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ON THE LIMITING DISTRIBUTION OF AND CRITICAL VALUES FOR THE MULTIVARIATE CRAMÉR-VON MISES STATISTIC

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Let $Y_1, Y_2, \cdots, Y_n$ $(n = 1, 2, \cdots)$ be independent random variables (r.v.'s) uniformly distributed over the d-dimensional unit cube, and let $\alpha_n(\cdot)$ be the empirical process based on this sequence of random samples. Let $V_{n,d}(\cdot)$ be the distribution function of the Cramér-von Mises functional of $\alpha_n(\cdot)$, and define $V_d(\cdot) = \lim_{n\to\infty} V_{n,d}(\cdot)$, $\Delta_{n,d} = \sup_{0<x<\infty} | V_{n,d}(x) - V_d(x) |$. We deduce that $\Delta_{n,d} = O(n^{-1})$, $d \geq 1$, and calculate also the "usual" levels of significance of the distribution function $V_d(\cdot)$ for $d = 2$ to 50, using expansion methods. Previously these were known only for $d = 1, 2, 3$.

1. Introduction. Let $Y_1, \cdots, Y_n$ be independent random variables (r.v.'s) uniformly distributed over the d-dimensional unit cube I^d $(d \geq 1)$, and let $E_n(y)$ be the empirical distribution function of $Y_1, \cdots, Y_n$, i.e., for $y = (y_1, \cdots, y_d) \in I^d$, $E_n(y)$ is the proportion of $Y_j = (Y_{j1}, \cdots, Y_{jd})$, $j = 1, \cdots, n$, whose components are less than or equal to the corresponding components of y, conveniently written as

$$E_n(y) = E_n(y_1, \cdots, y_d) = n^{-1} \sum_{j=1}^n \prod_{i=1}^d I_{[0,y_i]}(Y_{ji}), \tag{1.1}$$

where, for real numbers $a, u \in [0, 1]$,

$$I_{[0,a]}(u) = \begin{cases} 1 & \text{if } u \leq a \\ 0 & \text{if } u > a. \end{cases} \tag{1.2}$$

The corresponding uniform empirical process is

$$\alpha_n(y) = n^{1/2}\{E_n(y) - \lambda(y)\}, \qquad y \in I^d, \qquad d \geq 1, \tag{1.3}$$

where $\lambda(y) = \Pi_{i=1}^d y_i$.

This process occurs in the context of continuous distribution functions F on R^d in the following way. Let $\mathscr{F}$ be the class of continuous distribution functions on d-dimensional Euclidean space R^d $(d \geq 1)$, and let $\mathscr{F}_0$ be the subclass consisting of every member of $\mathscr{F}$ which is a product of its associated one-dimensional marginal distribution functions. Let $X_1, \cdots, X_n$ be independent random d-vectors with a common distribution function $F \in \mathscr{F}$, and let $F_n(x)$ be the empirical distribution of $X_1, \cdots, X_n$. That is, for $x = (x_1, \cdots, x_d) \in R^d$,

$$F_n(x) = F_n(x_1, \cdots, x_d) = n^{-1} \sum_{j=1}^n \prod_{i=1}^d I_{(-\infty,x_i]}(X_{ji}), \tag{1.4}$$

where, for all real numbers a and u,

$$I_{(-\infty,a]}(u) = \begin{cases} 1 & \text{if } u \leq a \\ 0 & \text{if } u > a. \end{cases} \tag{1.5}$$

Now consider the empirical process

$$\beta_n(x) = n^{1/2}\{F_n(x) - F(x)\}, \quad x = (x_1, \cdots, x_d) \in R^d, \quad d \geq 1. \tag{1.6}$$

Let $y_i = F_i(x_i)$ be the ith marginal distribution of $F \in \mathscr{F}$ and let $F_i^{-1}(\cdot)$ be its inverse.

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Now if $F \in \mathscr{F}_0$, then

$$(1.7)\quad \begin{aligned}\beta_n(x) &= n^{1/2}\{F_n(x) - \prod_{i=1}^d F_i(x_i)\} = n^{1/2}\{F_n(F_1^{-1}(y_1), \cdots, F_d^{-1}(y_d)) - \lambda(y)\} \\ &= n^{1/2}\{E_n(y) - \lambda(y)\} = \alpha_n(y), \qquad y = (y_1, \cdots, y_d) \in I^d, \qquad d \geq 1.\end{aligned}$$

Therefore, if $F \in \mathscr{F}_0$, then β_n is distribution free.

As to $\alpha_n(\cdot)$, the following results are known.

THEOREM A. *Let $X_1, \cdots, X_n$ ($n = 1, 2, \cdots$) be independent random d-vectors with a common distribution function $F \in \mathscr{F}_0$ and let $\alpha_n(\cdot)$ be as in* (1.7). *Then one can construct a probability space $(\Omega, \mathscr{A}, P)$ with $\{\alpha_n(y); y \in I^d (d \geq 1), n = 1, 2, \cdots\}$ and a sequence of Brownian bridges $\{B_n(y); y \in I^d\ (d \geq 1)\}$ on the space so that for any $\mu > 0$ there exist a $C > 0$ such that* (cf. Csörgő and Révész, 1975) *for each n*

$$(1.8)\qquad P\{\sup_{y\in I^d} | \alpha_n(y) - B_n(y)| > C(\log n)^{3/2} n^{-\frac{1}{2(d+1)}}\} \lessapprox n^{-\mu}, \qquad d \geq 1.$$

Further, if $d = 2$, then (cf. Tusnády, 1977) *for all n and x*

$$(1.9)\qquad P\{\sup_{y\in I^2} | \alpha_n(y) - B_n(y)| > n^{-1/2}(C \log n + x)\log n\} < Le^{-\lambda x},$$

where C, L, λ are positive absolute constants.

