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WEAK CONVERGENCE OF HIGH LEVEL CROSSINGS AND MAXIMA FOR ONE OR MORE GAUSSIAN PROCESSES

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Weak convergence of the multivariate point process of upcrossings of several high levels by a stationary Gaussian process is established. The limit is a certain multivariate Poisson process. This result is then used to determine the joint asymptotic distribution of heights and locations of the highest local maxima over an increasing interval. The results are generalized to upcrossings and local maxima of two dependent Gaussian processes. To prevent nuisance jitter from hiding the overall structure of crossings and maxima the above results are phrased in terms of ε-crossings and ε-maxima, but it is shown that under suitable regularity conditions the results also hold for ordinary upcrossings and maxima.

0. Introduction. The asymptotic Poisson distribution of the number of high level crossings by stationary Gaussian processes is well-known and has been established under weaker and weaker conditions by a number of authors, most recently by Berman (1971). The more general notion of ε-crossings was introduced and examined by Pickands (1969), and his results were later refined by Qualls and Watanabe (1972).

All these results deal exclusively with the number of crossings in one specific interval of increasing length. However, as will be shown in this paper, the asymptotic Poisson distribution is only one aspect of *weak convergence* of the point process of crossings (regular or ε-crossings) towards a Poisson point process. Another obvious consequence of the weak convergence is the asymptotic distribution of the locations of the crossings.

Moreover, it will be shown that the point process on R^2 of crossings of two (or, generally, n) levels converges weakly to a certain 2-variate (n-variate) Poisson process. This generalizes the results by Qualls (1969) in two directions. Firstly, it generalizes his result on the joint asymptotic distribution of the number of crossings of two or more levels to weak convergence; and secondly it generalizes his results from regular crossings to ε-crossings.

The technique used to obtain convergence towards the Poisson process is the following. By means of the theory for weak convergence of point processes in the formulation developed by Jagers (1972) and Kallenberg (1973) the problem is reduced to proving convergence of the joint distribution of the maxima in several disjoint intervals. This will be done by the usual methods to obtain

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asymptotic distributions of maxima for Gaussian processes as in Qualls and Watanabe (1972) and Leadbetter (1974a).

As one application we will—in Section 3—show that the weak limit theorem for multiple level crossings determines the joint asymptotic distribution of the *heights* and *locations* of the highest local maxima. To obtain sufficient generality we introduce the notion of *ε-maxima*, which by definition are separated at least a distance ε apart. For stationary *sequences* weak convergence has been studied by among others Leadbetter (1974b) and Welsh (1971, 1972, 1973). Welsh's results implicitly give a joint asymptotic distribution of heights and locations of maxima. The location of the maximum of a stationary process is also studied by Leadbetter (1974a).

As a generalization of a result by Lindgren (1974) we will also—in Section 4—prove related results for crossings and maxima of *dependent* Gaussian processes.

In Section 5, finally, we show that the results in Sections 2–4 hold under suitable regularity conditions if ordinary crossings and maxima are substituted for ε-crossings and ε-maxima.

1. Preliminaries. In the sequel we will consider a stationary and continuous Gaussian process X with mean zero and covariance function r. At zero r satisfies the condition

$$r(t) = 1 - |t|^\alpha C(t) + o(|t|^\alpha C(t)), \qquad t \to 0, \tag{1.1}$$

where $0 < \alpha \leqq 2$ and $C(t)$ varies slowly. Qualls and Watanabe (1972) noted that since $|t|^\alpha C(t)$ is determined only up to $o(|t|^\alpha C(t))$ we can choose a monotone version of $|t|^\alpha C(t)$ in some positive neighbourhood of zero. The inverse of $(2|t|^\alpha C(t))^{\frac{1}{2}}$ will be denoted by $G(t)$, and will be well-defined for small t. In particular, if $C(t) \equiv C$ then $G(t) = (t^2/2C)^{1/\alpha}$.

At infinity r will satisfy the mixing condition

$$r(t) \log t \to 0, \quad t \to \infty \qquad \text{or} \qquad \textstyle\int_{-\infty}^{\infty} r(t)^2\, dt < \infty, \tag{1.2}$$

as in Berman (1964), Pickands (1969), and Qualls and Watanabe (1972).

DEFINITION 1.1. A continuous function $f\colon R \to R$ has an ε-upcrossing of the level u at t_0 if

$$u = f(t_0) = \sup_{t_0-\varepsilon<t<t_0} f(t).$$

Note that this definition permits a local maximum at t_0. However, in our context these events have probability zero for any fixed level u, since maximum of a Gaussian process over a fixed bounded interval has continuous distribution; cf. Ylvisaker (1968).

We now state the extension by Qualls and Watanabe (1972) of Pickands' (1969) Poisson limit theorem for ε-upcrossings.

THEOREM 1.2 (Pickands; Qualls and Watanabe). *If X is a zero-mean, stationary* *and continuous Gaussian process with covariance function r satisfying* (1.1) *and* (1.2), *then there are levels* $u_{T,x}$ *such that the number of* ε*-upcrossings of* $u_{T,x}$ *by the process* X *in the interval* $[0, T]$ *is asymptotically Poisson distributed with mean value* x *when* T *tends to infinity. Furthermore*

$$TP(\sup_{0<t<1} X(t) > u_{T,x}) \rightarrow x$$

and the mean number of ε*-upcrossings during* $[0, T]$ *converges to* x.

REMARK 1.3. We can choose the levels $u_{T,x}$ as

$$u_{T,x} = (2\log T)^{\frac{1}{2}} - \frac{\log x + \frac{1}{2}\log\log T + \log G(1/(2\log T)^{\frac{1}{2}}) + \log(2\pi^{\frac{1}{2}}/H_\alpha)}{(2\log T)^{\frac{1}{2}}} \tag{1.3}$$

where H_α is a certain positive number given by Pickands and Qualls and Watanabe, but any level $u_{T,x} + o(1/(2\log T)^{\frac{1}{2}})$ will do as well in the theorem.

REMARK 1.4. Levels corresponding to different mean values $x > 0$, $y > 0$ in the limiting distribution will be arbitrarily close to each other when T tends to infinity, since

$$u_{T,x} - u_{T,y} = \frac{\log y/x}{(2\log T)^{\frac{1}{2}}}\,.$$

REMARK 1.5. An inverse relation of (1.3) up to $o(1/(2\log T)^{\frac{1}{2}})$ is

$$T_{u,x} = xue^{u^2/2}G\left(\frac{1}{u}\right)H_\alpha(2\pi)^{\frac{1}{2}}\,. \tag{1.4}$$

2. Weak convergence of the point process of ε-upcrossings. In this paper we will use the notion of point processes as random integer valued measures on R^n. Weak convergence of point processes is defined as in Jagers (1972) and Kallenberg (1973).

Now consider a stationary and continuous Gaussian process X with mean zero, unit variance and covariance function r satisfying (1.1) and (1.2). Let $x > y$ be positive numbers and let the levels $u_{T,x}$ and $u_{T,y}$ be defined by (1.3). Define the point processes ξ_T and η_T of time-normalized ε-upcrossings:

$$\begin{aligned} \xi_T(B) &= \#\ \ \varepsilon\text{-upcrossings of}\ \ u_{T,x}\ \ \text{by}\ \ X(t): t \in T\cdot B \\ \eta_T(B) &= \#\ \ \varepsilon\text{-upcrossings of}\ \ u_{T,y}\ \ \text{by}\ \ X(t): t \in T\cdot B\,, \end{aligned} \tag{2.1}$$

(B a linear Borel set). Furthermore, let ξ be a Poisson process with intensity x and let η be the binomial thinning of ξ with deletion probability $1 - y/x$, i.e. η is the Poisson processes with intensity y generated by independent deletion of points in the ξ-process. Then the product measure $\xi \times \eta$ is a point process, i.e. a random integer valued measure on R^2.

