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SLLNs AND CLTs FOR INFINITE PARTICLE SYSTEMS

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We consider initial point processes A_0 on Z^d where $A_0(x)$, $x \in Z^d$ are independent and satisfy certain technical conditions. The particles initially present are assumed to be translated independently according to recurrent random walks. Various limit theorems are then proved involving $S_n(B)$—the total occupation time of B by time n, and $L_n(B)$—the number of distinct particles in B by time n.

1. Introduction. Suppose that at time zero $A_0(x)$ particles are placed at $x \in Z^d$ (Z^d denotes the space of d-dimensional integers). The particles are then assumed to be translated independently by random walks all with the same transition function as a fixed random walk $\{\xi_n\}$.

Let B denote a finite nonempty subset of Z^d, $A_n(B)$ the number of particles in B at time n and $I_n(B)$ the number of particles which have entered B for the first time at time n. Then $S_n(B) = \sum_{k=1}^n A_k(B)$ is the total occupation time of B by time n of all the particles and $L_n(B) = \sum_{k=1}^n I_k(B)$ is the number of distinct particles in B by time n. It is these two quantities which will be studied in this paper.

Basically, the problems involved are determining a strong law of large numbers and central limit theorem for each of the two functionals. These problems were solved by Port (1966, 1967) in case the $A_0(x)$ are independent and identically distributed Poisson variables. Weiss (1971) considered these problems in the case where the Poisson assumption is dropped and the random walks are *transient*. The results here were stated for $d = 1$ but the same arguments give the theorems for arbitrary d.

A crucial result in obtaining the desired limit theorems in the above papers concerns the determination of the asymptotic variance of the functionals $S_n(B)$ and $L_n(B)$. An interesting and somewhat surprising result obtained in Weiss (1971) is that although results in Stone (1968) indicate that the asymptotic variance in the non-Poisson case should be the same as that in the Poisson case, this is not the situation when $d = 1$ and $E|\xi_1| < \infty$. However, the asymptotic variance is as expected in all other (transient) cases.

The purpose of this paper is to analyze the functionals $S_n(B)$ and $L_n(B)$ without the Poisson assumption in the *recurrent* case ($d = 1$ or 2). As in the transient

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case, we show that when $d = 2$ the asymptotic variance remains unchanged from the Poisson case. However, when $d = 1$ it is indicated that in general there will be no asymptotic variance and in several cases where there is (i.e., the domain of attraction of a stable law), this asymptotic variance differs from that when the particles are put down according to a Poisson process.

Specifically, we assume that $A_0(x)$, $x \in Z^d$ are independent random variables with finite sixth moments and that there are constants $\lambda > 0$, ν and M such that

$$\text{(1.1)} \qquad \begin{array}{lll} \text{(i)} & \mu_1(x) \to \lambda & \text{as } |x| \to \infty \\ \text{(ii)} & \mu_2(x) \to \nu & \text{as } |x| \to \infty \\ \text{(iii)} & \mu_j(x) \leqq M\,, & 1 \leqq j \leqq 6\,, \quad x \in Z^d\,, \end{array}$$

where $\mu_j(x) = E[A_0(x)(A_0(x) - 1) \cdots (A_0(x) - j + 1)]$. Note that (1.1) is satisfied whenever $A_0(x)$, $x \in Z^d$ are independent and identically distributed random variables with finite sixth moments.

2. Preliminaries and notation. Let $P_n(x, y)$ denote the n-step transition function of the random walk $\{\xi_k\}$ governing the motion of the particles. We define

$$G_n(x, y) = \textstyle\sum_{k=1}^{n} P_k(x, y)$$

and set $g_n = G_n(0, 0)$. Also, we denote the hitting time of B by V_B. That is,

$$V_B = \min\{n \geqq 1 : \xi_n \in B\}\,.$$

Moreover, we set $q_n = P_0(V_{\{0\}} > n)$. To continue, we define

$$N_n(B) = \textstyle\sum_{k=1}^{n} 1_B(\xi_k)$$

where 1_B denotes the indicator function of B. Finally, Z^d denotes the space of d-dimensional integers, R^d the space of d-dimensional reals and $|B|$ the cardinality of B.

