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THE OTHER LAW OF THE ITERATED LOGARITHM[1]

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Let $\{X_n\}$ be a sequence of independent, identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$. Define $S_n = X_1 + \cdots + X_n$, and $A_n = \max_{1 \leq k \leq n} |S_k|$. We prove that $\liminf A_n(n/\log\log n)^{-\frac{1}{2}} = \pi/8^{\frac{1}{2}}$ with probability one. This result was proved by Chung under the assumption of a finite third moment and under progressively weaker moment assumptions by Pakshirajan, Breiman, and Wichura. Chung posed the problem of whether it is enough to assume only the finiteness of the second moment in his review of Pakshirajan's paper in 1961. We showed earlier that $(n/\log\log n)^{\frac{1}{2}}$ is the correct normalization but were unable to show that the constant is necessarily $\pi/8^{\frac{1}{2}}$.

1. Introduction. Let $\{X_n\}$ be a sequence of real valued, independent, identically distributed random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Define $S_n = X_1 + \cdots + X_n$ and $A_n = \max_{1 \leq k \leq n} |S_k|$. We will prove the

THEOREM. *If $EX_1 = 0$ and $EX_1^2 = 1$, then*

$$\liminf_{n\to\infty} \frac{A_n}{(n/\log\log n)^{\frac{1}{2}}} = \frac{\pi}{8^{\frac{1}{2}}} \quad \text{a.s.}$$

This result was obtained by Chung (1948) under the additional assumption that $E|X_1|^3 < \infty$. In 1959, Pakshirajan showed that it was sufficient to assume that $E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. In his review of [6], Chung raised the question of whether the result is still valid if only the second moment is assumed finite as in the case of the Hartman–Wintner law of the iterated logarithm. Breiman (1967) proved that it is sufficient to assume that $E(X_1 \log\log |X_1|)^2 < \infty$ and Wichura [7] recently showed that even $EX_1^2 \log\log |X_1| < \infty$ is enough. In an earlier related paper [5], we proved the theorem except for the fact that the constant value of the lim inf might depend on the distribution of X_1.

2. Proof of theorem. First we will need a result about Brownian motion which is well known but we will include the proof for completeness.

LEMMA 1. *Let B_t be standard Brownian motion and $R_t = \max_{0 \leq s \leq t} |B_s|$. Let c be a positive constant and $t_n = cn^n$. Then*

$$\liminf_{n\to\infty} \frac{R_{t_n}}{(t_n/\log\log t_n)^{\frac{1}{2}}} \leq \frac{\pi}{8^{\frac{1}{2}}} \quad \text{a.s.}$$

(Actually, there is equality but we only need the inequality.)

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PROOF. The distribution of R_t is given by [4, page 206]:

$$P\left\{\frac{R_t}{t^{\frac{1}{2}}} < x\right\} = P\{R_1 < x\} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\{-\pi^2(2n+1)^2/8x^2\},$$

and so

$$\frac{4}{\pi}\{e^{-\pi^2/8x^2} - \tfrac{1}{3}e^{-9\pi^2/8x^2}\} \leqq P\left\{\frac{R_t}{t^{\frac{1}{2}}} < x\right\} \leqq \frac{4}{\pi} e^{-\pi^2/8x^2}. \tag{2.1}$$

Now let $U_n = \max_{t_{n-1} \leqq s \leqq t_n} |B_s - B_{t_{n-1}}|$. The U_n are independent and

$$P\left\{U_n < \frac{\pi}{8^{\frac{1}{2}}}\left(\frac{t_n}{\log\log t_n}\right)^{\frac{1}{2}}\right\} \geqq P\left\{\frac{R_{t_n - t_{n-1}}}{(t_n - t_{n-1})^{\frac{1}{2}}} < \frac{\pi}{8^{\frac{1}{2}}} \frac{1}{(\log\log t_n)^{\frac{1}{2}}}\right\} \sim \frac{4}{\pi} \frac{1}{n \log n}.$$

Thus, by Borel–Cantelli we have

$$\liminf_{n\to\infty} \frac{U_n}{(t_n/\log\log t_n)^{\frac{1}{2}}} \leqq \frac{\pi}{8^{\frac{1}{2}}} \quad \text{a.s.}$$

But, since $R_{t_n} \leqq R_{t_{n-1}} + U_n$ and

$$\limsup_{n\to\infty} \frac{R_{t_{n-1}}}{(t_n/\log\log t_n)^{\frac{1}{2}}}$$
$$= \limsup_{n\to\infty} \frac{R_{t_{n-1}}}{(t_{n-1}\log\log t_{n-1})^{\frac{1}{2}}} \frac{(t_{n-1}\log\log t_{n-1})^{\frac{1}{2}}}{(t_n/\log\log t_n)^{\frac{1}{2}}} = 0$$

by the standard law of the iterated logarithm, this is sufficient.