THEOREM 2.1. *If* X *is a zero-mean, stationary and continuous Gaussian process with covariance function* r *satisfying* (1.1) *and* (1.2), *then the point process* $\xi_T \times \eta_T$

defined by (2.1) *converges weakly to* $\xi \times \eta$ *where* ξ *is a Poisson process with intensity* x *and* η *is a binomial thinning of* ξ *with deletion probability* $1 - y/x$.

PROOF. To prove the weak convergence we will use the following criterion by Kallenberg (1973), Theorem 2.5. If ζ, ζ_T, $(T > 0)$ are point processes without multiple points on R^n such that for all bounded rectangles R, $P(\zeta(\partial R) > 0) = 0$, ($\partial R$ denotes the boundary of R), and if

(2.2) for all bounded rectangles R there is a sequence $(\{R_{mj} : j = 1, \cdots, k_m\})_{m=1}^{\infty}$ of partitions of R into rectangles with $\lim_{m\to\infty} \max_{1\le j\le k_m} \operatorname{diam}(R_{mj}) = 0$ such that $\lim_{m\to\infty} \limsup_{T\to\infty} \sum_{j=1}^{k_m} P(\zeta_T(R_{mj}) > 1) = 0$,

and

(2.3) for all finite unions U of rectangles $P(\zeta_T(U) = 0) \to P(\zeta(U) = 0)$, $T \to \infty$,

then ζ_T converges weakly to ζ, $T \to \infty$.

To prove the theorem we have to check that (2.2) and (2.3) hold. We will then make substantial use of the one-level version of the theorem, i.e. that $\xi_T \to \xi$ weakly, so we prove this first.

1. *Proof of* (2.2) *in the one-level case.* Here R is a real bounded interval and we take a sequence of partitions which for each $m = 1, 2, \cdots$ divides R into m subintervals $R_{m1}, \cdots, R_{mm}$ of equal lengths. Then

$$\max_{1\le j\le m} \operatorname{diam}(R_{mj}) \to 0, \quad m \to \infty$$

and

$$\begin{aligned}\lim_{m\to\infty} \limsup_{T\to\infty} & \sum_{j=1}^{m} P(\xi_T(R_{mj}) > 1) \\ &= \lim_{m\to\infty} \sum_{j=1}^{m} (1 - e^{-x|R_{mj}|} - x|R_{mj}|e^{-x|R_{mj}|}) \\ &= \lim_{m\to\infty} m\left(1 - e^{-(x|R|)/m} - \frac{x|R|}{m} e^{-(x|R|)/m}\right) = 0,\end{aligned}$$

where the first equality follows by Theorem 1.2.

2. *Proof of* (2.3) *in the one-level case.* Let $U = \bigcup_{i=1}^{n} R_i$. Then, as $T \to \infty$,

$$\begin{aligned}P(\{\xi_T(\textstyle\bigcup_{i=1}^{n} R_i) = 0\}) &= P(\textstyle\bigcap_{i=1}^{n} \{\xi_T(R_i) = 0\}) \\ &\to P(\textstyle\bigcap_{i=1}^{n} \{\xi(R_i) = 0\}) = P(\{\xi(\textstyle\bigcup_{i=1}^{n} R_i) = 0\}),\end{aligned}$$

by Lemma 2.3 below. Hence we have proved that $\xi_T \to \xi$ weakly when $T \to \infty$. Obviously we also have $\eta_T \to \eta$ weakly. To prove $\xi_T \times \eta_T \to \xi \times \eta$ we will now use this result.

3. *Proof of* (2.2) *in the two-level case.* We have $R = I \times J$ where I and J are bounded real intervals. For $m = 1, 2, \cdots$ we partition both I and J into m intervals of equal lengths $I_{m1}, \cdots, I_{mm}$ and $J_{m1}, \cdots, J_{mm}$, respectively, and then

partition R as follows,

$$\{R_{mi}: i = 1, \cdots, m^2\} = \{I_{mj} \times J_{mk}: j, k = 1, \cdots, m\}.$$

Then $\max_{1 \leq i \leq m^2} \operatorname{diam}(R_{mi}) \to 0$, $m \to \infty$ and

$$\begin{aligned} &P(\xi_T \times \eta_T(I_{mj} \times J_{mk}) > 1) \\ &\quad \leqq P(\xi_T(I_{mj}) > 1, \eta_T(J_{mk}) \geqq 1) + P(\xi_T(I_{mj}) \geqq 1, \eta_T(J_{mk}) > 1) \\ &\quad \leqq P(\xi_T(I_{mj}) > 1, \xi_T(J_{mk}) \geqq 1) + P(\xi_T(I_{mj}) \geqq 1, \xi_T(J_{mk}) > 1) + o(1), \end{aligned}$$

$T \to \infty$, since

$$\begin{aligned} &P(\eta_T(J_{mk}) \geqq 1, \xi_T(J_{mk}) = 0) \to 0, \\ &P(\eta_T(J_{mk}) > 1, \xi_T(J_{mk}) \leqq 1) \to 0, \quad T \to \infty, \end{aligned}$$

by (2.5) of Lemma 2.2 below. Thus

$$\begin{aligned} L &= \lim_{m\to\infty} \limsup_{T\to\infty} \textstyle\sum_{j=1}^m \sum_{k=1}^m P(\xi_T \times \eta_T(I_{mj} \times J_{mk}) > 1) \\ &\leqq \lim_{m\to\infty} \limsup_{T\to\infty} \textstyle\sum_{j=1}^m \sum_{k=1}^m P(\xi_T(I_{mj}) > 1, \xi_T(J_{mk}) \geqq 1) \\ &\qquad + \lim_{m\to\infty} \limsup_{T\to\infty} \textstyle\sum_{j=1}^m \sum_{k=1}^m P(\xi_T(I_{mj}) \geqq 1, \xi_T(J_{mk}) > 1). \end{aligned}$$

By the first part of the proof we know that $\xi_T \to \xi$ weakly so

$$\begin{aligned} L &\leqq \lim_{m\to\infty} \textstyle\sum_{j=1}^m \sum_{k=1}^m P(\xi(I_{mj}) > 1, \xi(J_{mk}) \geqq 1) \\ &\qquad + \lim_{m\to\infty} \textstyle\sum_{j=1}^m \sum_{k=1}^m P(\xi(I_{mj}) \geqq 1, \xi(J_{mk}) > 1) \\ &\leqq \lim_{m\to\infty} \{\textstyle\sum_{j=1}^m \sum_{k=1}^m 2P(\xi(I_{mj} \cup J_{mk}) \geqq 3) \\ &\qquad + \textstyle\sum_{j=1}^m \sum_{k=1}^m 2P(\xi(I_{mj} \cap J_{mk}) \geqq 1, \xi(I_{mj} \cup J_{mk}) > 1)\}. \end{aligned}$$

Now the first sum is of order $m^2 \cdot o(m^{-2})$, $m \to \infty$, and in the second there are at most $2m$ nonzero terms, each of order $o(m^{-1})$ since no more than $2m$ of the intersections $I_{mj} \cap J_{mk}$: $j, k = 1, \cdots, m$, are nonempty. Hence $L = 0$ and (2.2) is satisfied.