3. Statement of results. Suppose that at time zero we distribute particles in Z^d according to a point process A_0, satisfying the conditions stated in Section 1. The particles are then translated independently according to *recurrent* random walks with transition function $P(x, y)$. Throughout we assume that B is a finite nonempty subset of Z^d.

Our first results handle the case when $d = 2$. In this situation we have the following theorems.

THEOREM 1. *Let $d = 2$ and $S_n(B)$ denote the total occupation time of B by time n. Then*

$$ES_n(B) \sim \lambda|B|n\,, \tag{3.1}$$

$$\operatorname{Var} S_n(B) \sim 2\lambda|B|^2 n g_n\,, \tag{3.2}$$

$$P(\lim_{n\to\infty} S_n(B)/n = \lambda|B|) = 1\,, \tag{3.3}$$

and for each $t \in R$,

$$\lim_{n\to\infty} P\left(\frac{S_n(B) - ES_n(B)}{[\operatorname{Var} S_n(B)]^{\frac{1}{2}}} \leqq t\right) = \Phi(t)\,, \tag{3.4}$$

where Φ is the standard normal distribution function.

THEOREM 2. *Let $d = 2$ and $L_n(B)$ denote the number of distinct particles in B by time n. Then*

$$EL_n(B) \sim \lambda n q_n \,, \tag{3.5}$$

$$\operatorname{Var} L_n(B) \sim \lambda n q_n \,, \tag{3.6}$$

$$P(\lim_{n\to\infty} L_n(B)/EL_n(B) = 1) = 1 \,, \tag{3.7}$$

and for each $t \in R$,

$$\lim_{n\to\infty} P\left(\frac{L_n(B) - EL_n(B)}{[\operatorname{Var} L_n(B)]^{\frac{1}{2}}} \leqq t\right) = \Phi(t) \,. \tag{3.8}$$

We next consider the case $d = 1$. It can be shown that in general $\operatorname{Var} S_n(B)$ and $\operatorname{Var} L_n(B)$ will not have any asymptotic behavior as in the two-dimensional case. Therefore, we consider random walks in the domain of attraction of a stable law. To be specific, we assume that the random walk is aperiodic and is in the domain of attraction of a stable law, V_α, with density f_α ($1 < \alpha \leqq 2$). Of course, if the random walk has finite nonzero variance then this will be the case with $\alpha = 2$ and $V = \Phi$.

THEOREM 3. *Let $d = 1$ and the assumptions be as above. Then*

$$ES_n(B) \sim \lambda |B| n \tag{3.9}$$

and there is a slowly varying function $L(\cdot)$ such that

$$\lim_{n\to\infty} \frac{L(n)}{n^{2-1/\alpha}} \operatorname{Var} S_n(B) = |B|^2(2\lambda r_\alpha + (\nu - \lambda^2)s_\alpha) \,, \tag{3.10}$$

where

$$r_\alpha = f_\alpha(0)[(2 - \alpha^{-1})(1 - \alpha^{-1})]^{-1} \,, \tag{3.11}$$

and

$$s_\alpha = \int_0^1 \int_0^1 \int_{-\infty}^{\infty} f_\alpha\left(\frac{t}{x^{1/\alpha}}\right) f_\alpha\left(\frac{t}{y^{1/\alpha}}\right) x^{-1/\alpha} y^{-1/\alpha} \, dt \, dx \, dy \,. \tag{3.12}$$

Moreover,

$$P(\lim_{n\to\infty} S_n(B)/n = \lambda|B|) = 1 \,, \tag{3.13}$$

and for each $t \in R$,

$$\lim_{n\to\infty} P\left(\frac{S_n(B) - ES_n(B)}{[\operatorname{Var} S_n(B)]^{\frac{1}{2}}} \leqq t\right) = \Phi(t) \,. \tag{3.14}$$