Now we are ready to obtain the upper bound of the theorem. We use the Skorokhod embedding (see, e.g. [3, page 276]) to construct a standard Brownian motion B_t and a sequence $\{T_n\}$ of nonnegative, independent, identically distributed random variables with $ET_1 = 1$ such that the distribution of $S_1, S_2, \cdots$ is the same as that of $B_{T_1}, B_{T_1+T_2}, \cdots$. Thus we may work with the latter sequence. Take $\varepsilon > 0$ and let $t_n = (1+\varepsilon)n^n$,

$$\Omega_1 = \left\{R_{t_n} < (1+\varepsilon)\frac{\pi}{8^{\frac{1}{2}}}\left(\frac{t_n}{\log\log t_n}\right)^{\frac{1}{2}} \text{ i.o.}\right\},$$

$$\Omega_2 = \{T_1 + \cdots + T_n < (1+\varepsilon)n \text{ eventually}\},$$

and $\Omega_0 = \Omega_1 \cap \Omega_2$. By Lemma 1 and the law of large numbers, $P(\Omega_0) = 1$. Now if $\omega \in \Omega_0$ and n is sufficiently large,

$$A_{n^n} = \max_{1 \leqq j \leqq n^n} |B_{T_1+\cdots+T_j}| \leqq R_{t_n}$$

and infinitely often this is

$$\leqq (1+\varepsilon)\frac{\pi}{8^{\frac{1}{2}}}\left(\frac{t_n}{\log\log t_n}\right)^{\frac{1}{2}} \leqq (1+\varepsilon)^{\frac{3}{2}} \frac{\pi}{8^{\frac{1}{2}}}\left(\frac{n^n}{\log\log n^n}\right)^{\frac{1}{2}}.$$

This suffices for the upper bound.

The lower bound depends on

LEMMA 2. *If $c < \pi/8^{\frac{1}{2}}$, then there is an $\eta > 1$ and a positive constant C such that*

$$P\left\{A_n < c\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}}\right\} \leqq C\left(\frac{1}{\log n}\right)^{\eta}$$

for all n sufficiently large.

Assuming the lemma for the moment, let $n_k = [\alpha^k]$ for $\alpha > 1$. Then $\log n_k \sim k \log \alpha$ so that by Borel–Cantelli, with probability one,

$$A_{n_k} \geqq c\left(\frac{n_k}{\log\log n_k}\right)^{\frac{1}{2}}$$

for k sufficiently large. Then if $n_k \leqq n < n_{k+1}$, and k is large,

$$A_n \geqq A_{n_k} \geqq c\left(\frac{n_k}{\log\log n_k}\right)^{\frac{1}{2}} \geqq \frac{c}{\alpha}\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}} \quad \text{a.s.}$$

This is sufficient since c/α can be made arbitrarily close to $\pi/8^{\frac{1}{2}}$.

PROOF OF LEMMA 2. First, let

$$M_n = \max\nolimits_{1\leqq k\leqq n} S_k\,, \qquad m_n = \min\nolimits_{1\leqq k\leqq n} S_k$$
$$M = \max\nolimits_{0\leqq t\leqq 1} B_t\,, \qquad m = \min\nolimits_{0\leqq t\leqq 1} B_t\,.$$

Now, if $a < b$, we have by the invariance principle (see, e.g. [3, page 282])

$$P\left\{a < \frac{m_n}{n^{\frac{1}{2}}} \leqq \frac{M_n}{n^{\frac{1}{2}}} < b\right\} \to P\{a < m \leqq M < b\} \tag{2.2}$$

and the convergence is uniform in a, b since the limit distribution is continuous. If $\gamma = (b-a)/2$, then

$$P\{a < m \leqq M < b\} \leqq P\{R_1 < \gamma\}\,. \tag{2.3}$$

(This intuitive result may be obtained easily from the joint distribution of m and M (see, e.g. [1, page 79]), or it may be seen even more easily by noting that it is equivalent to saying that Brownian motion is less likely to leave $(-\gamma, \gamma)$ by time one, starting from zero, than if it starts at $x \neq 0$. The inequality is then obtained by starting a Brownian motion at zero and restarting it when it first hits the set $\{x, -x\}$.) Now by (2.2), (2.3), and (2.1) we see that for given $\gamma > 0$, there exists a ν_0 such that for all a, b satisfying $b - a = 2\gamma$ and all $\nu \geqq \nu_0$,

$$P\{a\nu^{\frac{1}{2}} < m_\nu \leqq M_\nu < b\nu^{\frac{1}{2}}\} \leqq 2e^{-\pi^2/8\gamma^2}\,. \tag{2.4}$$

Choose $\xi \in (c, \pi/8^{\frac{1}{2}})$, $\eta \in (1, \pi^2/8\xi^2)$ and then take β so large that

$$2^{1/\beta} < e^{-\eta+\pi^2/8\xi^2}\,. \tag{2.5}$$

Let $\nu = [\beta n/\log\log n]$ and $N = [\log\log n/\beta]$. Then, for large n,

$$\left\{A_n < c\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}}\right\} \subset \bigcap\nolimits_{k=1}^{N} E_k\,, \tag{2.6}$$

where

$$E_k = \bigcap_{j=1}^{\nu} \left\{ |S_{(k-1)\nu+j}| < c\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}} \right\}$$

$$\subset \bigcap_{j=1}^{\nu} \left\{ S_{(k-1)\nu+j} - S_{(k-1)\nu} \in \left(-\frac{\xi}{\beta^{\frac{1}{2}}}\nu^{\frac{1}{2}}, \frac{\xi}{\beta^{\frac{1}{2}}}\nu^{\frac{1}{2}}\right) - S_{(k-1)\nu} \right\}.$$

Thus, by (2.4) with $\gamma = \xi\beta^{-\frac{1}{2}}$, we have for all sufficiently large n

$$P(E_k \mid S_{(k-1)\nu}) \leqq 2e^{-\pi^2\beta/8\xi^2}.$$

Applying the Markov property to (2.6) in the usual way we then obtain

$$\begin{aligned} P\left\{ A_n < c\left(\frac{n}{\log\log n}\right)^{\frac{1}{2}} \right\} &\leqq (2e^{-\pi^2\beta/8\xi^2})^N \\ &\leqq C(2^{1/\beta}e^{-\pi^2/8\xi^2})^{\log\log n} \\ &< C(\log n)^{-\eta}, \end{aligned}$$

the last inequality being a consequence of (2.5).

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