4. *Proof of* (2.3) *in the two-level case.* Let $U = \bigcup_{i=1}^n R_i$ where $R_i = I_i \times J_i$, a cartesian product of real intervals. Then

$$\begin{aligned} &P(\{\xi_T \times \eta_T(\textstyle\bigcup_{i=1}^n R_i) = 0\}) \\ &\quad = P(\textstyle\bigcap_{i=1}^n \{\xi_T \times \eta_T(R_i) = 0\}) \\ &\quad = P(\textstyle\bigcap_{i=1}^n \{\xi_T(I_i) = 0 \text{ or } \eta_T(J_i) = 0\}) \\ &\quad = \textstyle\sum_{1 \leq j \leq 2^n} P(c_j(T)) - \sum_{1 \leq j_1 < j_2 \leq 2^n} P(c_{j_1}(T) \cap c_{j_2}(T)) + \cdots \\ &\qquad - P(c_1(T) \cap \cdots \cap c_{2^n}(T)), \end{aligned}$$

where $\{c_j(T): j = 1, \cdots, 2^n\} = \{\bigcap_{i=1}^n A_i(T): A_i(T) = \{\xi_T(I_i) = 0\}$ or $A_i(T) = \{\eta_T(J_i) = 0\}\}$. Each $c_j(T)$, $j = 1, \cdots, 2^n$ is an intersection $\bigcap_{i=1}^n A_i(T)$ where $A_i(T)$ is either $\{\xi_T(I_i) = 0\}$ or $\{\eta_T(J_i) = 0\}$. Now, by Lemma 2.3 below,

$$P(c_{j_1}(T) \cap c_{j_2}(T) \cap \cdots \cap c_{j_k}(T)) \to P(c_{j_1} \cap c_{j_2} \cap \cdots \cap c_{j_k}), \qquad T \to \infty,$$

where the events c_j are defined as $c_j(T)$ but from the $\xi \times \eta$-process. Hence

$$P(\xi_T \times \eta_T(\textstyle\bigcup_{i=1}^n R_i) = 0) \to P(\xi \times \eta(\bigcup_{i=1}^n R_i) = 0), \qquad T \to \infty,$$

which proves (2.3) and concludes the proof of the theorem.

During the course of the proof, we left two technical details unproved. We will now fill in the missing parts by means of two lemmas.

LEMMA 2.2. *Assume the hypothesis of Theorem* 2.1 *and let* I *be a bounded interval. Set* $t_1 = \inf\{t \in T \cdot I \colon X(t) \geqq u_{T,x}\}$ *and define recursively* $t_i = \inf\{t \geqq t_{i-1} + \varepsilon \colon X(t) \geqq u_{T,x}\}$, $i = 2, 3, \cdots$. *Let the number of* ε*-separated points over the level* $u_{T,x}$ *during* $T \cdot I$ *be*

$$N_T(I) = \max\{i \colon i = 0 \ \text{ or } \ t_i \in T \cdot I\}.$$

Then

$$P(N_T(I) > \xi_T(I)) \to 0, \qquad T \to \infty \tag{2.4}$$

and consequently

$$P(\eta_T(I) > \xi_T(I)) \to 0, \qquad T \to \infty. \tag{2.5}$$

PROOF. Since $N_T(I) \geqq \eta_T(I)$, relation (2.5) is an immediate consequence of (2.4). To prove (2.4) we define $\xi_T{}^n(I)$: $n = 1, 2, \cdots$ as the number of ε/n-upcrossings of the level $u_{T,x}$ by the process $X(t)$ during $T \cdot I$; in particular $\xi_T(I) = \xi_T{}^1(I)$. Note that it is enough to take $I = [0, 1]$. We then have

$$P(N_T(I) > \xi_T(I)) \leqq P(\xi_T{}^n(I) > \xi_T(I)) + P(N_T(I) > \xi_T{}^n(I)),$$

where

$$P(\xi_T{}^n(I) > \xi_T(I)) \leqq E(\xi_T{}^n(I) - \xi_T(I)) \to 0, \qquad T \to \infty,$$

by Theorem 1.2. Furthermore, if $M(B) = \sup\{X(t) \colon t \in B\}$,

$$\begin{aligned} P(N_T(I) > \xi_T{}^n(I)) &\leqq P\left(\bigcup_{k=0}^{[T/\varepsilon+1]} \left\{M\left(\left[k\varepsilon, \left(k + \frac{1}{n}\right)\varepsilon\right]\right) > u_{T,x}\right\}\right) \\ &\leqq (T/\varepsilon + 2)P(M([0, \varepsilon/n]) > u_{T,x}) \to x/n, \qquad T \to \infty, \end{aligned}$$

by Theorem 1.2. Since n is arbitrary, we have proved the lemma.

LEMMA 2.3. *Let* $I_1, \cdots, I_m, J_1, \cdots, J_n$ *be bounded real intervals and assume the hypothesis of Theorem* 2.1 *holds. Then*

$$\begin{aligned} &P(\textstyle\bigcap_{i=1}^m \{\xi_T(I_i) = 0\}, \bigcap_{j=1}^n \{\eta_T(J_j) = 0\}) \\ &\qquad \to P(\textstyle\bigcap_{i=1}^m \{\xi(I_i) = 0\}, \bigcap_{j=1}^n \{\eta(J_j) = 0\}), \qquad T \to \infty. \end{aligned}$$

PROOF. Let $I = \bigcup_{i=1}^m I_i$ and $J = \bigcup_{j=1}^n J_j$. We note that it is enough to prove the lemma for I and J disjoint, since if not so then

$$P(\{\xi_T(I) = 0\} \cap \{\eta_T(I \cap J) \geqq 1\}) \to 0 = P(\{\xi(I) = 0\} \cap \{\eta(I \cap J) \geqq 1\})$$

by Lemma 2.2. Further we note that it is sufficient to prove

$$\begin{aligned} &P(M(T \cdot I) < u_{T,x}, M(T \cdot J) < u_{T,y}) \\ &\qquad - P(M(T \cdot I) < u_{T,x})P(M(T \cdot J) < u_{T,y}) \to 0 \end{aligned} \tag{2.6}$$

as $T \to \infty$, where, as usual, $M(B) = \sup\{X(t) \colon t \in B\}$. This follows since e.g.

$$\begin{aligned} &P(\xi_T(I) = 0, M(T \cdot I) > u_{T,x}) \\ &\qquad \leqq \textstyle\sum_{i=1}^m P(M([T \cdot \inf I_i, T \cdot \inf I_i + \varepsilon]) > u_{T,x}) \to 0 \end{aligned}$$

by Theorem 1.2, and since, still by Theorem 1.2, $\xi_T(I)$ is asymptotically Poisson distributed when $m = n = 1$. For $m, n > 1$ we get the desired result by repeated applications of (2.6) with $x = y$.

The proof of (2.6) now proceeds in two steps. First we make a discrete approximation of the X-process and then get the asymptotic independence by standard methods, as e.g. in Leadbetter (1974a).

