □

THEOREM 5.1. *Let $\mathbf{X}$ be a collection in general position with $n \ge p + 1$ points in $\mathbb{R}^p$. Let $w(\cdot)$ satisfy* (W1) *and* (W2) *and let $\mathbf{t}_{0,n}(\mathbf{X}) \in \mathbb{R}^p$ and*

$\mathbf{C}_{0,n}(\mathbf{X}) \in \mathrm{PDS}(p)$ *be affine equivariant estimates of location and covariance that satisfy* (5.1). *Then*

$$\varepsilon^*(\mathbf{t}_{1,n}, \mathbf{X}) \geq \min\{\varepsilon^*(\mathbf{t}_{0,n}, \mathbf{X}), \varepsilon^*(\mathbf{C}_{0,n}, \mathbf{X})\} \tag{5.2}$$

*and*

$$\varepsilon^*(\mathbf{C}_{1,n}, \mathbf{X}) \geq \min\{\varepsilon^*(\mathbf{t}_{0,n}, \mathbf{X}), \varepsilon^*(\mathbf{C}_{0,n}, \mathbf{X})\}. \tag{5.3}$$

PROOF. Replace at most $m = n \min\{\varepsilon^*(\mathbf{t}_{0,n}, \mathbf{X}), \varepsilon^*(\mathbf{C}_{0,n}, \mathbf{X})\} - 1$ points of $\mathbf{X}$ and denote by $\mathbf{Y}_m = \{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ the new corrupted collection. Write $\mathbf{t}_{0,n}^*$ and $\mathbf{C}_{0,n}^*$ for $\mathbf{t}_{0,n}(\mathbf{Y}_m)$ and $\mathbf{C}_{0,n}(\mathbf{Y}_m)$, $\mathbf{t}_{1,n}^*$ and $\mathbf{C}_{1,n}^*$ similarly, and write $d_0^*(\mathbf{y}_i)$ for $\{(\mathbf{y}_i - \mathbf{t}_{0,n}^*)^{\mathrm{T}}(\mathbf{C}_{0,n}^*)^{-1}(\mathbf{y}_i - \mathbf{t}_{0,n}^*)\}^{1/2}$. Because $m \leq n\varepsilon^*(\mathbf{t}_{0,n}, \mathbf{X}) - 1$ and $m \leq n\varepsilon^*(\mathbf{C}_{0,n}, \mathbf{X}) - 1$ it follows that there exist constants $k_0$, $k_1$ and $k_2$, which only depend on $\mathbf{X}$, such that

$$\left\|\mathbf{t}_{0,n}^*\right\| \leq k_0 < \infty \quad \text{and} \quad 0 < k_1 \leq \lambda_p\left(\mathbf{C}_{0,n}^*\right) \leq \lambda_1\left(\mathbf{C}_{0,n}^*\right) \leq k_2 < \infty. \tag{5.4}$$

As $\mathbf{C}_{0,n}$ is an affine equivariant covariance estimator it holds that $m \leq \lfloor (n - p + 1)/2 \rfloor) - 1$. It then follows from (5.1) that the corrupted ellipsoid $E(\mathbf{t}_{0,n}^*, \mathbf{C}_{0,n}^*, c_0)$ still covers $\lfloor (n + p + 1)/2 \rfloor - m \geq p + 1$ original points of $\mathbf{X}$. Without loss of generality assume that these points are $\mathbf{x}_1, \ldots, \mathbf{x}_{p+1}$. We find that

$$\sum_{i=1}^{n} w\left(d_0^*(\mathbf{y}_i)\right) \geq \sum_{j=1}^{p+1} w\left(d_0(\mathbf{x}_j)\right) \geq (p + 1)w(c_0) > 0, \tag{5.5}$$

which means that the denominator of $\mathbf{t}_{1,n}^*$ and $\mathbf{C}_{1,n}^*$ will always be uniformly bounded away from 0.

We first show that $\|\mathbf{t}_{1,n}^*\|$ remains bounded. According to (2.3), for every $\mathbf{y}_i \in \mathbf{Y}_m$ we have $\|\mathbf{y}_i - \mathbf{t}_{0,n}^*\|^2 \leq (\mathbf{y}_i^* - \mathbf{t}_{0,n}^*)^{\mathrm{T}}(\mathbf{C}_{0,n}^*)^{-1}(\mathbf{y}_i - \mathbf{t}_{0,n}^*)\lambda_1(\mathbf{C}_{0,n}^*)$. Then from (5.4) and (W2) it follows that every $\mathbf{y}_i \in \mathbf{Y}_m$ with positive weight $w(d_0^*(\mathbf{y}_i))$ is bounded:

$$\|\mathbf{y}_i\| \leq \left\|\mathbf{y}_i - \mathbf{t}_{0,n}^*\right\| + \left\|\mathbf{t}_{0,n}^*\right\| \leq \sqrt{c_1 k_2} + k_0. \tag{5.6}$$

Together with (5.5) and (W1) this means that there exists a constant $A_0$, which only depends on $\mathbf{X}$, such that

$$\left\|\mathbf{t}_{1,n}^*\right\| \leq A_0 < \infty, \tag{5.7}$$

which proves (5.2).