THEOREM 4. *Let $d = 1$ and the assumptions be as above. Then there is a slowly varying function $L(\cdot)$ such that*

$$EL_n(B) \sim \lambda t_\alpha n^{1/\alpha} L(n) \,, \tag{3.15}$$

where

$$t_\alpha = [f_\alpha(0)\Gamma(1 - \alpha^{-1})\Gamma(1 + \alpha^{-1})]^{-1} \,. \tag{3.16}$$

Also,

$$\operatorname{Var} L_n(B) \sim (\lambda t_\alpha + (\nu - \lambda^2)t_\alpha{}^2\alpha^{-2}u_\alpha)n^{1/\alpha}L(n)\,, \tag{3.17}$$

where

$$u_\alpha = \int_0^1 \int_0^1 \int_{-\infty}^{\infty} f_\alpha\left(\frac{t}{x^{1/\alpha}}\right) f_\alpha\left(\frac{t}{y^{1/\alpha}}\right) \frac{(1-x)^{\alpha^{-1}-1}(1-y)^{\alpha^{-1}-1}}{x^{1/\alpha}y^{1/\alpha}}\, dt\, dx\, dy\,. \tag{3.18}$$

Moreover,

$$P(\lim_{n\to\infty} L_n(B)/EL_n(B) = 1) = 1\,, \tag{3.19}$$

and for each $t \in R$,

$$\lim_{n\to\infty} P\left(\frac{L_n(B) - EL_n(B)}{[\operatorname{Var} L_n(B)]^{\frac{1}{2}}} \leqq t\right) = \Phi(t)\,. \tag{3.20}$$

4. Proofs. We first obtain an expression for the characteristic function of $S_n(B)$.

LEMMA 1. *With the notation as above we have*

$$E(\exp(i\theta S_n(B))) = \prod_x E(E_x(\exp(i\theta N_n(B)))^{A_0(x)}\,. \tag{4.1}$$

PROOF. Let $\{\xi_{nx}^{(l)}\}$ be independent random walks with the same transition function as $\{\xi_n\}$ and with $\xi_{0x}^{(l)} = x$. Then,

$$S_n(B) = \sum_x \sum_{l=1}^{A_0(x)} \sum_{k=1}^{n} 1_B(\xi_{kx}^{(l)})\,,$$

and hence

$$E(\exp(i\theta S_n(B)) \mid A_0) = \prod_x E_x(\exp(i\theta N_n(B)))^{A_0(x)}\,,$$

and thus (4.1) holds.

Using (4.1) and standard characteristic function arguments we obtain the following facts:

$$ES_n(B) = \sum_x \mu_1(x)E_xN_n(B)\,, \tag{4.2}$$

$$\operatorname{Var} S_n(B) = \sum_x \mu_1(x)E_xN_n(B)^2 + \sum_x (\mu_2(x) - \mu_1(x)^2)[E_xN_n(B)]^2\,. \tag{4.3}$$

REMARK. In the Poisson case $\mu_2(x) = \mu_1(x)^2$ and so the second term on the right hand side of (4.3) does not arise.

We now commence with the proof of Theorem 1. To obtain (3.1) we use (4.2). Note that $\sum \mu_1(x)E_xN_n(B) = \sum_{y\in B} \sum_{k=1}^{n} \sum \mu_1(x)P_k(x, y)$ and so (1.1) and a standard summability argument establishes formula (3.1).

Next we prove (3.2). First of all some computations show that

$$\sum_x E_xN_n(B)^2 = n|B| + 2\sum_{y\in B} \sum_{i=1}^{n-1} G_i(y, B)\,. \tag{4.4}$$

