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ON CONVERGENCE IN r-MEAN OF NORMALIZED PARTIAL SUMS

BY S. W. DHARMADHIKARI[1] AND M. SREEHARI

M. S. University of Baroda

Suppose $S_n = \sum_1^n X_j$, where $\{X_n\}$ is a sequence of random variables. Under progressively weaker hypotheses, Pyke and Root (1968), Chatterji (1969) and Chow (1971) have proved that $E|S_n - b_n|^r = o(n)$, where $0 < r < 2$ and $\{b_n\}$ is properly chosen. This paper gives a fairly elementary proof of Chow's result under further weakened hypotheses.

Suppose $0 < r < 2$. Let $\{X_n\}$ be a sequence of random variables and write $S_n = \sum_1^n X_k$. Pyke and Root (1968) have proved that if the X_n's are i.i.d. with $E|X_1|^r < \infty$, then $E|S_n - b_n|^r = o(n)$, where $b_n = 0$ or $E(S_n)$ according as $r < 1$ or $r \geqq 1$. Chatterji (1969) extended this result to the case where the X_n's are stochastically dominated by a random variable X with $E|X|^r < \infty$ and where b_n is suitably modified when $r \geqq 1$. Chow (1971) proved Chatterji's result under the hypothesis of uniform integrability of the $|X_n|^r$. The purpose of this paper is to give a fairly elementary proof of Chow's result under weaker conditions. The proof arose out of a simple proof [3] of mean convergence in the law of large numbers for i.i.d. random variables. We will need the following improvement of the Minkowski inequality; see Von Bahr and Esseen (1965) and Chatterji (1969).

LEMMA. *Suppose* $1 \leqq r \leqq 2$ *and let* $X_1, \cdots, X_n$ *be random variables such that* $E|X_k|^r < \infty$ *for* $k \leqq n$ *and* $E(X_k \mid S_{k-1}) = 0$ a.s. *for* $k = 2, \cdots, n$. *Then* $E|S_n|^r \leqq 2^{2-r} \sum_1^n E|X_j|^r$.

THEOREM. *Let* $0 < r < 2$. *Suppose* $\{X_n\}$ *is a sequence of random variables such that* $\sup_n E|X_n|^r < \infty$. *For a sequence* $\{a_n\}$ *of positive constants, write* $A_n = [|X_n| \geqq a_n]$ *and suppose that* $E(|X_n|^r I_{A_n}) \to 0$. *If either* (i) $0 < r < 1$ *and* $\sum_1^n a_k^{1-r} = o(n^{1/r})$ *or* (ii) $1 \leqq r < 2$, $\{X_n\}$ *is a martingale difference sequence and* $\sum_1^n a_k^{2-r} = o(n^{2/r})$, *then* $E|S_n|^r = o(n)$.

PROOF. Let $M = \sup_n E|X_n|^r$, $V_n = X_n I_{A_n}$ and $U_n = X_n - V_n$. Denote by $\mathscr{B}_n$ the σ-field induced by $X_1, \cdots, X_n$. We take $\mathscr{B}_0$ to be the trivial σ-field. Write $\alpha_n = E(U_n \mid \mathscr{B}_{n-1})$ or 0 according as $r \geqq 1$ or $r < 1$. Let $Y_n = U_n - \alpha_n$, $Z_n = V_n + \alpha_n$, $T_n = \sum_1^n Y_k$ and $W_n = \sum_1^n Z_k$. Then $S_n = T_n + W_n$ and we would have $E|S_n|^r = o(n)$ as soon as we prove that $E|T_n|^r = o(n)$ and $E|W_n|^r = o(n)$.

First suppose that $1 \leqq r < 2$, $\sum_1^n a_k^{2-r} = o(n^{2/r})$ and that $\{X_n\}$ is a martingale

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[1] At the University of New Mexico for the year 1975–76.

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difference sequence. Then

$$\alpha_n = E(U_n \mid \mathscr{B}_{n-1}) = -E(V_n \mid \mathscr{B}_{n-1}) .$$

Therefore by Jensen's inequality, $|\alpha_n|^r \leqq E(|U_n|^r \mid \mathscr{B}_{n-1})$ a.s. and $|\alpha_n|^r \leqq E(|V_n|^r \mid \mathscr{B}_{n-1})$ a.s. Hence

$$E|\alpha_n|^r \leqq E|U_n|^r \leqq E|X_n|^r \leqq M \quad \text{and} \tag{1}$$

$$E|\alpha_n|^r \leqq E|V_n|^r = E(|X_n|^r I_{A_n}) \to 0 . \tag{2}$$

Because of (2), $E|Z_n|^r \leqq 2^r E|V_n|^r \to 0$ and further $\{Z_n\}$ is a martingale difference sequence. Therefore, by the lemma, $E|W_n|^r \leqq 2^{2-r} \sum_1^n E|Z_k|^r = o(n)$. Now $\{Y_n\}$ is also martingale difference sequence with $|Y_n| \leqq 2a_n$ a.s. and (1) shows that $E|Y_n|^r \leqq 2^r M$. Therefore, the Y_n's being uncorrelated, we have

$$E(T_n^2) = \sum_1^n E(Y_k^2) \leqq \sum_1^n (2a_k)^{2-r} E|Y_k|^r$$
$$\leqq 4M \sum_1^n a_k^{2-r} = o(n^{2/r}) .$$

Therefore $E|T_n|^r \leqq [E(T_n^2)]^{r/2} = o(n)$. This proves the assertion for the case $r \geqq 1$.

Suppose now that $0 < r < 1$ and $\sum_1^n a_k^{1-r} = o(n^{1/r})$. Set $\alpha_n = 0$, $Y_n = U_n$ and $Z_n = V_n$. By hypothesis $E|V_n|^r = E(|X_n|^r I_{A_n}) \to 0$. Therefore $E|W_n|^r \leqq \sum_1^n E|V_k|^r = o(n)$. Further, $E|U_n|^r \leqq E|X_n|^r \leqq M$. Hence

$$E|T_n| \leqq \sum_1^n E|U_k| \leqq \sum_1^n a_k^{1-r} E|U_k|^r \leqq M \sum_1^n a_k^{1-r} = o(n^{1/r}) .$$

Therefore $E|T_n|^r \leqq [E|T_n|]^r = o(n)$. This proves the assertion for the case $r < 1$ and completes the proof of the theorem.

REMARKS. (1) In the situations considered by Pyke and Root, Chatterji, and Chow, the conditions of the theorem will be satisfied if we take $a_k = (k/\log k)^{1/r}$.

(2) The following example shows that the conditions of our theorem can hold even if the $|X_n|^r$ are not uniformly integrable. Let $P(X_k = \pm k^{1/2r}) = (1/k^{\frac{1}{2}})$ and $P(X_k = 0) = 1 - (2/k^{\frac{1}{2}})$, and take $a_k = 2k^{1/2r}$. One can similary construct an example to show that Chatterji's hypothesis of stochastic domination is stronger than Chow's hypothesis of uniform integrability.

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DEPARTMENT OF STATISTICS
FACULTY OF SCIENCE
M. S. UNIVERSITY OF BARODA
BARODA 2, INDIA