For T sufficiently large we now define the set

$$A_T = \{kaG(1/(2\log T)^{\frac{1}{2}}) : k \text{ integer}\},$$

where a is a positive number which will tend to zero at a later stage. In particular, if $C(t) \equiv C$ we have $G(1/(2\log T)^{\frac{1}{2}}) = (4C\log T)^{-1/\alpha}$ in the definition of A_T. We are now going to estimate $\{X(t) : t \in R\}$ by the discrete process $\{X(t) : t \in A_T\}$. Writing

$$M_a(B) = M(A_T \cap B) = \sup\{X(t) : t \in A_T \cap B\}$$

we want to prove

$$\text{(2.7)} \qquad \lim_{a\downarrow 0} \limsup_{T\to\infty} |P(M_a(T\cdot I) < u_{T,x}, M_a(T\cdot J) < u_{T,y}) - P(M(T\cdot I) < u_{T,x}, M(T\cdot J) < u_{T,y})| = 0 .$$

But

$$\begin{aligned}
0 &\leqq P(M_a(T\cdot I) < u_{T,x}, M_a(T\cdot J) < u_{T,y}) - P(M(T\cdot I) < u_{T,x}, M(T\cdot J) < u_{T,y}) \\
&\leqq P(M_a(T\cdot I) < u_{T,x}, M(T\cdot I) \geqq u_{T,x}) + P(M_a(T\cdot J) < u_{T,y}, M(T\cdot J) \geqq u_{T,y}) \\
&\leqq T\cdot P(M_a(I) < u_{T,x}, M(I) \geqq u_{T,x}) + T\cdot P(M_a(J) < u_{T,y}, M(J) \geqq u_{T,y}) + o(1) \\
&\leqq \textstyle\sum_{i=1}^m T\cdot \{P(M_a(I_i) < u_{T,x}) - P(M(I_i) < u_{T,x})\} \\
&\qquad + \textstyle\sum_{j=1}^n T\cdot \{P(M_a(J_j) < u_{T,y}) - P(M(J_j) < u_{T,y})\} + o(1) ,
\end{aligned}$$

which tends to zero if first $T \to \infty$ and then $a \downarrow 0$, since by Lemma 2.3 and Theorem 2.1 in Qualls and Watanabe (1972) each term tends to zero. Thus (2.7) is proved and the first part of the proof of the lemma is concluded.

To prove the asymptotic independence assume first that the distance between I and J is $\gamma > 0$, i.e.

$$\gamma = \inf\{|s - t| : s \in I, t \in J\} > 0 ,$$

and proceed as in Leadbetter (1974a), Lemmas 6.1 and 6.6. By a well-known inequality—see Leadbetter (1974a), page 23, for references—we get

$$\begin{aligned}
P &= |P(M_a(T\cdot I) < u_{T,x}, M_a(T\cdot J) < u_{T,y}) \\
&\qquad - P(M_a(T\cdot I) < u_{T,x}) \cdot P(M_a(T\cdot J) < u_{T,y})| \\
&\leqq \sum \left\{|r(s_i - t_j)| \exp\left\{-\frac{u_{T,x}^2}{1 + |r(s_i - t_j)|}\right\} : s_i \in A_T \cap (T\cdot I),\right. \\
&\qquad \left. t_j \in A_T \cap (T\cdot J)\right\} .
\end{aligned}$$

We have then reduced the problem to the one-level case and can estimate the sum under the hypothesis (1.2). Omitting details we get $P \to 0$ as $T \to \infty$, and we are finished with the asymptotic independence for the discrete approximation if I and J are separated by $\gamma > 0$.

Thus we know, by (2.7), that

$$P(M(T \cdot I) < u_{T,x}, M(T \cdot J) < u_{T,y}) - P(M_a(T \cdot I) < u_{T,x}, M_a(T \cdot J) < u_{T,y})$$

is arbitrarily close to zero if T is sufficiently large and a is sufficiently small. But for any small a we have the asymptotic independence, i.e.

$$P(M_a(T \cdot I) < u_{T,x}, M_a(T \cdot J) < u_{T,y})$$

is arbitrarily close to

$$P(M_a(T \cdot I) < u_{T,x}) \cdot P(M_a(T \cdot J) < u_{T,y}),$$

which in turn, still by (2.7), approximates

$$P(M(T \cdot I) < u_{T,x}) \cdot P(M(T \cdot J) < u_{T,y}).$$

This shows (2.6), i.e. that

$$P(M(T \cdot I) < u_{T,x}, M(T \cdot J) < u_{T,y}) \\ - P(M(T \cdot I) < u_{T,x}) \cdot P(M(T \cdot J) < u_{T,y}) \to 0$$

as $T \to \infty$. As noted in the beginning of the proof this gives

$$P(\xi_T(I) = 0, \eta_T(J) = 0) \to P(\xi(I) = 0, \eta(J) = 0),$$

and the lemma is proved for I and J separated by a positive distance. Approximation of J by a sequence of sets J^n on distance $1/n$ from I will yield the final result.

We conclude this section by stating the n-level version of Theorem 2.1.

THEOREM 2.4. *Let X be a zero-mean, stationary and continuous Gaussian process with covariance function r satisfying* (1.1) *and* (1.2). *Let $0 < x_n < x_{n-1} < \cdots < x_1$ be real numbers and let $\xi_{T,x_i}(B)$ be the number of ε-upcrossings of the levels u_{T,x_i} by the process $\{X(t): t \in T \cdot B\}$. Further, let ξ_{x_1} be a Poisson process with intensity x_1 and define recursively the Poisson process ξ_{x_k} as the binomial thinning of $\xi_{x_{k-1}}$ with deletion probability $1 - x_k/x_{k-1}$, $k = 2, \cdots, n$. Then*

$$\xi_{T,x_1} \times \cdots \times \xi_{T,x_n} \to \xi_{x_1} \times \cdots \times \xi_{x_n}$$

weakly, when $T \to \infty$.

The proof of this theorem follows the same lines as the proof of Theorem 2.1 and is not given here.

3. Joint distribution of heights and locations of the highest local ε-maxima. In this section we are going to consider the joint asymptotic distribution of the heights and locations of maxima of the process X in an expanding interval $[0, T]$. Since the condition (1.1) on the covariance function r does not ensure a finite number of local maxima we will introduce the notion of local ε-maxima which, by definition, are separated by at least the distance ε.

Let
$$M_1(T) = \sup\{X(t) : t \in [0, T]\}$$
be the global maximum in the interval $[0, T]$ and call its location
$$S_1(T) = \inf\{t \in [0, T] : X(t) = M_1(T)\} .$$
Let $\varepsilon > 0$ be fixed.

DEFINITION 3.1. The second local ε-maximum and its location are
$$M_2(T) = \sup\{X(t) : t \in [0, T], t \notin (S_1(T) - \varepsilon, S_1(T) + \varepsilon)\} ,$$
$$S_2(T) = \inf\{t \in [0, T] : X(t) = M_2(T), t \notin (S_1(T) - \varepsilon, S_1(T) + \varepsilon)\} ,$$
respectively. The nth highest local ε-maximum is defined recursively.

Further justification of Definition 3.1 is given by the facts that the asymptotic results do not depend on ε and that, under further restrictions on the covariance function, the local ε-maxima will be shown to coincide asymptotically with the ordinary local maxima.

