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COMBINING INDEPENDENT NONCENTRAL CHI SQUARED OR F TESTS[1]

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The problem of combining several independent Chi squared or F tests is considered. The data consist of n independent Chi squared or F variables on which tests of the null hypothesis that all noncentrality parameters are zero are based. In each case, necessary conditions and sufficient conditions for a test to be admissible are given in terms of the monotonicity and convexity of the acceptance region. The admissibility or inadmissibility of several tests based upon the observed significance levels of the individual test statistics is determined. In the Chi squared case, Fisher's and Tippett's procedures are admissible, the inverse normal and inverse logistic procedures are inadmissible, and the test based upon the sum of the significance levels is inadmissible when the level is less than a half. The results are similar, but not identical, in the F case. Several generalized Bayes tests are derived for each problem.

1. Introduction. The problem of combining several independent tests of significance into one overall test has long been of interest. See Mosteller and Bush (1954) for an introduction and Oosterhoff (1969) and Rosenthal (1978) for more recent developments. In this paper we find a minimal complete class of tests in the problem of combining independent noncentral Chi squared tests, and necessary conditions and sufficient conditions for a test to be admissible in the problem of combining independent noncentral F tests. Among other results, we show that for almost all cases of interest, Fisher's and Tippett's procedures are admissible while the inverse normal, inverse logistic and sum of p_i's procedures are inadmissible.

The problems we treat fit into the following framework. Let $T_1, \cdots, T_n$ be real-valued test statistics, where for each i, T_i has density $f_i(t_i; \theta_i)$ with respect to Lebesgue measure. We wish to combine the n problems testing $\theta_i = 0$ versus $\theta_i > 0$ into one overall problem testing

$$H_0: \boldsymbol{\theta} = \mathbf{0} \text{ versus } H_A: \boldsymbol{\theta} \in \Omega_A \equiv \Omega - \{\mathbf{0}\}, \tag{1.1}$$

where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)$ and $\Omega = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \theta_i \geq 0\ \forall i\}$. Two types of tests for problem (1.1) are generalized Bayes tests and those based on the observed significance levels of the individual T_i's. Under general conditions, satisfied in our problems, the former tests are admissible. See Theorem 5.1 of Farrell (1968). The latter type arise when each f_i has monotone likelihood ratio in θ_i. The observed significance level for T_i when $T_i = t_i$ is observed is defined to be

$$p_i = p_i(t_i) \equiv P_0(T_i > t_i),$$

where P_0 is probability under H_0. Several tests based upon $(p_1, \cdots, p_n)$ have been

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proposed. These tests are called nonparametric or omnibus procedures since they are applicable for any f_i's with monotone likelihood ratio.

Fisher (1938) and Pearson (1933) suggested using the test based upon $\Pi\, p_i$, or equivalently, the test with rejection region

$$-2\,\Sigma \log p_i > \chi^2_{2n,\alpha}, \tag{1.2}$$

where $\chi^2_{\nu,\alpha}$ is the upper α point of a central Chi squared variable on ν degrees freedom. Tippett (1931) proposed the test which rejects H_0 when

$$\min_i\{p_i\} < 1 - (1-\alpha)^{1/n}. \tag{1.3}$$

Other procedures include the inverse normal (Liptak, 1958; and Stouffer according to Mosteller and Bush, 1954), inverse logistic (Mudholkar and George, 1977), inverse Chi squared (Yates, 1955 and Lancaster, 1961), and sum of p_i's (Edgington, 1972) procedures, which reject H_0 when

$$-\Sigma\,\Phi^{-1}(p_i) > \sqrt{n}\,\Phi^{-1}(1-\alpha), \tag{1.4}$$

$$-\Sigma \log[p_i/(1-p_i)] > b_\alpha, \tag{1.5}$$

$$\Sigma\, G^{-1}(1-p_i;\beta_i) > \chi^2_{\Sigma\beta_i,\alpha}, \tag{1.6}$$

and

$$\Sigma\, p_i < c_\alpha \tag{1.7}$$

respectively, where Φ is the standard normal distribution function and $G(\cdot;\beta)$ is the χ^2_β distribution function. Note that test (1.6) is equivalent to test (1.2) when all β_i's are 2, and that all but test (1.3) are of the form

$$\Sigma\, H_i^{-1}(p_i) < h_{1\alpha} \tag{1.8}$$

or

$$\Sigma\, H_i^{-1}(1-p_i) > h_{2\alpha} \tag{1.9}$$

for some distribution functions $H_1, \cdots, H_n$.

Several authors have studied admissibility of combination procedures. Birnbaum (1954) showed that given any nonparametric combination procedure which has an acceptance region A monotone increasing in the p_i's (i.e., if $(p_1^0, \cdots, p_n^0) \in A$ and $p_i \geq p_i^0$ for all i, then $(p_1, \cdots, p_n) \in A$), there exists a problem for which the procedure is most powerful against some alternative. In fact, Brown, Cohen and Strawderman (1976) have shown such tests form a complete class. Thus there is no hope in general deciding which of the nonparametric procedures is best. When each T_i has an exponential family density with natural parameter θ_i, the minimal complete class of tests is given by all tests with acceptance region (essentially) convex and monotone decreasing in T-space. See Birnbaum (1954, 1955) and Eaton (1970). In this case, Birnbaum (1954) strongly suggested that Fisher's procedure (1.2) is admissible, proved that Tippett's procedure (1.3) is admissible, and showed that tests based upon

$$\Pi\,(1-p_i) \tag{1.10}$$

or the rth ($r > 1$) smallest p_i are inadmissible.

Oosterhoff (1969) applies Birnbaum's ideas to find a minimal complete class of invariant tests in a (non-exponential) problem involving the combination of nonindependent noncentral t-tests. In this paper we apply the techniques of the above authors to the Chi squared and F situations. Ghia (1976) has conjectured minimal complete classes in these problems (proved when each variable is Chi squared on 1 degree of freedom) under the assumption that the null and alternative parameter spaces are topologically separated.