Next we show that $\lambda_p(\mathbf{C}_{1,n}^*)$ is uniformly bounded away from 0. Consider the numerator of $\mathbf{C}_{1,n}^*$ and write this as the sum $\mathbf{A} + \mathbf{B}$ of the matrices

$$\mathbf{A} = \sum_{i=1}^{p+1} w\left(d_0^*(\mathbf{x}_i)\right)\left(\mathbf{x}_i - \mathbf{t}_{1,n}^*\right)\left(\mathbf{x}_i - \mathbf{t}_{1,n}^*\right)^{\mathrm{T}},$$

$$\mathbf{B} = \sum_{i=p+2}^{n} w\left(d_0^*(\mathbf{y}_i)\right)\left(\mathbf{y}_i - \mathbf{t}_{1,n}^*\right)\left(\mathbf{y}_i - \mathbf{t}_{1,n}^*\right)^{\mathrm{T}},$$

where $\mathbf{x}_1, \dots, \mathbf{x}_{p+1}$ have positive weight, as they are inside $E(\mathbf{t}^*_{0,n}, \mathbf{C}^*_{0,n}, c_0)$. Since both $\mathbf{A}$ and $\mathbf{B}$ are symmetric nonnegative matrices it follows from Lemma 5.1 that $\lambda_p(\mathbf{A} + \mathbf{B}) \geq \lambda_p(\mathbf{A})$. Also, (2.3) implies that for $\alpha_1, \alpha_2 \geq \gamma > 0$ and $\mathbf{A}_1, \mathbf{A}_2$ symmetric nonnegative it holds that $\lambda_p(\alpha_1 \mathbf{A}_1 + \alpha_2 \mathbf{A}_2) \geq \gamma\lambda_p(\mathbf{A}_1 + \mathbf{A}_2)$. Because $w(d_0^*(\mathbf{x}_i)) \geq w(c_0)$ for every $i = 1, \dots, p+1$, it follows that $\lambda_p(\mathbf{A}) \geq w(c_0)\lambda_p(\mathbf{M})$, where $\mathbf{M} = \sum_{i=1}^{p+1}(\mathbf{x}_i - \mathbf{t}^*_{1,n})(\mathbf{x}_i - \mathbf{t}^*_{1,n})^{\mathrm{T}}$. Write $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$, where

$$\mathbf{M}_1 = \sum_{i=1}^{p+1} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^{\mathrm{T}},$$

$$\mathbf{M}_2 = (p+1)\left(\bar{\mathbf{x}} - \mathbf{t}^*_{1,n}\right)\left(\bar{\mathbf{x}} - \mathbf{t}^*_{1,n}\right)^{\mathrm{T}},$$

with $\bar{\mathbf{x}} = (p+1)^{-1}\sum_{i=1}^{p+1}\mathbf{x}_i$. Both $\mathbf{M}_1$ and $\mathbf{M}_2$ are symmetric and nonnegative, so that Lemma 5.1 yields $\lambda_p(\mathbf{M}) \geq \lambda_p(\mathbf{M}_1)$. The matrix $\mathbf{M}_1$ is proportional to the sample covariance matrix of $\mathbf{x}_1, \dots, \mathbf{x}_{p+1}$ and as $\mathbf{X}$ is in general position $\mathbf{M}_1$ must have a smallest eigenvalue $\lambda_p(\mathbf{M}_1) > 0$. We find that the smallest eigenvalue of the numerator of $\mathbf{C}^*_{1,n}$ is greater than $w(c_0)\lambda_p(\mathbf{M}_1) > 0$. Because of (W1) the denominator of $\mathbf{C}^*_{1,n}$ is bounded above by $nw(0)$. It follows that there exists a constant $A_1 > 0$, which only depends on $\mathbf{X}$, such that $\lambda_p(\mathbf{C}^*_{1,n}) \geq A_1$.

Finally, for any $\mathbf{v} \in \mathbb{R}^p$ it holds that $\lambda_1(\mathbf{v}\mathbf{v}^{\mathrm{T}}) \leq \operatorname{trace}(\mathbf{v}\mathbf{v}^{\mathrm{T}}) = \|\mathbf{v}\|^2$. Together with Lemma 5.1, inequality (5.5) and (W1), this implies that

$$\begin{aligned}\lambda_1(\mathbf{C}^*_{1,n}) &\leq \left((p+1)w(c_0)\right)^{-1} \sum_{i=1}^{n} w\left(d_0^*(\mathbf{y}_i)\right)\lambda_1\left(\left(\mathbf{y}_i - \mathbf{t}^*_{1,n}\right)\left(\mathbf{y}_i - \mathbf{t}^*_{1,n}\right)^{\mathrm{T}}\right)\\ &\leq \left((p+1)w(c_0)\right)^{-1} w(0) \sum_{i=1}^{n} \left\|\mathbf{y}_i - \mathbf{t}^*_{1,n}\right\|^2.\end{aligned}$$

Using (5.6) and (5.7), we find that there exists a constant $A_2$, which only depends on $\mathbf{X}$, such that $\lambda_1(\mathbf{C}^*_{1,n}) \leq A_2 < \infty$. This completes the proof. □

Other suggestions have been made to combine high asymptotic efficiency with high breakdown point. Results of Beran (1977) show that if you estimate the center of symmetry by minimum Hellinger distance you get high efficiency combined with breakdown point $1/2$ asymptotically. However, MHD estimation may not be practical in high dimensions because it depends on density estimation. Other minimum distance estimators are discussed by Donoho and Liu (1988). Yohai (1987) combined high breakdown point with high efficiency for regression estimators. Lopuhaä (1988) extended this approach to affine multivariate location estimators. Unfortunately, a similar approach for covariance estimators, i.e., first estimate the location parameter affinely with high breakdown point and then compute an $M$ estimate of covariance based on the recentered observations, would fail because covariance $M$ estimators have a low breakdown point [Tyler (1986)].

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