For illuminating comments concerning rates of approximation in higher dimensions, we refer to Tusnády (1977b), and for best possible rates of approximation in case of $d = 1$, we refer to Komlós, Major and Tusnády (1975), and Tusnády (1977a). We recall in passing that a Brownian bridge $\{B(y); y \in I^d\}$ is a separable Gaussian process with $EB(y) = 0$ and $EB(x)B(y) = \prod_{i=1}^d (x_i \wedge y_i) - (\prod_{i=1}^d x_i)(\prod_{i=1}^d y_i)$.

Given $F \in \mathscr{F}_0$, we are interested in the asymptotic distribution of the multivariate Cramér-von Mises statistic

$$(1.10)\qquad W_{n,d}^2 = \int_{R^d} \beta_n^2(x) \prod_{i=1}^d dF_i(x_i) = \int_{I^d} \alpha_n^2(y) \prod_{i=1}^d dy_i, \qquad d \geq 1,$$

where $\beta_n(x)$, $\alpha_n(y)$, $y_i = F_i(x_i)$ are as in (1.7). Naturally, say by (1.8), we have for $d \geq 1$ that

$$(1.11)\qquad h(\alpha_n(\cdot)) \to_{\mathscr{D}} h(B(\cdot)),$$

for every continuous functional h on the space of real valued functions on I^d endowed with the supremum topology, and whence also

$$(1.12)\qquad W_{n,d}^2 \to_{\mathscr{D}} W_d^2 = \int_{I^d} B^2(y)\, dy =_{\mathscr{D}} \int_{I^d} B_n^2(y)\, dy = W_d^2(n), \qquad d \geq 1.$$

Here, and in what follows, dy stands for $\prod_{i=1}^d dy_i$. For further results concerning the distance of $W_{n,d}^2$ and $W_d^2(n)$, we refer to Corollary 1 in Csörgő (1979).

Let $V_{n,d}(x)$ be the distribution function of $W_{n,d}^2$ of (1.10) and let $V_d(x)$ be that of W_d^2 of (1.12). Then (1.12) reads

$$(1.13)\qquad \lim_{n\to\infty} P\{W_{n,d}^2 \leq x\} = \lim_{n\to\infty} V_{n,d}(x) = V_d(x), \qquad d \geq 1.$$

Put $\Delta_{n,d} = \sup_{0<x<\infty} | V_{n,d}(x) - V_d(x)|$. Then we have the following.

THEOREM B (*Götze*, 1979). $\Delta_{n,1} = O(n^{-1+\varepsilon})$ *for any* $\varepsilon > 0$.

Earlier, S. Csörgő (1976) showed that $\Delta_{n,1} = O(n^{-1/2}\log n)$ and, on the basis of his complete asymptotic expansion for the Laplace transform of $W_{n,1}^2$, he conjectured that $\Delta_{n,1}$ is of order $1/n$. This conjecture was further studied by S. Csörgő and L. Stachó (1979) by giving a recursion formula for the exact distribution function $V_{n,1}$ of the r.v. $W_{n,1}^2$. They prove that the latter is $[n/2]$ times continuously differentiable, and reduce the problem of proving $\Delta_{n,1} = O(1/n)$ to that of showing the boundedness of the sequence $\{\int_{1/12n}^{n/3} | V_{n,1}^{(49)}(x)| \, dx\}_{n\geq 98}$, where $V_{n,1}^{(49)}$ stands for the 49th derivative of $V_{n,1}$. Their recursion formula is, in principle, also applicable to tabulating $V_{n,1}$ exactly. Actually Götze (1979) proved $\Delta_{n,1} = O(n^{-1})$ without explicitly stating it: his Remark 2.6 holds for $W_{n,1}^2 = n^{-1} \sum_{i,j=1}^{n} h(x_i, y_j)$ with $h(x, y) = 2^{-1}(x^2 + y^2) - (x \vee y) + 1/3$ (cf. Example 2.13 in Götze, 1979), and hence (2.5) of Theorem 2.3 in Götze (1979) implies

COROLLARY 1. $\Delta_{n,1} = O(n^{-1})$.

Naturally, much work has already been done to compile tables for $V_{n,1}$. A survey and comparison of these can be found in Knott (1974), whose results prove to be the most accurate so far. All these results and tables are based on some kind of an approximation of $V_{n,1}$. An extensive tabulation of V_1 (cf. (1.13)) can be found in the recent monograph of Martynov (1978), where the theory and applications of a wide range of univariate Cramér-von Mises type statistics are surveyed.