By the weak ratio ergodic theorem (see Spitzer (1964), page 10), the result that q_n is slowly varying (see Kesten and Spitzer (1963)) and the fact that $g_n \sim 1/q_n$ we deduce

$$\sum_x E_xN_n(B)^2 \sim 2|B|^2ng_n\,. \tag{4.5}$$

To continue we prove that

$$\sum_x [E_n N_n(B)]^2 = o(ng_n) . \tag{4.6}$$

Since B is finite we can assume without loss of generality that $B = \{0\}$. Let $\phi(\theta)$ denote the characteristic function of the random walk and C the square in R^2 with center at the origin and sides of length 2π. Then by standard harmonic analysis techniques we obtain the relations

$$\begin{aligned} \sum_x [G_n(0, x)]^2 &= (2\pi)^{-2} \int_C |\textstyle\sum_{k=1}^n \phi^k(\theta)|^2 \, d\theta \\ &\leqq (2\pi)^{-2} \textstyle\sum_{i,j=1}^n \int_C |\phi(\theta)|^{i+j} \, d\theta . \end{aligned} \tag{4.7}$$

Now, (see Spitzer (1964), page 73), there is a constant $K > 0$ such that

$$(2\pi)^{-2} \int_C |\phi(\theta)|^i \, d\theta \leqq K/i \qquad (i \geqq 1) ,$$

and this fact along with (4.7) implies that the term on the left hand side of (4.7) is $O(n)$. Using $g_n \uparrow \infty$ now yields (4.6). Taking (1.1), (4.5) and (4.6) into account we get (3.2).

In order to prove (3.3) and (3.4) we will need the following estimates which can be obtained from the facts that $G_n(x, B) = O(g_n)$, uniformly in x, and (4.5).

LEMMA 2. *Let the notation be as before. Then*

$$\sum E_x|N_n(B) - E_x N_n(B)|^3 = O(ng_n^{\,2}) , \tag{4.8}$$

$$\sum E_x|N_n(B) - E_x N_n(B)|^4 = O(ng_n^{\,3}) , \tag{4.9}$$

$$\sup_x E_x|N_n(B) - E_x N_n(B)|^3 = O(g_n^{\,3}) , \tag{4.10}$$

$$\sup_x \operatorname{Var}_x N_n(B) = O(g_n^{\,2}) . \tag{4.11}$$

Now, let $S_{nx}(B)$ denote the total occupation time of B by time n of the particles starting at x. Using the independence of the $S_{nx}(B)$, $x \in Z^2$, we find that

$$\begin{aligned} E[S_n(B) - ES_n(B)]^4 &= \sum E[S_{nx}(B) - ES_{nx}(B)]^4 \\ &\quad + 3 \textstyle\sum_{x \neq y} \operatorname{Var} S_{nx}(B) \operatorname{Var} S_{ny}(B) . \end{aligned} \tag{4.12}$$

Using (4.9), (1.1), (3.2) and $g_n = o(n)$, we deduce from (4.12) that

$$E[S_n(B) - ES_n(B)]^4 = O(n^2 g_n^{\,2}) . \tag{4.13}$$

By Chebyschev's inequality, for any $t > 0$,

$$P(|S_n(B) - ES_n(B)| > nt) \leqq E[S_n(B) - ES_n(B)]^4/n^4 t^4 \tag{4.14}$$

and the term on the right is that of a convergent series because of (4.13) and the fact that g_n varies slowly. Applying the Borel–Cantelli lemma along with (3.1) we get (3.3).

To prove (3.4) let ψ_n and ψ_{nx} denote the characteristic functions of $[S_n(B) - ES_n(B)]/[\operatorname{Var} S_n(B)]^{\frac{1}{2}}$ and $[S_{nx}(B) - ES_{nx}(B)]/[\operatorname{Var} S_n(B)]^{\frac{1}{2}}$, respectively. Then $\psi_n = \prod \psi_{nx}$. Now, we can write

$$\psi_{nx}(\theta) = 1 - \frac{\theta^2}{2} \frac{\operatorname{Var} S_{nx}(B)}{\operatorname{Var} S_n(B)} + R_{nx}(\theta) , \tag{4.15}$$

where

$$(4.16)\qquad |R_{nx}(\theta)| \leqq \frac{|\theta|^3}{3!} E|S_{nx}(B) - ES_{nx}(B)|^3/[\operatorname{Var} S_n(B)]^{\frac{3}{2}}$$