THEOREM 3.2. *If X is a zero-mean, stationary and continuous Gaussian process with covariance function r satisfying* (1.1) *and* (1.2) *then*

$$(3.1) \qquad P(a_T(M_1(T) - b_T) \leqq \mu_1, S_1(T) \leqq \sigma_1 T, a_T(M_2(T) - b_T) \leqq \mu_2, S_2(T) \leqq \sigma_2 T) \to \sigma_1\sigma_2 e^{-e^{-\mu_2}}(1 + e^{-\mu_2} - e^{-\mu_1}) , \qquad T \to \infty ,$$

for $0 \leqq \sigma_1, \sigma_2 \leqq 1$, $\mu_2 \leqq \mu_1$, *where*
$$a_T = (2 \log T)^{\frac{1}{2}}$$
and
$$b_T = (2 \log T)^{\frac{1}{2}} - \frac{\frac{1}{2} \log \log T + \log G(1/(2 \log T)^{\frac{1}{2}}) + \log (2\pi^{\frac{1}{2}}/H_\alpha)}{(2 \log T)^{\frac{1}{2}}} .$$

PROOF. Set $y = e^{-\mu_1}$ and $x = e^{-\mu_2}$. Then (3.1) follows if we prove that, with $u_{T,y}$ and $u_{T,x}$ defined by (1.3),
$$P_T = P(M_1(T) \leqq u_{T,y}, S_1(T) \leqq \sigma_1 T, M_2(T) \leqq u_{T,x}, S_2(T) \leqq \sigma_2 T)$$
tends to $\sigma_1\sigma_2 e^{-x}(1 + x - y)$ when $T \to \infty$. To achieve this we will consider crossings of finer and finer grids of levels. For notational convenience we assume that e.g. $\sigma_1 \leqq \sigma_2$ and then split P_T into four parts as follows,
$$\begin{aligned} P_T = {} & P(u_{T,x} < M_1(T) \leqq u_{T,y}, S_1(T) \leqq \sigma_1 T, M_2(T) \leqq u_{T,x}, S_2(T) \leqq \sigma_1 T) \\ & + P(M_1(T) \leqq u_{T,x}, S_1(T) \leqq \sigma_1 T, M_2(T) \leqq u_{T,x}, S_2(T) \leqq \sigma_1 T) \\ & + P(u_{T,x} < M_1(T) \leqq u_{T,y}, S_1(T) \leqq \sigma_1 T, M_2(T) \leqq u_{T,x}, \\ & \quad \sigma_1 T < S_2(T) \leqq \sigma_2 T) \\ & + P(M_1(T) \leqq u_{T,x}, S_1(T) \leqq \sigma_1 T, M_2(T) \leqq u_{T,x}, \sigma_1 T < S_2(T) \leqq \sigma_2 T) \\ = {} & P_{T,1} + P_{T,2} + P_{T,3} + P_{T,4}, \quad \text{say.} \end{aligned}$$

We first evaluate $P_{T,1}$. Let $n < m$ be arbitrary, fixed integers—later on we will let them go to infinity—and put $x_k = (k/n)x$, $k = n, n+1, \cdots, m$. Then

$$P_{T,1} \geqq \sum_{k=n}^{m-1} P(u_{T,x} < M_1(T) \leqq u_{T,y}, S_1(T) \leqq \sigma_1 T, \\ u_{T,x_{k+1}} < M_2(T) \leqq u_{T,x_k}, S_2(T) \leqq \sigma_1 T) . \tag{3.2}$$

Now recall the notation $\xi_{T,x}(I)$ from Theorem 2.4 and write $N_{T,x}$ for the number of ε-separated points above the level $u_{T,x}$ in the interval $[0, \sigma_1 T]$; cf. Lemma 2.2. Then the sum in (3.2) is not less than

$$\sum_{k=n}^{m-1} P(\xi_{T,x_{k+1}}([\sigma_1, 1]) = 0, \xi_{T,x_{k+1}}([0, \sigma_1]) = 2, \\ \xi_{T,x_k}([0, \sigma_1]) = 1, \xi_{T,x}([0, \sigma_1]) = 1, \xi_{T,y}([0, \sigma_1]) = 0) \\ - P(N_{T,x_m} > \xi_{T,x_m}([0, \sigma_1])) - P(M([\sigma_1 T - \varepsilon, \sigma_1 T]) > u_{T,x_m}) . \tag{3.3}$$

To see that (3.3) is a true lower bound check Figure 3.1 and reason like this. When $M([\sigma_1 T - \varepsilon, \sigma_1 T]) < u_{T,x_m}$ and $\xi_{T,x_m}([\sigma_1, 1]) = 0$ there are no local ε-maxima above u_{T,x_m} in $[\sigma_1 T, T]$, and we have, for all levels $u_{T,z} \geqq u_{T,x_m}$, that

$$\xi_{T,z}([0, \sigma_1]) \leqq \# \quad \text{local } \varepsilon\text{-maxima above} \quad u_{T,z} \quad \text{during} \quad [0, \sigma_1 T] \leqq N_{T,z} .$$

Thus either

$$N_{T,z} - \xi_{T,z}([0, \sigma_1]) > 0$$

or

$$\xi_{T,z}([0, \sigma_1]) = \# \quad \text{local } \varepsilon\text{-maxima above} \quad u_{T,z} \quad \text{during} \quad [0, \sigma_1 T] .$$

But $N_{T,z} - \xi_{T,z}([0, \sigma_1])$ is increasing in z, i.e. decreasing in the level, so

$$N_{T,x_m} = \xi_{T,x_m}([0, \sigma_1])$$

implies that

$$\xi_{T,z}([0, \sigma_1]) = \# \quad \text{local } \varepsilon\text{-maxima above} \quad u_{T,z} \quad \text{during} \quad [0, \sigma_1 T]$$

for *all* levels $u_{T,z}$ above u_{T,x_m}. This shows that (3.3) is a lower bound for the sum in (3.2).

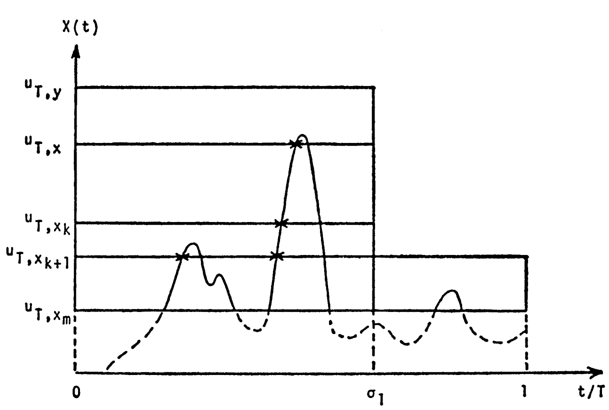

FIG. 3.1. Approximate determination of heights and locations of maxima through knowledge of the crossings of a fine grid of levels.