As to higher dimensions, $d \geq 2$, no analytic results appear to be known about the exact distribution of $V_{n,d}$ (cf. (1.13)). The characteristic function of V_d (cf. (1.13)) is known (cf. Dugue, 1969; Durbin, 1970), and also it is known that (cf. Anderson and Darling, 1952; Rosenblatt, 1952) W_d^2 may be written in the form

$$W_d^2 = \sum_{k=1}^{\infty} \mu_k^{-1} X_k^2, \qquad d \geq 1, \tag{1.14}$$

where the X_k are independent standard normal random variables and the μ_k are the eigenvalues of the integral equation

$$\int_{I^d} E\{B(x_1)B(x_2)\} f(x_2) \, dx_2 = \mu f(x_1) \tag{1.15}$$

with eigenfunctions f and kernel $EB(x_1)B(x_2)$. Whence, in order to tabulate V_d $(d \geq 2)$, just as in the case of V_1 (cf. Durbin and Knott, 1972) one may try working with a numerical inversion of the characteristic function of V_d, or one may try to calculate a number of the necessary eigenvalues for (1.14). Unfortunately, both methods turn out to be quite difficult to follow directly. Durbin (1970) succeeded in solving the latter problem for $d = 2$, as did Krivyakova, Martynov and Tyurin (1977) for $d = 3$. In a similar vein, due to (1.14), one could also try to approximate critical values of the distribution function V_d via a Zolotarev (1961) or Hoeffding (1964) type tail expansion of the latter. Unfortunately, this route also requires a number of the eigenvalues for (1.14) and, as just noted, the calculation of these is difficult in higher dimensions.

Using the characteristic function of Dugue (1969), in this paper we obtain a recursive equation for the cumulants of W_d^2, and then use the Cornish-Fisher asymptotic expansion to calculate its critical values for $d = 2, 3, \cdots, 50$ at various levels of rejection probabilities. These critical values are within 3% of Durbin's values for $d = 2$ and of those of Krivyakova, Martynov and Tyurin for $d = 3$. As far as we know, there exist no other tables for $d \geq 4$. Details, as to how to calculate approximate significance points for all $d > 1$ and tables for the "usual" levels of significance for $d = 2$ to 50, are given in Section 3. Proofs for the statements of the latter are given in Section 4.

As to the question of convergence of the Cornish-Fisher expansion for the distribution function of the r.v. W_d^2, we do not have more evidence than the good numerical agreement with Durbin (1970) for $d = 2$ and with Krivyakova, Martynov and Tyurin (1977) for $d = 3$ (cf. Remark 3.2 and Table 1 for details). In addition, we note also that errors in our tables for higher dimensions should be further reduced due to the fact that the cumulants K_n of W_d^2 are $O(e^{-d})$ (cf. Corollary 7 and Remark 3.2). Therefore our tables should be quite accurate for all the dimensions calculated, indeed improving as d increases. Given that other methods did not work for us, and that no other tables seem to be available for $d \geq 4$, we decided to proceed with the rigorous calculation of cumulants of V_d for the sake of the Cornish-Fisher formal expansion of the latter, having observed the good agreement with existing tables for $d = 2$ and 3.

Since nothing appears to be known about the exact distribution function $V_{n,d}$ for $d \geq 2$, it is desirable to have an analogue of Corollary 1 for $\Delta_{n,d}$ when $d \geq 2$. A complete solution to this problem is again contained in Götze (1979), as outlined in our next section.

2. On Rates of Convergence for $V_{n,d}$ ($d \geq 2$). As a point of reference here and for further use in the sequel, we first quote the following.

THEOREM C (*Dugue*, 1969). *The characteristic function $\phi(t)$ of the* r.v. $W_d^2 (d \geq 1)$ *is*

$$\phi(t) = E \exp(itW_d^2) = \lim_{n\to\infty} E \exp(itW_{n,d}^2) = \left\{ i2^{d-1} \frac{d}{dt} C_d(t) \right\}^{-1/2}, \qquad d \geq 1, \tag{2.1}$$

where

$$C_1(t) = \cos\{(2it)^{1/2}\}, \tag{2.2}$$

and

$$C_d(t) = \textstyle\prod_{j=1}^{\infty} C_{d-1}\left\{\dfrac{t}{(j-½)^2\pi^2}\right\}, \qquad d \geq 2. \tag{2.3}$$

COROLLARY 2 (*following Durbin*, 1970). *For the characteristic function $\phi(t)$ of the* r.v. W_d^2 ($d \geq 1$) *we also have the following forms*

$$\phi(t)^{-2} = -2^d \frac{d}{du} C_d(t), \qquad u = 2it \ (d \geq 1), \tag{2.4}$$

where

$$C_1(t) = \textstyle\prod_{j=1}^{\infty}\left\{1 - \dfrac{2it}{(j-½)^2\pi^2}\right\}, \tag{2.5}$$

and $C_d(t)$ ($d \geq 2$) is as in (2.3). *Whence*

$$\phi^{-2}(t) = P(t)S(t) \tag{2.6}$$

where

$$P(t) = \textstyle\prod_{j_1=1}^{\infty} \cdots \prod_{j_d=1}^{\infty}\left\{1 - \dfrac{2it}{(j_1-½)^2 \cdots (j_d-½)^2\pi^{2d}}\right\}, \qquad d \geq 1,$$

$$S(t) = \sum_{j_1=1}^{\infty} \cdots \sum_{j_d=1}^{\infty} \frac{(j_1-½)^{-2} \cdots (j_d-½)^{-2}\pi^{-2d}}{\left\{1 - \dfrac{2it}{(j_1-½)^2 \cdots (j_d-½)^2\pi^{2d}}\right\}} = -\frac{1}{2i}\frac{d}{dt}\log P(t).$$

PROOF. By (2.2) we have (2.5), since (cf. formula (4.3.90) in Abramowitz and Stegun, 1964)

$$\cos\{(2it)^{1/2}\} = \textstyle\prod_{j=1}^{\infty}\left\{1 - \dfrac{2it}{(j-½)^2\pi^2}\right\}. \tag{2.7}$$

By (2.3) and differentiation we also get (2.4).