Marden (1978) and Marden and Perlman (1981) exhibit the minimal complete class for the F case.

In Sections 2 and 3 we observe $X_1^2, \cdots, X_n^2$ independent, where

$$X_i^2 \sim \chi_{\nu_i}^2(\tau_i^2), \tag{1.11}$$

i.e., X_i^2 is a noncentral Chi squared variable with ν_i degrees of freedom and noncentrality parameter τ_i^2. Here we base problem (1.1) on $\mathbf{X} \equiv (X_1, \cdots, X_n)$ where $\boldsymbol{\theta} = \tau \equiv (\tau_1, \cdots, \tau_n)$.

Bhattacharya (1961) performed Monte Carlo experiments to compare the powers of tests (1.2), (1.3) and the test with rejection region

$$\Sigma X_i^2 > \chi_{\Sigma\nu_i}^2 \text{ (sum test).} \tag{1.12}$$

Koziol and Perlman (1978) numerically compared the powers of the three tests Bhattacharya treated, plus test (1.4) and the tests with the following rejection regions:

$$\Sigma X_i > c_\alpha \text{ (sum of Chi's test),} \tag{1.13}$$

and

$$\Sigma b_i \exp(c_i X_i^2) > d_\alpha, \tag{1.14}$$

where $b_i > 0$ and $c_i > 0$ for all i. They also derive some Bayes tests, including the sum test (1.12) and test (1.14). Their calculations reveal that the inverse normal test performs poorly while Fisher's procedure (1.2) and the sum test (1.12) have good power over a wide range of alternatives. Tippett's procedure (1.3) and test (1.14) have good power only along the axes of Ω, and the sum of Chi's test (1.13) has good power only when each $\nu_i = 1$ and the alternatives are near the line $(1, 1, \cdots, 1)$.

They note that when all ν_i's ≥ 2, Fisher's test (1.2) is "relatively minimax" in the sense that it minimizes the maximum shortcoming in power at certain points τ among tests they consider. When all ν_i's are 1, the sum test (1.12) has this property. They consequently propose the test with rejection region

$$\Sigma_{i\in I_1} X_i^2 - 2\Sigma_{i\in I_2} \log p_i(X_i) > \chi_{n_1+2n_2,\alpha}^2 \tag{1.15}$$

where $I_1 = \{i \mid \nu_i = 1\}$, $I_2 = \{i \mid \nu_i > 1\}$, and $n_k = \#I_k$. This test is expected to perform well with respect to the relatively minimax criterion.

We are able to adapt Birnbaum's results on exponential families to this case (a la Oosterhoff (1969), Theorems 1.4.1 and 1.4.3) since the density of $\mathbf{X}$ is close to exponential with natural parameter τ (see (2.5)). In Section 2 it is shown that a necessary and sufficient condition for a test to be admissible is that its acceptance region is convex and monotone decreasing in the $\mathbf{X}$-space. Using this result, we show in Section 3 that tests (1.2), (1.3), (1.12), (1.13), (1.14) and (1.15) are admissible for all levels α, tests (1.4) and (1.5) are inadmissible for $0 < \alpha < 1$, and test (1.7) is inadmissible for $0 < \alpha < 1 - 1/n!$ We conjecture that test (1.6) is admissible for all α, since both special cases (1.2) and (1.12) (when $\nu_i = \beta$) are. L. Brown and W. Strawderman (Koziol and Perlman, 1976, page 23) have already shown that Tippett's procedure (1.3) is admissible. Thus our results complement those of Koziol and Perlman. For tests with reasonable power overall, Fisher's procedure, the sum test, and test (1.15) are recommended. They are admissible, so cannot be dominated, are easy to calculate, and have easily obtained cutoff points.

In Sections 4 and 5 we follow a similar program when we base (1.1) on $F_1, \cdots, F_n$ independent, where

$$F_i \sim \frac{\chi_{\nu_i}^2(\Delta_i)}{\chi_{\mu_i}^2}, \tag{1.16}$$

the numerator and denominator Chi squared being independent. Now $\boldsymbol{\theta} = \boldsymbol{\Delta} \equiv (\Delta_1, \cdots, \Delta_n)$.

Zelen (1957) and Zelen and Joel (1959) look at the test for $n = 2$ based upon $p_1^d p_2$ and

Pape (1972) considers tests for $n \geq 2$ based on $\Pi\, p_i^{d_i}$, where the F's arise from incomplete block designs. They suggest weights d or d_i. Monti and Sen (1976) propose using the test based upon $\Sigma\, \gamma_i Y_i$ (see (1.17)) and show that for the proper weights γ_i, the test has locally optimal properties.

Using the methods in Section 2, we show in Section 4 that a sufficient condition for a test to be admissible is that its acceptance region is convex and monotone decreasing in the space of $\mathbf{Y} \equiv (Y_1, \cdots, Y_n)$, where

$$Y_i = \frac{F_i}{1 + F_i}. \tag{1.17}$$

A necessary condition is that the acceptance region be convex and monotone decreasing the space of $\mathbf{Y}^* \equiv (Y_1^*, \cdots, Y_n^*)$, where

$$Y_i^* = Y_i^{r_i}, \tag{1.18}$$

and r_i is defined in (4.2).

Section 5 contains some generalized Bayes tests, including those which reject H_0 when

$$\Sigma\, F_i > f_\alpha, \tag{1.19}$$

$$\Sigma\, Y_i > y_\alpha, \tag{1.20}$$

and

$$\prod (1 + F_i) > g_\alpha. \tag{1.21}$$

We apply the results of Section 4 to show that Fisher's procedure (1.2) is admissible for $0 < \alpha < 1$ if and only if all ν_i's ≥ 2; Tippett's procedure (1.3) is always admissible; the inverse normal (1.4) and inverse logistic (1.5) procedures are inadmissible for $0 < \alpha < 1$ if either $n > 2$ or $n = 2$ and $\nu_1 < r_1\mu_2$, and are admissible if $n = 2$, $0 < \alpha < ½$, both ν_i's ≥ 2 and both μ_i's ≤ 2; and the sum of p_i's procedure (1.7) is inadmissible for $0 < \alpha < 1 - 1/n!$ if some $\mu_i > 2$, and admissible if all ν_i's ≥ 2 and all μ_i's ≤ 2.