For convenience, set $\Delta_{nx}(\theta) = \psi_{nx}(\theta) - 1$. Then (1.1), (4.11) and (3.2) imply that $\sup_x |\Delta_{nx}(\theta)| \to 0$. Hence, for large n we can write $\log(1 + \Delta_{nx}(\theta)) = \Delta_{nx}(\theta) + \Lambda_{nx}(\theta)|\Delta_{nx}(\theta)|^2$, where $\sup_x |\Lambda_{nx}(\theta)| \leqq 1$. Using (1.1) and (4.8) we obtain that $\sum \Delta_{nx}(\theta) \to -\theta^2/2$ and $\sum |\Delta_{nx}(\theta)| = O(1)$. Hence, $\sum \log(1 + \Delta_{nx}(\theta)) \to -\theta^2/2$ and so (3.4) follows from the continuity theorem. The proof of Theorem 1 is now complete.

We turn to the proof of Theorem 2. First of all results of Port (1965) imply that

$$(4.17)\qquad \textstyle\sum_x P_x(V_B \leqq n) \sim \sum_{k=0}^{n-1} q_k .$$

Computations similar to the ones above show that $EL_n(B) = \sum \mu_1(x)P_x(V_B \leqq n)$ and from this, (4.17), and the fact that q_n is slowly varying we deduce (3.5).

To obtain (3.6) we first do some calculations to get

$$(4.18)\qquad \operatorname{Var} L_n(B) = \textstyle\sum \mu_1(x)P_x(V_B \leqq n) + \sum (\mu_2(x) - \mu_1(x)^2)[P_x(V_B \leqq n)]^2 .$$

The first term on the right is just $EL_n(B)$ and so we restrict our attention to the second term on the right. Specifically, we will prove that

$$(4.19)\qquad \textstyle\sum_x [P_x(V_B \leqq n)]^2 = o(nq_n) .$$

Now, a last entrance decomposition shows

$$(4.20)\qquad \begin{aligned} \textstyle\sum_x [P_x(V_B \leqq n)]^2 = {} & \textstyle\sum_{u,v \in B} \sum_{i,j=1}^{n} \sum_x P_i(0, x)P_j(u - v, x) \\ & \times P_u(V_B > n - i)P_v(V_B > n - j) . \end{aligned}$$

Kesten and Spitzer (1963) proved that for each finite nonempty set B there is a function $L_B(\cdot)$ such that for $y \in Z^2$, $P_y(V_B > n) \sim q_n L_B(y)$ and moreover $\sum_{y \in B} L_B(y) = 1$. Thus, in (4.19) we can assume $B = \{0\}$. Using harmonic analysis techniques as before we find that

$$(4.21)\qquad \textstyle\sum_{i,j=1}^{n} \sum_x P_i(0, x)P_j(0, x)q_{n-i}q_{n-j} = O\left(\sum_{i,j=1}^{n-1} \frac{q_{n-1}q_{n-j}}{i + j}\right) + o(1) .$$

Since q_n is slowly varying and decreasing, we have for any $\varepsilon > 0$, $q_{n-i}/q_n = O((n/(n - i))^\varepsilon)$. We then obtain for $\varepsilon < 1$,

$$\begin{aligned} \frac{1}{nq_n} \textstyle\sum_{i,j=1}^{n-1} \frac{q_{n-i}q_{n-j}}{i + j} &= O\left(q_n n^{-2} \sum_{i,j=1}^{n-1} \left(1 - \frac{i}{n}\right)^{-\varepsilon} \left(1 - \frac{j}{n}\right)^{-\varepsilon} \left(\frac{i}{n} + \frac{j}{n}\right)^{-1}\right) \\ &= O(q_n) \to 0 , \end{aligned}$$

and hence (4.19) holds. Using (4.18) and (4.19) we obtain (3.6).

The proofs of (3.7) and (3.8) are similar to those of (3.3) and (3.4) and therefore will be omitted.

We now turn to the proof of Theorem 3. The proof of (3.9) is the same as that of (3.1) and so we proceed to (3.10).