It follows from Theorem 2.9 and the calculations of Example (2.13) of Götze (1979) that not only does one have Corollary 1, but also that asymptotic expansions of arbitrary order of $V_{n,1}$ exist; cf. also S. Csörgő (1976). Commenting on an earlier version of our paper, Dr. Götze (private correspondence, 1980) pointed out to us that the same is true for the distribution function $V_{n,d}$ ($d \geq 2$) (cf. (1.13)). Namely for $W_{n,d}^2$ ($d \geq 2$), whose limit in distribution is (1.14), it follows from the just-quoted result of Dugue (1969) that infinitely many μ_k^{-1} of the latter are nonzero and hence (2.4) of Theorem 2.9 in Götze (1979) is satisfied. As to the remaining smoothness condition (2.8) of Theorem 2.9 in Götze (1979), one has

$$W_{n,d}^2 = n^{-1} \textstyle\sum_{i=1,j=1}^{n} h(x_i, y_j) \tag{2.8}$$

where

$$h(x, y) = \prod_{p=1}^{d} \{1 - (x_p \vee y_p)\} - \prod_{p=1}^{d} 2^{-1}(1 - x_p^2) - \prod_{p=1}^{d} 2^{-1}(1 - y_p^2) + 3^{-d}.$$

Since $x \mapsto h(x, y)$ is differentiable if $x_p \neq y_p$ for every p and fulfills the conditions of Lemma 2.2 of Bhattacharya and Ghosh (1978), it follows from this lemma that

$$\limsup_{|t|>c>0} \left| \int_{I^d} \exp\{ith(x, y)\}\, dx_1 \cdots dx_d \right| < 1 - \delta, \qquad \delta > 0.$$

for every $y \in I^d$, and condition (2.8) of Theorem 2.9 in Götze (1979) also follows, resulting in an asymptotic expansion of arbitrary order for $V_{n,d}$.

This result that there exist asymptotic expansions of arbitrary order of $V_{n,d}$ is, at present, of theoretical interest only, since there exist so far no expressions for the limiting distribution V_d of $V_{n,d}$ and for its first few approximations in terms of power series (defining the distribution function) instead of their characteristic functions. Hence it is of interest to note that Remark 2.6 in Götze (1979) holds also for $h(x, y)$ of (2.8) above, and hence (2.5) of Theorem 2.3 in Götze (1979) also implies the following.

COROLLARY 3. *Let $\Delta_{n,d}$ $(d \geq 1)$ be as in Section 1. Then*

$$\Delta_{n,d} = O(n^{-1}), \qquad d \geq 1.$$

3. Calculation of the Critical Value of the *d*-Dimensional Cramér-von Mises Distribution. Our first goal is to calculate the cumulants of the r.v. W_d^2 (cf. (1.12)). Our starting point is Theorem C via its Corollary 1. To obtain the required cumulants, we need the following lemma.

LEMMA 1. *For*

$$|u| < \left(\frac{\pi}{2}\right)^d, \quad \text{with} \quad u = 2it, \quad -\frac{d}{du} \log C_d(t) = \sum_{n=0}^{\infty} L_{n+1}^d u^n,$$

where

$$L_n = \sum_{j=1}^{\infty} \{(j - ½)^2 \pi^2\}^{-n} = \left(\frac{2}{\pi}\right)^{2n} \sum_{j=0}^{\infty} (1 + 2j)^{-2n} = \left(\frac{2}{\pi}\right)^{2n} \lambda(2n),$$

and $\lambda(m)$ is a tabulated function (cf., e.g., formula (23.2.20) *in Abramowitz and Stegun,* 1964).

As mentioned in the Introduction, all the required proofs of this section are given in Section 4.

THEOREM 1. *Using the above nomenclature, the cumulant function* $\log \phi(t)$ *of the* r.v. *W_d^2 is the solution of the differential equation*:

$$2\frac{d}{du} \log \phi(t) = Z_1(t) - Z_2(t)/Z_1(t),$$

where

$$Z_1(t) = \sum_{n=0}^{\infty} L_{n+1}^d u^n, \quad Z_2(t) = \frac{d}{du} Z_1(t), \quad u = 2it, \quad |u| < \left(\frac{\pi}{2}\right)^d.$$

COROLLARY 4. *The characteristic function $\phi(t)$ of the* r.v. *W_d^2 is given by*

$$\phi^{-2}(t) = 2^d \sum_{n=0}^{\infty} L_{n+1}^d u^n \exp\left(-\sum_{n=1}^{\infty} L_n^d \frac{u^n}{n}\right), \quad u = 2it, \quad |u| < \left(\frac{\pi}{2}\right)^d.$$

We note that the above series converge rapidly, so it is easy to calculate the values of $\phi(t)$ in the manner indicated.

COROLLARY 5. *The numerical values of the cumulants K_n of the* r.v. W_d^2 *are calculated, in sequence, from the system of equations*

$$K_n = 2^{n-1}(Z_n - X_{n+1}),$$

where

$$Z_{n+1} = n!L_{n+1}^d, \quad X_{n+2} = \{Z_{n+2} - \textstyle\sum_{j=1}^n \binom{n}{j} X_{n+2-j}Z_{j+1}\}/Z_1.$$

COROLLARY 6. *The numerical values of the moments M_n of the* r.v. W_d^2 *are calculated, in sequence, from the system of equations*

$$M_n = 2^n P_{n+1},$$

where

$$2P_{n+2} = Q_{n+2} + \textstyle\sum_{j=1}^n \binom{n}{j} Q_{n+2-j}P_{j+1}, \quad Q_{n+1} = Z_n - X_{n+1}$$

and Z_n, X_n are defined in Corollary 5.