Now the last two terms in (3.3) tend to zero by Lemma 2.2 and Theorem 1.2, respectively, so by applying Theorem 2.4 to the terms in the first sum we get that

$$\begin{aligned}\liminf_{T\to\infty} P_{T,1} &\geqq \textstyle\sum_{k=n}^{m-1} e^{-x_{k+1}(1-\sigma_1)} \cdot \dfrac{1}{2!}(x_{k+1}\sigma_1)^2 e^{-x_{k+1}\sigma_1} \\ &\quad\times \binom{2}{1}\dfrac{x_k}{x_{k+1}}\left(1-\dfrac{x_k}{x_{k+1}}\right)\cdot\dfrac{x}{x_k}\cdot\left(1-\dfrac{y}{x}\right) \\ &= \sigma_1^2(x-y)e^{-(n+1)x/n}\cdot\dfrac{x}{n}\,\dfrac{1-e^{-(m-n)x/n}}{1-e^{-x/n}}\,,\end{aligned}$$

which tends to $\sigma_1^2(x-y)e^{-x}$ if first $m\to\infty$ and then $n\to\infty$. Hence we have that

$$\liminf_{T\to\infty} P_{T,1} \geqq \sigma_1^2(x-y)e^{-x}\,. \tag{3.4}$$

We can now get an upper bound for $P_{T,1}$ in a similar way:

$$\begin{aligned}P_{T,1} \leqq \textstyle\sum_{k=n}^{m-1} P(&\xi_{T,x_k}([\sigma_1,1])=0,\ \xi_{T,x_{k+1}}([0,\sigma_1])=2, \\ &\xi_{T,x_k}([0,\sigma_1])=1,\ \xi_{T,x}([0,\sigma_1])=1,\ \xi_{T,y}([0,\sigma_1])=0) \\ &+ \textstyle\sum_{k=n}^{m-1} P(\xi_{T,x_{k+1}}([0,\sigma_1])>2,\ \xi_{T,x_k}([0,\sigma_1])=1) \\ &+ P(N_{T,x_m} > \xi_{T,x_m}([0,\sigma_1])) + P(M([\sigma_1 T-\varepsilon,\sigma_1 T]) > u_{T,x_m})\,.\end{aligned} \tag{3·5}$$

As above, Theorem 2.4 gives that

$$\limsup_{T\to\infty} P_{T,1} \leqq \sigma_1^2(x-y)e^{-x}\,,$$

which together with (3.4) gives that

$$\lim_{T\to\infty} P_{T,1} = \sigma_1^2(x-y)e^{-x}\,.$$

The limits of $P_{T,2}$, $P_{T,3}$ and $P_{T,4}$ can be calculated in a quite similar way. They are $\sigma_1^2e^{-x}$, $(\sigma_1\sigma_2-\sigma_1^2)(x-y)e^{-x}$, and $(\sigma_1\sigma_2-\sigma_1^2)e^{-x}$, respectively. Adding the four terms gives the desired result that

$$\lim_{T\to\infty} P_T = \sigma_1\sigma_2 e^{-x}(1+x-y)\,,$$

and the theorem is proved.

REMARK 3.3. One can note that the asymptotic distribution of heights and locations in Theorem 3.2 coincides with the asymptotic distribution of heights and locations of the suitably normalized maxima in a *sequence* of independent standard normal variables.

4. High level crossings and maxima for dependent processes. High level crossings for two or more dependent processes occur in some practical situations, e.g. in stochastic models for alarm systems in which a stochastic process X_1 is predicted by means of some other process X_2. A *catastrophe* and an *alarm* then correspond to high level crossings by $X_1(t)$ and $X_2(t)$, respectively.

One remarkable feature of dependent Gaussian processes is that regardless of how high the correlation—short of perfect correlation—the number of high level crossings by the two processes are asymptotically independent. This has been shown by Lindgren (1974) for covariance functions that are twice differentiable.

The technique used in Section 2 of this paper is directly applicable in the two-process case and gives, as will be shown below, the weak convergence of the flows of ε-upcrossings towards two independent Poisson processes.

Let X_1 and X_2 be two stationary, continuous Gaussian processes with mean zero, whose covariance functions r_1 and r_2 satisfy conditions (1.1) and (1.2) with

$$r_k(t) = 1 - |t|^{\alpha_k} C_k(t) + o(|t|^{\alpha_k} C_k(t)) , \qquad t \to 0 ,$$

where $0 < \alpha_k \leqq 2$ and $C_k(t)$ is slowly varying, $k = 1, 2$. The exponents α_1 and α_2 may very well be different.

Also suppose that the cross-correlation function

$$r_{12}(t) = C(X_1(s), X_2(s + t))$$

satisfies

$$\sup_t |r_{12}(t)| < 1 \tag{4.1}$$

and

$$r_{12}(t) \log t \to 0 , \quad t \to \infty , \qquad \text{or} \qquad \textstyle\int_{-\infty}^{\infty} r_{12}(t)^2\, dt < \infty . \tag{4.2}$$

Incidentally, we note that (4.2) implies that $r_{12}(t) \to 0$, $t \to \infty$.

Let x_1 and x_2 be positive numbers and define the levels u_{T,x_1} and u_{T,x_2} by (1.3):

$$u_{T,x_k} = (2 \log T)^{\frac{1}{2}} - \frac{\log x_k + \frac{1}{2} \log \log T + \log G_k(1/(2 \log T)^{\frac{1}{2}}) + \log (2\pi^{\frac{1}{2}}/H_{\alpha_k})}{(2 \log T)^{\frac{1}{2}}} \tag{4.3}$$

where G_k and H_{α_k} are as in the preliminaries. For $\varepsilon > 0$ let

$$\xi_T{}^k(B) = \# \ \ \varepsilon\text{-upcrossings of the level } \ u_{T,x_k} \ \text{ by } \ X_k(t) : t \in T \cdot B$$

define the time-normalized point process of ε-upcrossings, and let further ξ^1 and ξ^2 be independent Poisson processes with intensities x_1 and x_2, respectively. Then we have the following theorem.

THEOREM 4.1. *If X_1 and X_2 are zero-mean, stationary and continuous Gaussian processes with covariance functions r_1 and r_2 satisfying* (1.1) *and* (1.2) *and with cross-correlation function r_{12} satisfying* (4.1) *and* (4.2) *then the point process on R^2, $\xi_T{}^1 \times \xi_T{}^2$, of ε-upcrossings converges weakly towards the product $\xi^1 \times \xi^2$ of the two independent Poisson processes.*

PROOF. The point is, as it was in the proof of Theorem 2.1, that the conditions (2.2) and (2.3) for weak convergence of point processes can be deduced from the standard Poisson limit theorem amended with a proof of the asymptotic independence of extremes, i.e. a proof of Lemma 2.3 or its alternative formulation (2.6).

Starting with the asymptotic independence we again let $U = \bigcup_{i=1}^n R_i$, $R_i = I_i^1 \times I_i^2$ be a finite union of two-dimentional rectangles, and set $I^1 = \bigcup_{i=1}^n I_i^1$, $I^2 = \bigcup_{i=1}^n I_i^2$. Also define

$$M^k(B) = \sup\{X_k(t) : t \in B\}$$

for $k = 1, 2$. Writing $u_{T,k}$ instead of u_{T,x_k} we have to show that

$$\begin{aligned}(4.4)\qquad P(M^1(T \cdot I^1) < u_{T,1},\, M^2(T \cdot I^2) < u_{T,2}) \\ - P(M^1(T \cdot I^1) < u_{T,1})P(M^2(T \cdot I^2) < u_{T,2}) \to 0\end{aligned}$$

as $T \to \infty$. Approximating over the discrete sets

$$A_T^k = \left\{\nu a G_k\left(\frac{1}{(2\log T)^{\frac{1}{2}}}\right) : \nu \quad \text{integer}\right\}, \quad k = 1, 2,$$

and letting

$$M_a^k(B) = \sup\{X_k(t) : t \in A_T^k \cap B\},$$

it follows as in the proof of Lemma 2.3 that

$$\begin{aligned}\lim_{a\downarrow 0} \limsup_{T\to\infty} |P(M_a^1(T \cdot I^1) < u_{T,1},\, M_a^2(T \cdot I^2) < u_{T,2}) \\ - P(M^1(T \cdot I^1) < u_{T,1},\, M^2(T \cdot I^2) < u_{T,2})| = 0.\end{aligned}$$