By the local central limit theorem for stable laws (see Gnedenko and Kolmogorov (1968)) we have

$$\lim_{n\to\infty} |B_n P_n(0, x) - f_\alpha(x/B_n)| = 0 \tag{4.22}$$

uniformly in x and this implies $P_n(0, x) \sim f_\alpha(0)B_n^{-1}$. Now, we can write $B_n = n^{1/\alpha}L(n)$ where L is slowly varying. It follows that

$$\textstyle\sum_{k=1}^n G_k(y, z) \sim n^{2-1/\alpha}[(1-\alpha^{-1})(2-\alpha^{-1})L(n)]^{-1}f_\alpha(0)$$

and from this and (4.4) we deduce that

$$L(n) \textstyle\sum_x \mu_1(x)E_x N_n(B)^2 \sim 2\lambda|B|^2 r_\alpha n^{2-1/\alpha} \tag{4.23}$$

where r_α is given by (3.1).

To complete the proof of (3.10) it is necessary to establish the following result.

$$L(n) \textstyle\sum_{i,j=1}^n \sum_x P_i(0, x)P_j(y, x) \sim n^{2-1/\alpha}s_\alpha \tag{4.24}$$

where s_α is given by (3.12).

The following lemma shows that we can without loss of generality assume $L(n)$ is a constant, say a. The result follows from the fact that if L is slowly varying then $L(\gamma x) \sim L(x)$ as $x \to \infty$, uniformly in γ, for γ bounded away from 0 and ∞.

LEMMA 3. *Let L be a slowly varying function and let $\delta > 0$. Then for each $\varepsilon > 0$ there is an N such that $n \geqq N$ and $\delta n \leqq i, j \leqq n$ implies $|L(j)/L(i) - 1| < \varepsilon$.*

Now, some computations show that we can take $y = 0$ in (4.24). The idea of the proof of (4.24) is to use (4.22). To use this in estimating the left hand side of (4.24) it is necessary to decompose the sum into three parts in the following manner:

$$\begin{aligned}\textstyle\sum_{i,j=1}^n \sum_x P_i(0, x)P_j(0, x) &\\ = \textstyle\sum_{i,j=\delta n}^n \sum_{|x|\leqq\beta B_n}[\quad] &+ \textstyle\sum_{i,j=\delta n}^n \sum_{|x|>\beta B_n}[\quad] + \sum_{i,j=1}^{\delta n}\sum_x[\quad]\,.\end{aligned} \tag{4.25}$$

Here β and δ are fixed (but arbitrary) positive constants.

The first term on the right hand side of (4.25) can be handled by using the local limit theorem, whereas the second two terms on the right can be shown to be negligible with respect to the first. We indicate briefly how this is done.

To handle the second term on the right we use the fact that $P_n(0, x) = O(B_n^{-1})$ uniformly in x (see (4.22)), along with some computations to get

$$\begin{aligned}\textstyle\sum_{i,j=\delta n}^n \sum_{|x|>\beta B_n}[\quad] &= O(\textstyle\sum_{i,j=\delta n}^n B_j^{-1}P(|\xi_i| > 2B_i(\beta-1)))\\ &= O\left(n^{2-1/\alpha}\left[1 - V_\alpha\left(\frac{\beta-1}{2}\right) + V_\alpha\left(\frac{1-\beta}{2}\right)\right]\right),\end{aligned}$$

and this last term can be made small relative to $n^{2-1/\alpha}$ by choosing β large. The third term on the right is easily dispensed with since some computations indicate that it is $O(n^{2-1/\alpha}\delta^{1-1/\alpha})$ and this can be made small relative to $n^{2-1/\alpha}$ by taking δ small.