COROLLARY 7. *The mean μ_d and the variance σ_d^2 of the* r.v. W_d^2 *are*

$$\mu_d = 2^{-d} - 3^{-d}, \qquad \sigma_d^2 = 2.3^{-d}\left\{2^{-d} - 2\left(\frac{5}{2}\right)^{-d} + 3^{-d}\right\}.$$

COROLLARY 8. *From Corollary* 7, *as d increases, μ_d and σ_d^2 tend to zero and the V_d distribution* (*cf.* (1.24)) *concentrates as a unit mass at the origin.*

COROLLARY 9. *From Corollary* 5 *and using the Cornish-Fisher asymptotic expansion,* (*cf., e.g.,* Abramowitz and Stegun, 1964 (26.2.49)) *a table of critical values of V_d is calculated and summarized in Table* 1.

REMARK 3.1. The critical values of Table 1 were calculated by digital computer, using the recursion formula of Corollary 5 to calculate the first 6 cumulants, and then using

TABLE 1.
Critical Values for d-Dimensional Cramér-Von Mises Statistic

DIM d	Mult*	Probability of Exceeding Critical Value									Source
		.25	.10	.05	.025	.01	.005	.0025	.001	.0005	
1			.347	.461		.743			1.168		K–S
2			.25533	.32611		.50166	.58 app		.77 app	.85 app	D
2		.165427	.255847	.330883	.409203	.514715	.594055	.671778	.770489	.841437	C
3		.101916	.149276	.188768	.230000	.285406	.326865	.367230	.417995	.454015	C
3			.1489	.1860		.2779	.3191		.4166	.4592	KMT
4		.056837	.079600	.098645	.118543	.145260	.165216	.184595	.208871	.226002	C
5		.030197	.040609	.049338	.058458	.070690	.079807	.088637	.099652	.107381	C
6		.015621	.020235	.024108	.028151	.033564	.037586	.041467	.046280	.049630	C
7		.007956	.009957	.011637	.013390	.015729	.017460	.019122	.021167	.022576	C
8		.004015	.004869	.005587	.006334	.007328	.008059	.008758	.009608	.010186	C
9		.002015	.002376	.002678	.002993	.003410	.003715	.004004	.004352	.004584	C
10		.001008	.001159	.001285	.001417	.001590	.001716	.001834	.001974	.002066	C
10	10^{-2}	.10078	.11588	.12854	.14167	.15899	.17154	.18340	.19743	.20662	C
15	10^{-4}	.31106	.32907	.34403	.35933	.37896	.39266	.40487	.41804	.42537	C
20	10^{-5}	.09620	.09822	.09988	.10155	.10364	.10504	.10622	.10735	.10782	C
25	10^{-7}	.29910	.30133	.30312	.30490	.30709	.30853	.30969	.31072	.31103	C
30	10^{-9}	.93268	.93511	.93702	.93891	.94122	.94274	.94397	.94505	.94538	C
35	10^{-10}	.29120	.29147	.29167	.29187	.29212	.29229	.29243	.29256	.29261	C
40	10^{-12}	.90965	.90998	.91020	.91041	.91068	.91087	.91103	.91120	.91129	C
45	10^{-13}	.28424	.28427	.28430	.28432	.28435	.28437	.28439	.28441	.28443	C
50	10^{-15}	.88821	.88824	.88827	.88829	.88832	.88835	.88837	.88840	.88842	C

Sources: K-S Kendall and Stuart (1967) from Anderson and Darling (1952); D Durbin (1970); KMT Krivyakova, Martynov, Tyurin (1977); C Present work.
** Each line in the table is to be multiplied by the factor given in this column.*

these in the Cornish-Fisher asymptotic expansion. The coefficients of the latter are tabulated in Abramowitz and Stegun (1964, (26.2.51)). The computer program is fairly brief, and is available for us upon request.

REMARK 3.2. For the two-dimensional case, $d = 2$, the critical values for the V_d distribution are compared with those calculated by Durbin (1970) as follows.

CRITICAL VALUES				
P	**Durbin**	**Present work**	**Error**	**%**
.10	0.25533	0.25585	.00052	0.20365
.05	0.32611	0.33088	.00477	1.46269
.01	0.50166	0.51471	.01305	2.60136
.005	0.58 app.	0.5940	.0140	2.4137
.001	0.77 app.	0.7705	.0005	0.065
.0005	0.85 app.	0.8414	.0086	−1.012

The error is attributed to the use of the Cornish-Fisher expansion, which calculates the critical values as a function of the standardized cumulants K_n/σ^n. We used the first 6 cumulants. For higher orders, K_n/σ^n becomes very large, as follows.

n =	3	4	5	6	7	8	9	10
K_n/σ^n =	2.390	9.271	48.78	322.23	2557	23682	250682	2985300

For higher dimensions, $d > 2$, the errors are reduced because the cumulants K_n decrease as d increases, $K_n = O(e^{-d})$.

COROLLARY 10. *Each of the critical values given in Table 1 can be written as*

$$C(p, d) = \mu_d + \sigma_d \cdot W(p, d)$$

where μ_d and σ_d may be calculated from Corollary 4.

Thus the values $W(p, d)$ are shift- and scale-free critical values, and they are listed in Table 2.