2. Chi squared case: Conditions. Consider problem (1.1) based on $\mathbf{X}$ as in (1.11) and immediately thereafter. Let the space of $\mathbf{X}$ be denoted $\mathscr{X} \equiv \{x \in \mathbb{R}^n \mid x_i > 0 \text{ for all } i\}$. We assume the reader is familiar with the standard terminology of decision theory as in Ferguson (1967). Test functions on $\mathscr{X}$ will be denoted $\phi(\mathbf{x})$, and the risk function is given by

$$r_\tau(\phi) = E_0(\phi)[1 - E_\tau(\phi)] \quad \text{if} \quad \tau = 0[\tau \in \Omega_A].$$

Define $\mathscr{C}$ to be the class of sets $C \subseteq \mathscr{X}$ satisfying the following two conditions:

(i) C is monotone decreasing, i.e., $\mathbf{x} \in C$ and $y_i \leq x_i \forall i$, $\mathbf{y} \in \mathscr{X}$ implies that $\mathbf{y} \in C$.

(ii) C is closed and convex in $\mathscr{X}$.

Let

$$\Phi = \{\phi \mid \phi = 1 - I_C,\, C \in \mathscr{C}\} \quad \text{and} \quad \bar{\Phi} = \{\bar{\phi} \mid \bar{\phi} = \phi \text{ a.e. } [\mu],\, \phi \in \Phi\},$$

where I_B is the indicator function of the set $B \subseteq \mathscr{X}$, and μ is Lebesgue measure on $\mathscr{X}$. The main result of this section follows.

THEOREM 2.1. *$\bar{\Phi}$ is a minimal complete class of tests for problem* (1.1) *based on* $\mathbf{X}$.

The proof of Theorem 2.1 is presented in two parts. Part I is a combination of Theorems 1.4.1 and 1.4.3 of Oosterhoff (1969).

PROOF OF THEOREM 2.1. *Part I*: *$\bar{\Phi}$ is complete.* We start with Theorem 5.8 of Wald (1950), which states that the set of all proper Bayes tests and their weak* limits constitute an essentially complete class. Any proper Bayes test for problem (1.1) is essentially of the form

$$\phi = \begin{cases} 1 & \text{if } \int_{\Omega_A} R_\tau(\mathbf{x})\pi(d\tau) > d \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for some proper measure π on Ω_A and finite constant d, where

$$R_\tau(\mathbf{x}) = \prod R_{\tau_i}(x_i; \nu_i), \tag{2.2}$$

and $R_\tau(x; \nu) = f_\tau(x; \nu)/f_0(x; \nu)$, $f_\tau(x; \nu)$ being the density of a $(X_\nu^2(\tau^2))^{1/2}$ variable. From Anderson (1958), pages 112–113, we have

$$R_\tau(x; \nu) = \exp(-\tau^2/2) \textstyle\sum_{k=0}^{\infty} \dfrac{\Gamma(\nu/2)\Gamma(1/2 + k)}{\Gamma(\nu/2 + k)\Gamma(1/2)} \dfrac{(\tau^2 x^2)^k}{(2k)!} \tag{2.3}$$

$$= \exp(-\tau^2/2) \int_{-1}^{1} e^{\tau u x} h_\nu(du), \tag{2.4}$$

where h_ν is a probability measure on $[-1, 1]$ whose density with respect to Lebesgue measure is proportional to $(1 - u^2)^{(\nu-3)/2} I_{(-1,1)}$ when $\nu \geq 2$, and which equals ½ $(\delta_{-1} + \delta_1)$ when $\nu = 1$, where δ_y is the measure putting a point mass 1 at y. By (2.2) and (2.4),

$$R_\tau(\mathbf{x}) = \exp(-\Sigma\tau_i^2/2) \int_{\mathscr{U}} \exp(\Sigma\tau_i u_i x_i) h(d\mathbf{u}), \tag{2.5}$$

where $h(d\mathbf{u}) \equiv \Pi\, h_{\nu_i}(du_i)$ is a probability measure on $\mathscr{U}$, the closed unit cube in $\mathbb{R}^n$. Equation (2.5) shows that $R_\tau(\mathbf{x})$ is convex in x, and equation (2.3) shows that $R_\tau(x)$ is strictly increasing in each x_i. Thus $\{\mathbf{x} \mid \int R_\tau(\mathbf{x})\pi(d\tau) \leq d\} \in \mathscr{C}$, and ϕ of (2.1) is in Φ. The argument in the proof of Theorem 3.1 of Eaton (1970), which refers to Birnbaum (1955) and Matthes and Truax (1967), shows that any weak* limit of tests in Φ must be in $\bar{\Phi}$. Thus $\bar{\Phi}$ is essentially complete by Wald's Theorem.

We argue that the family of densities considered here is complete (in the sense of Lehmann (1959), page 113), which implies that $\bar{\Phi}$ is complete. Each X_i can be thought of as arising from a sample of ν_i independent normal variates with variance 1 and means unconstrained except to have norm τ_i. If τ is allowed to range over Ω, the family of densities of the totality of these normal variates is complete, hence the family of densities of $\mathbf{X}$ is.