It is only the asymptotic independence of the maxima of the discrete processes that requires some extra calculation, i.e. that

$$\begin{aligned}(4.5)\qquad P(M_a^1(T \cdot I^1) < u_{T,1},\, M_a^2(T \cdot I^2) < u_{T,2}) \\ - P(M_a^1(T \cdot I^1) < u_{T,1})P(M_a^2(T \cdot I^2) < u_{T,2}) \to 0,\end{aligned}$$

when $T \to \infty$. By the same inequality as referred to in the proof of Lemma 2.3 the difference in (4.5) is bounded by a constant times

$$\begin{aligned}(4.6)\qquad \sum \Big\{\frac{|r_{12}(s_i - t_j)|}{(1 - r_{12}(s_i - t_j)^2)^{\frac{1}{2}}} \exp(-\tfrac{1}{2}Q_{ij}) : \\ s_i \in A_T^1 \cap (T \cdot I^1),\, t_j \in A_T^2 \cap (T \cdot I^2)\Big\},\end{aligned}$$

where

$$Q_{ij} = \frac{1}{1 - r_{12}(s_i - t_j)^2}(u_{T,1}^2 - 2r_{12}(s_i - t_j)u_{T,1}u_{T,2} + u_{T,2}^2).$$

Putting $u_T = \min(u_{T,1}, u_{T,2})$ we see that

$$Q_{ij} \geqq \frac{u_T^2}{1 - r_{12}(s_i - t_j)^2}(2 - 2r_{12}(s_i - t_j)) \geqq \frac{2u_T^2}{1 + |r_{12}(s_i - t_j)|},$$

and, since furthermore, $|r_{12}(s_i - t_j)|$ is bounded away from 1 by condition (4.1) we get that (4.6) is bounded by a constant times

$$\sum |r_{12}(s_i - t_j)| \exp\left\{-\frac{u_T^2}{1 + |r_{12}(s_i - t_j)|}\right\},$$

where s_i and t_j range over the discrete sets $A_T^1 \cap (T \cdot I^1)$ and $A_T^2 \cap (T \cdot I^2)$, respectively. This sum is of the same type as the bound in Lemma 2.3, the main differences being that the ranges of s_i and t_j are not necessarily compatible and that the sets $T \cdot I^1$ and $T \cdot I^2$ may be nondisjoint. We have therefore to exercise a little more finesse when estimating it.

Let the number of s_i's and t_j's be bounded by

$$n_k = \frac{T \cdot A}{aG_k(1/(2\log T)^{\frac{1}{2}})}, \qquad k = 1, 2,$$

where

$$A = 2\sup\{|t| : t \in I^1 \cup I^2\}.$$

For each fixed s_i we sum over t_j, extending the sum over the set $\{t_j : t_j \in A_T{}^2, |s_i - t_j| \leqq T \cdot A\}$. If we write

$$\rho_j = \sup\{|r_{12}(s - t_j)| : 0 \leqq s \leqq aG_2(1/(2\log T)^{\frac{1}{2}})\},$$

we can use stationarity to obtain the new bound

$$n_1 \sum \left\{\rho_j \exp\left\{-\frac{u_T{}^2}{1+\rho_j}\right\} : t_j \in A_T{}^2, |t_j| \leqq T \cdot A\right\}. \tag{4.8}$$

We now split the sum in (4.8) into two parts,

$$\textstyle\sum_{|t_j| \leqq q(T)} + \sum_{|t_j| > q(T)}, \tag{4.9}$$

where $q(T) \to \infty$ when $T \to \infty$ with a rate to be specified later. Since, by (4.1), there is a constant $c > 0$ such that $\rho_j \leqq 1 - c$ for all j, we can estimate the first sum by

$$n_1 \frac{q(T)}{aG_2(1/(2\log T)^{\frac{1}{2}})} \exp\left\{-\frac{u_T{}^2}{2-c}\right\}.$$

Writing $G(T) = \min(G_1(1/(2\log T)^{\frac{1}{2}}), G_2(1/(2\log T)^{\frac{1}{2}}))$ this is not greater than

$$\frac{AT^2 e^{-u_T^2}}{a^2 \log T \cdot (G(T))^2} \cdot \frac{q(T)\log T}{T} \cdot \exp\left(u_T{}^2 \cdot \frac{1-c}{2-c}\right). \tag{4.10}$$

Now it follows from Remarks 1.3 and 1.4 that

$$\frac{T^2 e^{-u_{T,k}^2}}{\log T \cdot (G_k(1/(2\log T)^{\frac{1}{2}}))^2}$$

is bounded for $k = 1, 2$, and that $u_{T,1}^2 - u_{T,2}^2 = \text{constant} + o(1)$. This implies that the first factor in (4.10) is bounded and we get the bound

$$\frac{q(T)\log T}{T} \cdot T^{2\cdot(1-c)/(2-c)} = q(T)\log T \cdot T^{-c/(2-c)},$$

which tends to zero when $T \to \infty$ if, for example, $q(T) = T^\beta$ with $0 < \beta < c/(2-c)$, which concludes the estimation of the first sum in (4.9).

To estimate the second sum we can proceed exactly as in Section 2. There is no difficulty in substituting G_1 and G_2 in the appropriate places and to replace $u_{T,x}^2$ by $u_{T,1}^2$ and $u_{T,2}^2$ since, as noted, $u_{T,1}^2 - u_{T,2}^2$ is bounded.

This remark finishes the proof of (4.4). It then follows as in the proof of Theorem 2.1 that condition (2.3) is satisfied.

To prove (2.2) we again proceed as in the proof of Theorem 2.1 and partition as in Step 3 of that proof. We get

$$P(\xi_T^1 \times \xi_T^2(I_{mj}^1 \times I_{mk}^2) > 1) \leqq P(\xi_T^1(I_{mj}^1) \geqq 1, \xi_T^2(I_{mk}^2) > 1) \\ + P(\xi_T^1(I_{mj}^1) > 1, \xi_T^2(I_{mk}^2) \geqq 1).$$

That the terms to the right are of the order $O(m^{-2}) \cdot O(m^{-1}) = o(m^{-2})$ follows from the independent-maximum result as it does for one process, but we omit the details.

REMARK 4.2. An immediate consequence of Theorem 4.1 is that the heights and locations of the highest local ε-maxima in the two processes are asymptotically independent.

5. Ordinary crossings and maxima under regularity conditions. The notions of ε-upcrossings and ε-maxima were introduced to prevent nuisance jitter from hiding the overall structure of crossings and extremes for nondifferentiable processes. In this section we will see that if the number of upcrossings per time unit has finite expectation then Theorem 2.1 will hold with ordinary upcrossings substituted for ε-upcrossings. Similarly, Theorem 3.2 will hold for ordinary maxima if the expected number of local maxima is finite.