TABLE 2.
"Scale-Free" Critical values W(P, d) for d-Dimensional Cramér-Von Mises Statistic

DIM d	Mean μ_d	S.D. σ_d	Probability of Exceeding Critical Value									Source[1]
			.25	.10	.05	.025	.01	.005	.0025	.001	.0005	
1				1.2095	1.9742		3.8659			6.7169		K–S
2	.138889	.095581		1.2182	1.9588		3.7954	4.614		6.602	7.439	D
2	.138889	.095581	.27765	1.22365	2.00870	2.82810	3.93200	4.76208	5.57524	6.60798	7.35025	C
3	.087963	.050212	.27788	1.22108	2.00758	2.82874	3.93218	4.75786	5.56174	6.57274	7.29010	C
4	.050154	.024163	.27655	1.21863	2.00684	2.83032	3.93605	4.76191	5.56396	6.56861	7.27759	C
5	.027135	.011069	.27669	1.21738	2.00600	2.82994	3.93504	4.75870	5.55651	6.55166	7.24999	C
6	.0142533	.004914	.27835	1.21715	2.00503	2.82772	3.92902	4.74733	5.53702	6.51627	7.19802	C
7	.0073552	.0021368	.28127	1.21772	2.00393	2.82398	3.91873	4.72882	5.50669	6.46372	7.12288	C
8	.0037538	.00091524	.28522	1.21888	2.00269	2.81893	3.90484	4.70427	5.46717	6.39650	7.02779	C
9	.0019023	.00038775	.29001	1.22052	2.00126	2.81272	3.88788	4.67463	5.41994	6.31709	6.91621	C
10	.00095963	.00016293	.29552	1.22251	1.99961	2.80543	3.86826	4.64069	5.36636	6.22782	6.79146	C
10	.95963 (−3)*	.16293 (−3)	.29552	1.22251	1.99962	2.80543	3.86826	4.64070	5.36636	6.22782	6.79146	C
15	.30448 (−4)	.19910 (−5)	.33044	1.23517	1.98669	2.75491	3.74111	4.42907	5.04215	5.70386	6.07200	C
20	.95339 (−6)	.23121 (−7)	.37165	1.24761	1.96418	2.68586	3.58893	4.19423	4.70429	5.19339	5.39920	C
25	.29801 (−7)	.26423 (−9)	.41326	1.25641	1.93308	2.60667	3.43616	3.97889	4.42022	4.80808	4.92824	C
30	.93132 (−9)	.30041 (−11)	.45178	1.26103	1.89700	2.52558	3.29687	3.80080	4.21036	4.57001	4.68075	C
35	.29104 (−10)	.34095 (−13)	.48562	1.26233	1.85977	2.44858	3.17655	3.66123	4.06750	4.45290	4.60928	C
40	.90949 (−12)	.38675 (−15)	.51437	1.26156	1.82434	2.37915	3.07560	3.55407	3.97452	4.41679	4.64938	C
45	.28422 (−13)	.43863 (−17)	.53828	1.25981	1.79253	2.31879	2.99201	3.47137	3.91394	4.42512	4.74441	C
50	.88818 (−15)	.49743 (−19)	.55789	1.25789	1.76521	2.26767	2.92296	3.40586	3.87164	4.45030	4.85281	C

Critical value = $\mu_d + \sigma_d \cdot W(P, d)$
[1] *Same sources as Table 1.*
* *By .95963 (−3) we mean .95963 × 10^{-3}.*

4. Proofs of Statements in Section 3.

PROOF OF LEMMA 1. By Corollary 2 to Theorem C, we have

$$C_d(t) = \prod_{n=1}^{\infty} C_{d-1}\left\{\frac{t}{(n-\tfrac{1}{2})^2\pi^2}\right\}, \quad C_1(t) = \prod_{n=1}^{\infty}\left\{1 - \frac{2it}{(n-\tfrac{1}{2})^2\pi^2}\right\}.$$

Let $u = 2it$. Then

$$\log C_1(t) = \sum_{n=1}^{\infty} \log\left\{1 - \frac{u}{(n-\tfrac{1}{2})^2\pi^2}\right\}.$$

Thus

$$\log C_d(t) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \log\left\{1 - \frac{u}{(n_1-\tfrac{1}{2})^2 \cdots (n_d-\tfrac{1}{2})^2\pi^{2d}}\right\}$$

$$= \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \log(1 - uA_{n_1\cdots n_d}),$$

where

$$A_{n_1\cdots n_d} = \{(n_1-\tfrac{1}{2})^2 \cdots (n_d-\tfrac{1}{2})^2\pi^{2d}\}^{-1},$$

and so

$$\frac{d}{du}\log C_d(t) = -\sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} A_{n_1\cdots n_d}(1 - uA_{n_1\cdots n_d})^{-1}.$$

This can be expanded in powers of u provided that $|uA_{n_1\cdots n_d}| < 1$, that is, if $|u| < \left(\frac{\pi}{2}\right)^{2d}$.

Then

$$-\frac{d}{du}\log C_d(t) = \sum_{n=0}^{\infty} S_{n+1}u^n,$$

where

$$S_n = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} A^n_{n_1\cdots n_d} = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \{(n_1-\tfrac{1}{2})^2 \cdots (n_d-\tfrac{1}{2})^2\pi^{2d}\}^{-n}$$

$$= [\sum_{j=1}^{\infty} \{(j-\tfrac{1}{2})^2\pi^2\}^{-n}]^d.$$

Thus

$$-\frac{d}{du}\log C_d(t) = \sum_{n=0}^{\infty} L^d_{n+1}u^n, \quad L_n = \sum_{j=1}^{\infty} [(j-\tfrac{1}{2})^2\pi^2]^{-n}.$$

Hence the required result. We may write

$$L_n = \left(\frac{2}{\pi}\right)^{2n} \sum_{j=0}^{\infty} (1+2j)^{-2n}$$

with

$$\sum_{j=0}^{\infty} (1+2j)^{-2n} = \lambda(2n),$$

a function tabulated in Abramowitz and Stegun (1964, (23.2.20)) and related to the Riemann Zeta function. (This relationship might be useful in any attempt to find an expression for the V_d distribution in terms of standard functions. The series converges very rapidly so that we simply sum it directly.)