Part II. All $\phi \in \Phi$ are admissible. We adapt the proof of the theorem in Stein (1956). Our proof is very similar to part b) of the proof of Theorem 4.6 in Ghia (1976). We could also have used Theorem 1 of Nandi (1963). For a set $B \subseteq \mathscr{X}$, define $D(B)$ to be the set of all test functions ϕ such that $\phi = 1$ a.e. $[\mu]$ on B^c, i.e., the acceptance region of ϕ is essentially contained in B. The following lemma contains the main step in the proof.

LEMMA 2.2. *For $C \in \mathscr{C}$, if $\phi \in D(C)$ and $\psi \notin D(C)$, then there exists $\tau \in \Omega_A$ such that $r_\tau(\phi) < r_\tau(\psi)$.*

Take $\phi \in \bar{\Phi}$, and let C be the set in $\mathscr{C}$ corresponding to ϕ. Clearly $\phi \in D(C)$. Lemma 2.2 shows that for any test ψ such that $E_0(\phi) = E_0(\psi)$ and $\mu(\{\phi \neq \psi\}) > 0$, since $\psi \notin D(C)$, ϕ dominates ψ at some alternative. Thus ϕ is admissible.

PROOF OF LEMMA 2.2. Note that C can be expressed as an intersection of halfspaces of $\mathscr{X}$ of the form $H = \{\mathbf{x} \in \mathscr{X} \mid \Sigma\gamma_i x_i \leq c\}$ for $\gamma \in \Omega_A$ and $c > 0$. Thus for some H, $\phi \in D(H)$ and $\psi \notin D(H)$. Use the fact that $\mu(\{\Sigma\gamma_i x_i = c\}) = 0$ to obtain for $s > 0$ that

$$\text{(2.6)} \quad \exp(s^2\Sigma\gamma_i^2/2)\exp(-sc)[r_{s\gamma}(\psi) - r_{s\gamma}(\phi)] = \int_{\text{int}(H)} (\phi - \psi)T(\mathbf{x}, s)f_0(\mathbf{x})\mu\,(d\mathbf{x}) + \int_{H^c} (1 - \psi)T(\mathbf{x}, s)f_0(\mathbf{x})\mu\,(d\mathbf{x}),$$

where from (2.5),

$$\text{(2.7)} \qquad T(\mathbf{x}, s) = \int_{\mathscr{U}} \exp(s(\Sigma\gamma_i u_i x_i - c))h\,(d\mathbf{u}).$$

We will show that as $s \to \infty$, the limit of the left-hand side of (2.6) is $+\infty$, which proves the lemma. Start by taking the limit of the first integral of the right-hand side of (2.6). Since each $u_i \le 1$, (2.7) shows that $| T(\mathbf{x}, s) | \le 1$ and $\lim_{s\to\infty} T(\mathbf{x}, s) = 0$ for $\mathbf{x} \in \text{int}(H)$. Thus application of the Dominated Convergence Theorem shows the limit is zero. Now take the limit infimum of the second integral. Since $u_1, \cdots, u_n$ can be simultaneously arbitrarily close to 1 with positive h-measure, (2.7) shows that for $\mathbf{x} \in H^c$, $\lim_{s\to\infty} T(\mathbf{x}, s) = +\infty$. Because $1 - \phi_2 \ge 0$, Fatou's Lemma and the fact that $\mu(\{1 - \phi_2 > 0\} \cap H^c) > 0$ implies that the limit infimum of this term is $+\infty$. Hence the result.

3. Chi squared case: Specific tests. We refer the reader to Koziol and Perlman (1978) for some generalized Bayes tests. The above authors and Monti and Sen (1976) have shown that the test based on $\Sigma\gamma_i^2 X_i^2/\nu_i$ is locally most powerful for alternatives $\tau = \lambda(\gamma_1, \cdots, \gamma_n)$ as $\lambda \to 0$. Such tests are admissible. Each test we consider below satisfies the monotonicity condition (i) for tests in $\bar{\Phi}$, hence by Theorem 2.1, admissibility is equivalent to convexity of the acceptance region. It is easy to see that Tippett's procedure (1.3), the sum test (1.12), the sum of Chi's test (1.13) and test (1.14) have convex acceptance regions, hence are admissible. To show the same for Fisher's procedure (1.2), write

$$\text{(3.1)} \quad \begin{aligned} -\frac{d}{dx}\log p_i(x) &= -\frac{d}{dx}\int_{x^2}^{\infty} \exp(-w/2)w^{\nu_i/2-1}\,dw \\ &= 2x\left[\int_{x^2}^{\infty} \exp(-(w - x^2)/2)(w/x^2)^{\nu_i/2-1}\,dw\right]^{-1} \\ &= 2x\left[\int_{0}^{\infty} \exp(-w/2)(1 + w/x^2)^{\nu_i/2-1}\,dw\right]^{-1} \\ &= 2\left[\int_{0}^{\infty} \exp(-w/2)(w + x^2)^{-1/2}\,dw\right]^{-1} \quad \text{if} \quad \nu_i = 1. \end{aligned}$$

Line 3 (line 4 for $\nu_i = 1$) of (3.1) shows that $-$ (d/dx) *log* p_i *is increasing in* x, so that $-\Sigma \log p_i$ is convex in $\mathbf{x}$, hence the acceptance region is convex. Equation (3.1) also can be used to show that test (1.15) is admissible.