We now consider the first term on the right hand side of (4.25). Using the local limit theorem we get

$$\sum_{i,j=\delta n}^{n} \sum_{|x|\leq \beta B_n} [\quad] \sim \sum_{i,j=\delta n}^{n} \sum_{|x|\leq \beta B_n} f_\alpha\left(\frac{x}{B_i}\right) f_\alpha\left(\frac{x}{B_j}\right) \frac{1}{B_i B_j}\,. \tag{4.26}$$

Using the mean value theorem and the boundedness of $f_\alpha{}'$ it can be shown that the sum $\sum_{|x|\leq \beta B_n}$ on the right of (4.26) can be replaced by the integral $\int_{|t|\leq \beta B_n}$. A Riemann sum approximation shows that

$$\sum_{i,j=\delta n}^{n} \int_{-\infty}^{\infty} f_\alpha\left(\frac{t}{B_i}\right) f_\alpha\left(\frac{t}{B_j}\right) \frac{dt}{B_i B_j}$$
$$\sim \frac{n^{2-1/\alpha}}{a} \int_\delta^1 \int_\delta^1 \int_{-\infty}^{\infty} f_\alpha\left(\frac{t}{x^{1/\alpha}}\right) f_\alpha\left(\frac{t}{y^{1/\alpha}}\right) x^{-1/\alpha} y^{-1/\alpha}\, dt\, dx\, dy\,,$$

and the integral on the right can be made arbitrarily close to s_α by making δ sufficiently small. The proof of (4.24) is completed by verifying that

$$\sum_{i,j=\delta n}^{n} \int_{|t|>\beta B_n} f_\alpha\left(\frac{t}{B_i}\right) f_\alpha\left(\frac{t}{B_j}\right) \frac{dt}{B_i B_j} = O(n^{2-1/\alpha} \int_{|u|>(\beta-1)/2} f_\alpha(u)\, du)$$

which can be made small relative to $n^{2-1/\alpha}$ by choosing β large. Using (4.23) and (4.24) it follows that (3.10) holds.

The proofs of (3.13) and (3.14) were given in Weiss (1972) under the assumption that the random walks are in the domain of *normal* attraction of a stable law and that lim inf Var $S_n(B)/n^{2-1/\alpha} > 0$. In view of (3.10) this latter assumption is valid and in fact the limit exists and is given explicitly by the right hand side of (3.10).

Finally, we prove Theorem 4. First recall from (4.22) that

$$L(n)P_n(0, 0) \sim f_\alpha(0)n^{-1/\alpha}\,. \tag{4.27}$$

For convenience set $p_n = P_n(0, 0)$ and let $Q(t) = \sum_{n=0}^{\infty} q_n t^n$, $P(t) = \sum_{n=0}^{\infty} p_n t^n$ for $0 \leqq t < 1$. Note that $Q(t) = [(1-t)P(t)]^{-1}$. From (4.27) and Karamata's Tauberian theorem (see Feller (1966), page 243), $P(t) \sim (1-t)^{\alpha^{-1}-1}H((1-t)^{-1})$, as $t \to 1^-$, where $H(s) = f_\alpha(0)\Gamma(1-\alpha^{-1})/L(s)$. Thus, we conclude that as $t \to 1^-$,

$$Q(t) \sim (1-t)^{-1/\alpha}L((1-t)^{-1})[f_\alpha(0)\Gamma(1-\alpha^{-1})]^{-1}\,,$$

and using the converse part of the above Tauberian theorem we deduce that

$$\sum_{k=0}^{n-1} q_k \sim n^{1/\alpha}L(n)[f_\alpha(0)\Gamma(1-\alpha^{-1})\Gamma(1+\alpha^{-1})]^{-1}\,, \tag{4.28}$$

and in fact since q_n is monotone

$$q_n \sim n^{1/\alpha-1}L(n)[f_\alpha(0)\Gamma(1-\alpha^{-1})\Gamma(\alpha^{-1})]^{-1}\,. \tag{4.29}$$

In view of (4.17) this establishes (3.15).

To prove (3.17) we use similar techniques as in the proof of (3.10). We will not give the details here but will point out that the crucial facts needed in the verification (other than those needed to verify (3.10)) are (i) for each $y \in Z$,

$P_y(V_B > n) \sim \alpha^{-1} t_\alpha L_B(y) L(n) n^{1/\alpha - 1}$, which follows from the aforementioned result of Kesten and Spitzer along with (4.29); and (ii) the fact that $\sum_{y \in B} L_B(y) = 1$.

The proofs of (3.19) and (3.20) are omitted since they are similar to those of (3.3) and (3.4).

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