PROOF OF THEOREM 1. From Corollary 1 to Theorem C

$$\phi(t)^{-2} = -2^d \frac{d}{du} C_d(t), \qquad u = 2it.$$

From Lemma 1

$$-\frac{d}{du}\log C_d(t) = Z_1(t) = \sum_{n=0}^{\infty} L^d_{n+1}u^n.$$

Thus
$$-\frac{d}{du}C_d(t) = Z_1(t)C_d(t).$$

Combining, we have
$$\phi(t)^{-2} = 2^d Z_1(t)C_d(t),$$

or

$$2^{-d} = \phi^2(t)Z_1(t)C_d(t). \tag{4.1}$$

Differentiate with respect to $u = 2it$ to obtain
$$0 = \frac{d}{du}\{\phi^2(t)\}Z_1(t)C_d(t) + \phi^2(t)\left\{\frac{d}{du}Z_1(t)\right\}C_d(t) + \phi^2(t)Z_1(t)\left\{\frac{d}{du}C_d(t)\right\}.$$

Substitute
$$\frac{d}{du}C_d(t) = -Z_1(t)C_d(t),$$

and write
$$Z_2(t) = \frac{d}{du}Z_1(t).$$

Then

$$0 = C_d(t)\left[\frac{d}{du}\{\phi^2(t)\}Z_1(t) + \phi^2(t)\{Z_2(t) - Z_1^2(t)\}\right]. \tag{4.2}$$

From the definition of $C_d(t)$, we have $C_d(0) = 1$ and so $C_d(t) \neq 0$ for small t. Since
$$Z_1(t) = \sum_{n=0}^{\infty} L_{n+1}^d u^n,$$

so
$$Z_1(0) = L_1^d = \left\{\left(\frac{2}{\pi}\right)^2 \lambda(2)\right\}^d.$$

But $\lambda(2) = \pi^2/8$, hence $Z_1(0) = 2^{-d}$ and so $Z_1(t) \neq 0$ for small t. Hence by (4.1), $\phi^2(0) = 1$ as it should be since $\phi(t)$ is a characteristic function. Hence $\phi^2(t) \neq 0$ for small t. Consequently we can cancel $C_d(t)$, $Z_1(t)$ and $\phi^2(t)$ in (4.2) to obtain
$$0 = \phi^{-2}(t)\frac{d}{du}\phi^2(t) + \{Z_2(t)/Z_1(t)\} - Z_1(t),$$

and so
$$2\frac{d}{du}\log\phi(t) = Z_1(t) - Z_2(t)/Z_1(t),$$

the required result.

REMARK 4.1. Having expressed the cumulant function in terms of power series in t with known coefficients, the remaining results follow directly.

PROOF OF COROLLARY 4. By Theorem 1, and after integrating $Z_1(t)$ term by term to give
$$Z_0(t) = \sum_{n=1}^{\infty} L_n^d \frac{u^n}{n},$$

we have
$$\frac{d}{du}\log\phi^{-2}(t) = -\frac{d}{du}Z_0(t) + \frac{d}{du}\log Z_1(t).$$

Integrating the latter gives
$$\log\phi^{-2}(t) = -Z_0(t) + \log Z_1(t) + K,$$

thus

$$\phi^{-2}(t) = CZ_1(t)\exp\{-Z_0(t)\},$$

where C, K are constants of integration. From above $\phi^2(0) = 1$, $Z_0(0) = 0$, $Z_1(0) = 2^{-d}$. Thus $C = 2^d$, and

$$\phi^{-2}(t) = 2^d \textstyle\sum_{n=0}^{\infty} L_{n+1}^d u^n \exp\left(-\sum_{n=1}^{\infty} L_n^d \dfrac{u^n}{n}\right),$$

the required result.

PROOF OF COROLLARY 5. The cumulants K_n of the r.v. W_d^2 are defined by

$$\log\phi(t) = \textstyle\sum_{n=1}^{\infty} K_n \dfrac{(it)^n}{n!} = \sum_{n=1}^{\infty} K_n 2^{-n}\dfrac{u^n}{n!}, \qquad u = 2it.$$

Thus

$$K_n = 2^n \frac{d^n}{du^n}\log\phi(t)\Big|_{u=0}.$$

From Theorem 1 we have

$$2\frac{d}{du}\log\phi(t) = Z_1(t) - \frac{Z_2(t)}{Z_1(t)}.$$

Thus

$$K_1 = Z_1(0) - \frac{Z_2(0)}{Z_1(0)}.$$

Write

$$X_2(t) = \frac{Z_2(t)}{Z_1(t)}, \quad X_{n+2}(t) = \frac{d^n}{du^n}X_2(t), \quad Z_{n+1}(t) = \frac{d^n}{du^n}Z_1(t).$$

Then for $n \geq 1$,

$$2\frac{d^n}{du^n}\log\phi(t) = Z_n(t) - X_{n+1}(t).$$

Thus

$$K_n = 2^{n-1}\{Z_n(0) - X_{n+1}(0)\},$$

as required. Since, by definition,

$$Z_1(t) = \textstyle\sum_{n=0}^{\infty} L_{n+1}^d u^n,$$

then

$$Z_{n+1}(0) = n!\, L_{n+1}^d,$$

as required.