Consider test (1.8) when each H_i is a distribution function whose support contains (a, ∞) for some $a \in [-\infty, \infty)$, so that

$$\text{(3.2)} \qquad H_i^{-1}(p_i) \to \infty \qquad \text{as} \qquad x_i \to 0,$$

since $p_i \to 1$. Suppose this test has a convex acceptance region A. We show that if $h_{1\alpha} < \infty$, then α, the level of the test, is 0, which implies that the test is inadmissible for $0 < \alpha < 1$. Take any $\mathbf{x} \in \mathscr{X}$. By (1.8) and (3.2), since $h_{1\alpha} < \infty$, we can take $\varepsilon > 0$ small enough so that $\mathbf{x}^1 \equiv (nx_1, \varepsilon, \cdots, \varepsilon)$; $\mathbf{x}^2 \equiv (\varepsilon, nx_2, \varepsilon, \cdots, \varepsilon)$, $\cdots$; $\mathbf{x}^n \equiv (\varepsilon, \cdots, \varepsilon, nx_n)$ are in A. By the assumed convexity of A, $(1/n)\Sigma\mathbf{x}^i \equiv \mathbf{x} + (\varepsilon, \cdots, \varepsilon) \in A$. By the monotonicity of A, $\mathbf{x} \in A$. Thus $A = \mathscr{X}$, proving $\alpha = 0$. The inverse normal procedure (1.4), inverse logistic procedure (1.5) and test based on $\Sigma G^{-1}(p_i; \beta_i)$ (which reduces to (1.10) when each $\beta_i = 2$) are of this form, hence inadmissible for $0 < \alpha < 1$.

A similar argument will show that the sum of p_i's test (1.7) is inadmissible when $0 < c_\alpha < n - 1$, since, given any value for one coordinate of a point $\mathbf{x}$, the other coordinates can be chosen small enough so that $\mathbf{x}$ is in the acceptance region. Because $c_{\alpha^*} = n - 1$ when $\alpha^* = 1 - 1/n!$, the test (1.7) is inadmissible when $0 < \alpha < 1 - 1/n!$

Finally, note that weighting the summands by positive constants in any of tests (1.2), (1.4), (1.5), (1.12), (1.13) or (1.15) does not change the admissibility or inadmissibility of the test.

4. F case: Conditions. We test (1.1) based upon the variables $F_1, \cdots, F_n$ defined in (1.16). We work with Y and Y^* defined in (1.17) and (1.18) respectively, where $r = r(\nu, \mu)$ is defined as follows. Let ${}_1F_1(z; a, b)$ be the confluent hypergeometric function

$$(4.1) \qquad {}_1F_1(z; a, b) = \frac{\Gamma(b)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(b+k)} \frac{z^k}{k!}.$$

See Abramowitz and Stegun (1964), Chapter 13. Define r by

$$(4.2) \qquad r(\nu, \mu) = \inf\{s > 0 \mid \inf_{z>0} \frac{\partial}{\partial z} \frac{\partial}{\partial z^s} \log {}_1F_1(z; (\nu + \mu)/2, \nu/2) \geq 0\}.$$

In Marden and Perlman (1981) we show that

$$(4.3) \qquad \max(1/2, \nu/(\nu + \mu)) < r < 1.$$

Table 2.1 of Marden and Perlman (1980) contains some values of r.

The space of $\mathbf{Y}$ is $\mathscr{Y} \equiv \{\mathbf{y} \in R^n \mid 0 < y_i < 1 \forall i\}$, and for $\mathbf{Y}^*$ is $\mathscr{Y}^* \equiv \{y^* \in R^n \mid 0 < y_i^* < 1 \forall i\}$. Define $\mathscr{D}$ to be the class of all sets $C \subseteq \mathscr{Y}$ which are closed, monotone decreasing, and convex in $\mathscr{Y}$. Let $\mathscr{D}^*$ be the class of all sets $C^* \subseteq \mathscr{Y}$ whose images in $\mathscr{Y}^*$ under (1.18) are closed, monotone decreasing, and convex in $\mathscr{Y}^*$. A test $\phi(y)$ of (1.1) is a function on $\mathscr{Y}$. Let $\mathbf{\Psi}(\Psi^*)$ be the class of (nonrandomized) tests with acceptance regions in $\mathscr{D}(\mathscr{D}^*)$. Define $\bar{\Psi} = \{\phi \mid \phi = \psi \text{ a.e. } [\mu] \text{ for some } \psi \in \psi\}$, and Ψ^* similarly, where μ is Lebesgue measure on $\mathscr{Y}$. The main result of this section follows.

THEOREM 4.1. I) *$\bar{\Psi}^*$ is a complete class of tests for problem* (1.1) *based on* $\mathbf{Y}$. II) *All tests in Ψ* (*hence in $\bar{\Psi}$*) *are admissible for problem* (1.1) *based on* $\mathbf{Y}$.

PROOF. *Part* I. A proper Bayes test relative to the proper measure π on Ω_A is of the form (2.1) with $R_\tau(\mathbf{x})$ replaced by $Q_\Delta(\mathbf{y}) \equiv f_\Delta(\mathbf{y})/f_0(\mathbf{y})$, $f_\Delta(\mathbf{y})$ being the density of $\mathbf{Y}$ when $\mathbf{\Delta} \in \Omega$ obtains. From Anderson (1958), page 114, it is seen that

$$(4.4) \qquad Q_\Delta(\mathbf{y}) = \prod \exp(-\Delta_i/2)\, {}_1F_1(\Delta_i y_i/2; a_i, b_i),$$

where $a_i = (\nu_i + \mu_i)/2$, $b_i = \nu_i/2$, and ${}_1F_1$ is given in (4.1). Lemma 2.6 (c) of Marden and Perlman (1980) shows that $\log {}_1F_1(z; a, b)$ is convex in z^r for r in (4.2). Also, $Q_\Delta(\mathbf{y})$ is clearly increasing in each y_i. Thus the acceptance region of ϕ must be in $\mathscr{D}^*$, and $\phi \in \Psi^*$. As in Part I of the proof of Theorem 2.1 for Φ, any weak* limit of tests in Ψ^* is in $\bar{\Psi}^*$, hence $\bar{\Psi}^*$ is an essentially complete class of tests by Wald's Theorem. An argument as in Part I of the proof of Theorem 2.1 can be made to show that the family of densities of $\mathbf{Y}$ is complete, where each Y_i is considered to arise from ν_i independent normal variates with variance σ_i^2 and an independent $\sigma_i^2 X_{\mu i}^2$ variate. Thus $\bar{\Psi}^*$ is a complete class.