We first state a version of Theorem 2.1 for, what is called by Berman (1971), a standard process. Suppose that the covariance function r satisfies

$$r(t) = 1 - \lambda_2 t^2/2 + o(t^2), \qquad t \to 0, \tag{5.1}$$

so that the average number of ordinary crossings per time unit is finite. Define, for $0 < y \leqq x$,

$$\begin{aligned} \xi_T^0(B) &= \# \text{ upcrossings of the level } u_{T,x} \text{ by } X(t): t \in T \cdot B \\ \eta_T^0(B) &= \# \text{ upcrossings of the level } u_{T,y} \text{ by } X(t): t \in T \cdot B \end{aligned} \tag{5.2}$$

where

$$u_{T,z} = (2 \log T)^{\frac{1}{2}} - \frac{\log z + \log (2\pi/\lambda_2^{\frac{1}{2}})}{(2 \log T)^{\frac{1}{2}}}.$$

Then the following theorem is a consequence of Theorem 2.1 and the fact that as $T \to \infty$ every regular upcrossing of $u_{T,x}$, $u_{T,y}$ is also an ε-upcrossing.

THEOREM 5.1. *Let X be a zero-mean, stationary and continuous Gaussian process with covariance function r satisfying* (5.1) *and* (1.2). *Then $\xi_T^0 \times \eta_T^0$ defined by* (5.2) *converges weakly towards $\xi \times \eta$ where ξ is a Poisson process with intensity x and η is a binomial thinning of ξ with deletion probability* $1 - y/x$.

As noted above, condition (5.1) implies that the number of upcrossings during a finite interval has finite expectation. Now assume that the process is continuously sample differentiable and that its normalized derivative is a standard process, i.e.

$$-r''(t) = \lambda_2 - \lambda_4 t^2/2 + o(t^2)\,, \qquad t \to 0\,, \tag{5.3}$$

where λ_4 is the fourth spectral moment. Then the expected number of local maxima per time unit is $(2\pi)^{-1}(\lambda_4/\lambda_2)^{\frac{1}{2}}$.

Define the overall maximum $M_1{}^0(T) = \sup\{X(t) : t \in [0, T]\}$ and its location $S_1{}^0(T) = \inf\{t \in [0, T] : X(t) = M_1{}^0(T)\}$ as in Section 3, and let $M_2{}^0(T)$ and $S_2{}^0(T)$ be the height and location of the second local maximum of $X(t)$ in $[0, T]$. The following theorem takes the place of Theorem 3.2.

THEOREM 5.2. *If X is a stationary, zero-mean Gaussian process with continuously differentiable sample paths and with covariance function r that satisfies* (5.3) *and* (1.2), *then*

$$P(a_T(M_1{}^0(T) - b_T) \leqq \mu_1, S_1{}^0(T) \leqq \sigma_1 T, a_T(M_2{}^0(T) - b_T) \leqq \mu_2, S_2{}^0(T) \leqq \sigma_2 T)$$
$$\to \sigma_1\sigma_2 e^{-e^{-\mu_2}}(1 + e^{-\mu_2} - e^{-\mu_1})\,, \qquad T \to \infty\,,$$

where

$$a_T = (2\log T)^{\frac{1}{2}}$$
$$b_T = (2\log T)^{\frac{1}{2}} - \log(2\pi/\lambda_2^{\frac{1}{2}})/(2\log T)^{\frac{1}{2}}\,.$$

PROOF. The main step in the proof of Theorem 3.2 is the use of Theorem 2.4 together with the transition from the local ε-maxima $M_1(T)$ and $M_2(T)$ to the number of ε-upcrossings, as is demonstrated in inequalities (3.2)—(3.3) and (3.5). The same approximation technique works here. Only replace the number of ε-upcrossings $\xi_{T,z}$ by the number of ordinary upcrossings $\xi^0_{T,z}$ in the appropriate places. The term $P(N_{T,x_m} > \xi_{T,x_m}([0, \sigma_1]))$, which tends to zero according to Lemma 2.2, and that appears in (3.3) and (3.5), should then be replaced by the probability that

$$N^0_{T,x_m} = \#\ \text{ interior local maxima of }\ X(t)\ \text{ above }\ u_{T,x_m}\ \text{ in }\ [0, \sigma_1 T]$$

and

$$\xi^0_{T,x_m} = \xi^0_{T,x_m}([0, \sigma_1]) = \#\ \text{ upcrossings of }\ u_{T,x_m}\ \text{ in }\ [0, \sigma_1 T]$$

differ by at least one, and the probability $P(X(0) > u_{T,x_m})$. Thus we have to show that

$$P(|N^0_{T,x_m} - \xi^0_{T,x_m}| \geqq 1) \to 0$$

as $T \to \infty$. This can be seen as follows. For any $z > 0$ we have that either

$$N^0_{T,z} - \xi^0_{T,z} \geqq 0$$

or

$$N^0_{T,z} - \xi^0_{T,z} = -1 \qquad \text{and} \qquad X(\sigma_1 T) > u_{T,z}\,.$$

Thus

$$P(|N^0_{T,z} - \xi^0_{T,z}| \geqq 1) \leqq E(|N^0_{T,z} - \xi^0_{T,z}|)$$
$$\leqq E(N^0_{T,z} - \xi^0_{T,z}) + 2P(X(\sigma_1 T) > u_{T,z})\,.$$

Here

$$E(\xi^0_{T,z}) = \frac{\sigma_1 T}{2\pi} \lambda_2^{\frac{1}{2}} \exp(-\tfrac{1}{2}u^2_{T,z}) = \sigma_1 z \cdot (1 + o(1)), \qquad T \to \infty,$$

while the expected number of interior local maxima above the level $u = u_{T,z}$ is

$$\begin{aligned} E(N^0_{T,z}) = \frac{\sigma_1 T}{2\pi} (\lambda_4/\lambda_2)^{\frac{1}{2}}\{1 - \Phi(u(\lambda_4/(\lambda_4 - \lambda_2^2))^{\frac{1}{2}}) \\ + \lambda_2(2\pi/\lambda_4)^{\frac{1}{2}}\phi(u)\Phi(u\lambda_2/(\lambda_4 - \lambda_2^2)^{\frac{1}{2}})\}, \end{aligned}$$

cf. Cramér and Leadbetter (1967), page 247, equation (11.6.14).

Since, for some constant $K > 0$,

$$\begin{aligned} \frac{\sigma_1 T}{2\pi} (\lambda_4/\lambda_2)^{\frac{1}{2}}\{1 - \Phi(u(\lambda_4/(\lambda_4 - \lambda_2^2))^{\frac{1}{2}})\} &\leqq K \cdot \frac{T}{u} \phi(u(\lambda_4/(\lambda_4 - \lambda_2))^{\frac{1}{2}}) \\ &\leqq K \cdot \frac{T}{u} \phi(u) \to 0, \qquad T \to \infty, \end{aligned}$$

and

$$\begin{aligned} \frac{\sigma_1 T}{2\pi} (\lambda_4/\lambda_2)^{\frac{1}{2}} \cdot \lambda_2(2\pi/\lambda_4)^{\frac{1}{2}}\phi(u)\Phi(u\lambda_2/(\lambda_4 - \lambda_2^2)^{\frac{1}{2}}) &= \frac{\sigma_1 T}{2\pi} \lambda_2^{\frac{1}{2}} \exp(-\tfrac{1}{2}u^2)(1 + o(1)) \\ &= \sigma_1 z \cdot (1 + o(1)), \qquad T \to \infty, \end{aligned}$$

we have that

$$E(N^0_{T,z} - \xi^0_{T,z}) \to 0, \qquad T \to \infty.$$

We can now finish the proof as in Theorem 3.2, estimating the remaining probabilities by means of Theorem 5.1.

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