By definition, $X_2(t) = Z_2(t)/Z_1(t)$, so that $X_2(t)Z_1(t) = Z_2(t)$. Then, by repeated differentiation with respect to u, we get for $n \geq 1$

$$X_{n+2}(t)Z_1(t) + \textstyle\sum_{j=1}^{n}\dbinom{n}{j} X_{n+2-j}(t)Z_{j+1}(t) = Z_{n+2}(t).$$

We now evaluate this expression for $t = 0$ $(u = 0)$ and, for compactness, write

$$X_n = X_n(0), \qquad Z_n = Z_n(0).$$

Thus

$$X_{n+2} = \left\{Z_{n+2} - \textstyle\sum_{j=1}^{n}\dbinom{n}{j} X_{n+2-j}Z_{j+1}\right\}\Big/ Z_1,$$

as required with

$$Z_{n+1} = Z_{n+1}(0) = n!\, L_{n+1}^d.$$

PROOF OF COROLLARY 6. In a formal way, the moments M_n of the r.v. W_d^2 may be expressed as

$$\phi(t) = 1 + \textstyle\sum_{n=1}^{\infty} M_n \frac{(it)^n}{n!} = 1 + \sum_{n=1}^{\infty} M_n 2^{-n} \frac{u^n}{n!}, \qquad u = 2it.$$

Then
$$M_n = 2^n \frac{d^n}{du^n} \phi(t) \bigg|_{t=0}$$

provided that this limit exists. We have, by Theorem 1,

$$2 \frac{d}{du} \log \phi(t) = Z_1(t) - X_2(t),$$

where

$$X_2(t) = \frac{Z_2(t)}{Z_1(t)} = \frac{\left\{\frac{d}{du} Z_1(t)\right\}}{Z_1(t)}.$$

Write

$$P_1(t) = \phi(t),\ P_{n+1}(t) = \frac{d^n}{du^n} P_1(t),\ Q_1(t) = 2 \log P_1(t) = 2 \log \phi(t),\ Q_{n+1}(t) = \frac{d^n}{du^n} Q_1(t).$$

Then
$$2 \frac{d}{du} \log \phi(t) = Q_2(t) = Z_1(t) - X_2(t),$$

and by repeated differentiation with respect to u,

$$Q_{n+1}(t) = Z_n(t) - X_{n+1}(t), \qquad n \geq 1.$$

Also,

$$Q_1(t) = 2 \log P_1(t), \qquad Q_2(t) = 2 \frac{d}{du} \log P_1(t) = 2 P_2(t)/P_1(t),$$

so that
$$Q_2(t) P_1(t) = 2 P_2(t).$$

Then, by repeated differentiation,

$$Q_{n+2}(t) P_1(t) + \textstyle\sum_{j=1}^{n} \binom{n}{j} Q_{n+2-j}(t) P_{j+1}(t) = 2 P_{n+2}(t)$$

which generates successive values of $P_n(t)$, and

$$M_n = 2^n \frac{d^n}{du^n} \phi(t) \bigg|_{t=0} = 2^n P_{n+1}(0),$$

giving the required result when we write $P_n(0)$ as P_n, $Q_n(0)$ as Q_n. Notice that $P_1 = \phi(0) = 1$.

PROOF OF COROLLARY 7. The values of the first two cumulants are $K_1 = \mu_d$, $K_2 = \sigma_d^2$, the mean and variance of the V_d distribution. From Corollary 5

$$\mu_d = K_1 = Z_1 - X_2, \quad \sigma_d^2 = K_2 = 2(Z_2 - X_3)$$

and

$$Z_1 = L_1^d, \qquad Z_2 = L_2^d, \qquad Z_3 = 2L_3^d.$$

By Lemma 5

$$L_1 = \left(\frac{2}{\pi}\right)^2 \lambda(2), \quad L_2 = \left(\frac{2}{\pi}\right)^4 \lambda(4), \quad L_3 = \left(\frac{2}{\pi}\right)^6 \lambda(6).$$

By Abramowitz and Stegun (1964, (23.2.11) through (23.2.31)), $\lambda(2) = \pi^2/8$, $\lambda(4) = \pi^4/96$ and

$$\lambda(6) = (1 - 2^{-6})\zeta(6) = (1 - 2^{-6})\frac{(2\pi)^6}{2.6!}|B_6| = (1 - 2^{-6})\frac{(2\pi)^6}{2.6!}\cdot\frac{1}{42} = \frac{\pi^6}{15.2^6}.$$

Thus

$$L_1 = 1/2, \quad L_2 = 1/6, \quad L_3 = 1/15, \quad Z_1 = 2^{-d}, \quad Z_2 = 6^{-d}, \quad Z_3 = 2.15^{-d},$$

and

$$X_2 = Z_2/Z_1 = 6^{-d}/2^{-d} = 3^{-d},$$

$$X_3 = (Z_3 - X_2Z_2)/Z_1 = (2.15^{-d} - 3^{-d}6^{-d})/2^{-d} = 2\cdot\left(\frac{15}{2}\right)^{-d} - 3^{-2d}.$$

Thus $\mu_d = K_1$ and $\sigma_d^2 = K_2$ are as required.

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