Part II. For $B \subseteq \mathscr{Y}$, define $D(B)$ to be the class of all tests ϕ such that $\phi = 1$ a.e. $[\mu]$ on B^c. This part of the theorem will be established by proving the analogue of Lemma 2.2. (Replace $(\mathscr{C}, \tau)$ there with $(\mathscr{D}, \mathbf{\Delta})$ here.) Follow the proof of Lemma 2.2 with $(\mathbf{x}, \mathscr{X})$ replaced by $(\mathbf{y}, \mathscr{Y})$ until Equation (2.6), which becomes

(4.5) $$s^{-\sigma}\exp(s\Sigma\gamma_i/2)\exp(-sc/2)\,[r_{s\gamma}(\psi)-r_{s\gamma}(\phi)] = \int_{\mathrm{int}(H)}(\phi-\psi)V(\mathbf{y},s)f_0(\mathbf{y})\,d\mathbf{y} + \int_{H^c}(1-\psi)V(\mathbf{y},s)f_0(\mathbf{y})\,d\mathbf{y}$$

where, by (4.4),

(4.6) $$V(\mathbf{y},s) = s^{-\sigma}\exp(-sc/2)\prod_{i=1}^{n} {}_1F_1(s\gamma_i y_i/2;\,a_i,\,b_i)$$

and $\sigma = \Sigma(a_i - b_i)$. Abramowitz and Stegun (1964), Equation 13.14, states that when $a \geq b > 0$, as $z \to \infty$,

(4.7) $${}_1F_1(z;\,a,\,b) \sim [\Gamma(b)/\Gamma(a)]e^z z^{a-b}.$$

Hence from (4.7), $\lim_{s\to\infty} V(\mathbf{y}, s) = 0(\infty)$ as $\mathbf{y} \in \mathrm{int}(H)(H^c)$.

Now an argument similar to the one below Equation (2.7) applied to Equation (4.5) will prove the analogue of the lemma. To justify use of the Dominated Convergence Theorem in the first integral here, note that for $\mathbf{y} \in \mathrm{int}(H)$, since $e^{-z}{}_1F_1(z;\,a,\,b)$ is increasing in z when $a \geq b > 0$ and each $y_i < 1$,

$$|\,V(\mathbf{y},s)\,| \leq s^{-\sigma}\prod_{i=1}^{n}\exp(-s\gamma_i/2)\,{}_1F_1(s\gamma_i/2;\,a_i,\,b_i).$$

The right-hand term above is continuous and finite in $s > 0$, and has limit zero as $s \to \infty$ by (4.8), hence is bounded in $s \geq \varepsilon$ for any $\varepsilon > 0$.

REMARK 4.2 There are tests not in Ψ which are admissible. In fact, take any $\mathbf{\Delta} \in \Omega_A$ with at least two nonzero components. The test rejecting H_0 when $Q_{\mathbf{\Delta}}(\mathbf{y}) > c$ is admissible, but it can be shown that log $Q_{\mathbf{\Delta}}(\mathbf{y})$ is strictly concave in $\mathbf{y}$, hence the acceptance region is not in $\mathscr{D}$, so that the test is not in $\bar{\Psi}$. It is an open question whether all tests in $\bar{\Psi}^*$ are admissible.

5. F case: Specific tests. In Section 5.1 we present some generalized Bayes tests. In Section 5.2 we consider the nonparametric tests except for Tippett's procedure (1.3), which is clearly in $\bar{\Psi}^*$, hence admissible. Marden and Perlman (1981) contains additional results in the inverse normal or logistic cases when $n = 2$ and $(\nu_1, \mu_1) = (\nu_2, \mu_2)$, and in the sum of p_i's case when $n = 2$ or when $n > 2$ and $(\nu_i, \mu_i) = (\nu, \mu)$ for all i. We remark that Monti and Sen (1976) have shown that the test based on $\Sigma\gamma_i(\nu_i + \mu_i)\,Y_i/\nu_i$ is locally most powerful along $\mathbf{\Delta} = \lambda\gamma$ as $\lambda \to 0$.

5.1 Bayes tests. We follow Koziol and Perlman (1978) in considering Type I and II priors. Given measures $\pi_1, \cdots, \pi_n$ on $(0, \infty)$, the corresponding Type I and II priors on Ω_A are $\prod\pi_i\,(d\Delta_i)$ and $\Sigma c_i\delta_i(\mathbf{\Delta})\pi_i\,(d\mathbf{\Delta})$, respectively, where the c_i's are positive constants and $\delta_i(\mathbf{\Delta}) = 1$ if $\Delta_i > 0$ and $\Delta_j = 0$ for $j \neq i$, and 0 otherwise. The former measure treats the Δ_i's as independent, the latter places all its mass along the axes of Ω_A. Consider the finite measures on $(0, 2s_i)$, $s_i > 0$, with density $\exp(\Delta_i/2)f(\Delta_i/2s_i;\,b_i,\,a_i - b_i)/2s_i$ where f is the beta density

(5.1) $$f(w;\,m_1,\,m_2) = \beta(m_1,\,m_2)w^{m_1-1}(1-w)^{m_2-1} \quad \text{for} \quad w \in (0,\,1)$$

and

$$\beta(m_1,\,m_2) = \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_2)}.$$

Note that $f(y_i;\,\nu_i/2,\,\mu_i/2)$ is the null density of Y_i. From (4.1), (4.4) and (5.1), the Bayes tests corresponding to the Type I and II priors constructed from these measures are, respectively, based on

$$\Sigma c_i\exp(s_iY_i) \quad \text{and} \quad \Sigma s_iY_i.$$

Note test (1.20) is of the latter type. Now take a gamma mixture of each of the above measures on $(0, 2s_i)$, i.e., let s_i have density

$$[\Gamma(\beta_i)]^{-1}\beta_i^{\alpha_i}\exp(-\beta_i s_i)s_i^{\alpha_i-1} \quad \text{for } s_i > 0, \quad \alpha_i > 0, \quad \beta_i > 0.$$

These mixtures produce Type I and II priors yielding Bayes tests based on, respectively,

$$\Sigma c_i(\beta_i - Y_i)^{-\alpha_i} \quad \text{and} \quad \prod (\beta_i - Y_i)^{-\alpha_i},$$

where the statistic is taken to be $+\infty$ if $Y_i \geq \beta_i$. Letting $(\alpha_i, \beta_i, c_i) = (1, 1, 1)$ for all i, by (1.17) one obtains (1.19) and (1.21).

5.2 Nonparametric tests. Let ϕ be one of the tests (1.2), (1.4), (1.5) or (1.7). It has an acceptance region in $\mathscr{Y}$ of the form $C = \{y \in \mathscr{Y} \mid \Sigma w_i(y_i) \leq c\}$ where each w_i is strictly increasing with continuous second derivative. Let C^* be the acceptance region in $\mathscr{Y}^*$, i.e.,

$$C^* = \{\mathbf{y}^* \in \mathscr{Y}^* \mid \textstyle\sum u_i(y_i^*) \leq c\} \quad \text{where} \quad u_i(y_i^*) = w_i((y_i^*)^{1/r_i}).$$

We denote the boundary of $C(C^*)$ in $\mathscr{Y}(\mathscr{Y}^*)$ by $\partial C(\partial C^*)$.

LEMMA 5.1. *Suppose ϕ is as above.*

a) *If $n = 2$, then a sufficient condition for ϕ to be admissible is that*

$$\textstyle\sum_{i=1}^2 w_i''(y_i)/(w_i'(y_i))^2 \geq 0 \quad \text{for all} \quad \mathbf{y} \in \partial C. \tag{5.2}$$

b) *For arbitrary $n \geq 2$, a necessary condition for ϕ to be admissible is that for each (k, ℓ), $k \neq \ell$,*

$$u_k''(y_k^*)/(u_k'(y_k^*))^2 + u_\ell''(y_\ell^*)/(u_\ell'(y_\ell^*))^2 \geq 0 \quad \text{for all} \quad \mathbf{y}^* \in \partial C^*. \tag{5.3}$$

PROOF. a) By Theorem 4.1, if C is convex in $\mathscr{Y}$, then ϕ is admissible. Since the w_i's are continuous and strictly increasing, the function h defined on $\mathscr{Y}_1 \equiv \{y_1 \mid (y_1, y_2) \in \partial C$ for some $y_2\}$ via

$$w_1(y_1) + w_2(h(y_1)) = c \tag{5.4}$$

is well defined. Furthermore, since the w_i's have continuous second derivatives, h does, too. Now C is convex if and only if $h''(y_1) \leq 0$ for all $y_1 \in \mathscr{Y}_1$. Differentiate both sides of (5.4) with respect to y_1 to obtain that $h'(y_1) = -w_1'(h(y_1))/w_2'(y_1)$. Differentiate again to show that $-h''(y_1)w_2'/(w_1')^2$ equals the left-hand side of (5.2). Since $w_i' > 0$, $-h''(y_1) \leq 0$ for all $y_1 \in \mathscr{Y}_1$ if and only if (5.2) holds, hence part a) is proved.

b) Without loss of generality, take $(k, \ell) = (1, 2)$. Suppose ϕ is admissible and has level $0 < \alpha < 1$. By Theorem 4.1, ϕ must equal a.e. $[\mu]$ a test with acceptance region D^* which is closed and convex in $\mathscr{Y}^*$. Thus $\mu((C^* - D^*) \cup (D^* - C^*)) = 0$. However, since C^* and D^* are both closed, it must be that $\text{int}(C^*) \subseteq D^*$ and $\text{int}(D^*) \subseteq C^*$. Furthermore, each of C^* and D^* equals the closure of its (nonempty) interior, C^* because the w_i's are continuous and strictly increasing and D^* because it is convex. Thus $D^* = C^*$, i.e., C^* is convex, which implies that for any fixed $y_3^{*0}, \cdots, y_n^{*0}$, the set $C^{*0} = \{(y_1^*, y_2^*) \mid (y_1^*, y_2^*, y_3^{*0}, \cdots, y_n^{*0}) \in \partial C^*\}$ is convex. An argument as in the proof of part a) shows that C^{*0} is convex if and only if (5.3) holds, which proves part b).

We treat the tests individually. Here, $p_i(y_i) = p(y_i; \nu_i, \mu_i)$ where

$$p(y; \nu, \mu) = p(y) = \int_y^1 f(w; \nu/2, \mu/2)\, dw,$$

and we let $\beta_i = \beta(\nu_i/2, \mu_i/2)$; see (5.1). For convenience, define $t_i(y_i, s_i) = t(y_i, s_i; \nu_i, \mu_i)$, where

$$t(y, s; \nu, \mu) \equiv t(y, s) = [(\nu/2 - s)(1 - y) - (\mu/2 - 1)y]y^{-1}(1 - y)^{-1}.$$

Fisher's procedure (1.2). Suppose all ν_i's ≥ 2. Since

$$-\frac{d}{dy}\log p(y) = (1-y)^{-1}\left[\left[\int_0^1 (u + y^{-1}(1-u))^{\nu/2-1}u^{\mu/2-1}\,du\right]\right]^{-1}$$

is strictly increasing in y for $\nu \geq 2$, the acceptance region of test (1.2) is convex in $\mathbf{y}$, hence by Theorem 4.1, the test is admissible. Note that the weighted version of this test is also admissible. Now suppose $\nu_1 = 1$. On the boundary ∂C, $p_1p_2 = \exp(-c)/(p_3 \cdots p_n)$. Thus since $c > 0$, we can find $y_3^0, \cdots, y_n^0 \in (0, 1)$ small enough such that for $(y_1, y_2, y_3^0, \cdots, y_n^0) \in \partial C$, as $y_1 \to 0$, $y_2 \to y_2^0 \in (0, 1)$. But

$$\lim_{y_1\to 0,\, y_2\to y_2^0} y_1^{\nu_1/2} \textstyle\sum_{i=1}^2 \dfrac{u_i''(y_i^*)}{(u_i'(y_i^*))^2} = \lim_{y_1\to 0,\, y_2\to y_2^0} y_1^{\nu_1/2} \sum_{i=1}^2 \left[\dfrac{p_i(y_i)}{f_i(y_i)} t_i(y_i; r_i) + 1\right]$$

$$= \frac{\nu_1 - 2r_1}{2\beta_1} < 0$$

since $\nu_1 = 1$ and $r_1 > ½$ by (4.3). Thus condition (5.3) is violated, which by Lemma 5.1 b) proves the test is inadmissible.

Inverse normal procedure (1.4). When $n > 2$, for any arbitrarily small y_1 and y_2, we can find $y_3, \cdots, y_n$ such that $\mathbf{y} \in \partial C$. Using l' Hospital's rule, it can be shown that

$$\text{(5.5)} \qquad \lim_{y\to 0} \frac{u''(y^*)}{(u'(y^*))^2\Phi^{-1}(p(y))} = \lim_{y\to 0} \frac{\Phi'(\Phi^{-1}(p(y)))t(y, r)}{f(y)\Phi^{-1}(p(y))} - 1 = -2r/\nu.$$

Thus for small enough y_1 and y_2, (5.3) is violated for $(k, \ell) = (1, 2)$, and so the test is inadmissible. Now let $n = 2$, $\alpha < ½$ (implying $c > 0$), both ν_i's ≥ 2, and both μ_i's ≤ 2, so that $t_i(y_i, 1) \geq 0$. Hence

$$\text{(5.6)} \qquad \textstyle\sum_{i=1}^2 \dfrac{w_i''(y_i)}{(w_i'(y_i))^2} = \sum_{i=1}^2 \dfrac{\Phi'(\Phi^{-1}(p_i(y_i)))}{f_i(y_i)} t_i(y_i, 1) - \Phi^{-1}(p(y_1)) - \Phi^{-1}(p_2(y_2))$$

is positive on ∂C, since on ∂C, $-\Phi^{-1}(p_1) - \Phi^{-1}(p_2) = c > 0$.

Thus the test is admissible by Lemma 5.1 a). Finally, suppose $n = 2$ and $\nu_1 < r_1\mu_2$. As $y_1 \to 0$ for $\mathbf{y}$ on ∂C, $y_2 \to \infty$, hence

$$\text{(5.7)} \qquad \lim_{y_1\to 0,\, y\in\partial C} \textstyle\sum_{i=1}^2 \dfrac{u_i''(y_i^*)}{(u_i'(y_i^*))^2} \Big/ \Phi^{-1}(p_1(y_1)) = -2r_1/\nu_1 + 2/\mu_2 < 0,$$

which violates (5.3), showing the test inadmissible.

Inverse logistic procedure (1.5). Follow the arguments as for the inverse normal procedure, except replace (5.5) by

$$\lim_{y\to 0} \frac{u''(y^*)}{(u'(y^*))^2} = \lim_{y\to 0} \left[\frac{p(y)(1-p(y))}{f(y)} t(y, r) + 1 - 2p(y)\right] = -2r/\nu;$$

replace (5.6) by

$$\textstyle\sum_{i=1}^2 \dfrac{w_i''(y_i)}{(w_i'(y_i))^2} = \sum_{i=1}^2 \dfrac{p_i(y_i)(1-p_i(y_i))}{f_i(y_i)} t_i(y_i, 1) + 2(1 - p_1(y_1) - p_2(y_2))$$

and note that on ∂C, $p_1p_2/(1 - p_1 - p_2 + p_1p_2) = e^{-c} < 1$, hence $1 - p_1 - p_2 > 0$; and replace (5.7) by

$$\lim_{y_1\to 0,\, y\in\partial C} \textstyle\sum_{i=1}^2 \dfrac{u_i''(y_i^*)}{(u_i'(y_i^*))^2} = -2r_1/\nu_1 + 2/\mu_2 < 0.$$

Sum of p_i's test (1.7). When all ν_i's ≥ 2 and μ_i's ≤ 2,

$$-\frac{d^2}{dy_i^2} p_i(y_i) = f_i(y_i)t_i(y_i, 1) > 0$$

since $t_i(y_i, 1) > 0$. Thus $-\sum p_i(y_i)$ is convex, implying that $C \in \mathscr{C}$, and hence by Theorem 4.1, implying that the test is admissible. Now suppose $\mu_1 > 2$ and $0 < \alpha < 1 - 1/n!$ The latter condition implies that $0 < c < n - 1$, so that we can find $y_3^0, \cdots, y_n^0$ such that $0 < c^0 \equiv c - (p_3(y_3^0) + \cdots + p_n(y_n^0)) < 1$. The points $(y_1, y_2, y_3^0, \cdots, y_n^0)$ on ∂C include all points (y_1, y_2) such that $p_1(y_1) + p_2(y_2) = c^0$. Letting $y_1 \to 1$ along this curve, we have $y_2 \to y_2^0 \in (0, 1)$ and

$$\lim_{y_1\to 1, y_2\to y_2^0}(1 - y_1)^{\mu_1/2} \sum_{i=1}^2 \frac{u_i''(y_i^*)}{(u_i'(y_i^*))^2} = \lim_{y_1\to 1, y_2\to y_2^0}(1 - y_1)^{\mu_1/2} \sum_{i=1}^2 \frac{t_i(y_i, r_i)}{f_i(y_i)}$$

$$= (\mu_1 - 2)/2\beta_1 < 0.$$

Hence by Lemma 5.1 b), the test is inadmissible.

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