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A LOCAL TIME FOR A STORAGE PROCESS[1]

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Content process X of a continuous store satisfies $X_t = X_0 + A_t - \int_0^t r(X_s)\,ds$, $t \geqq 0$. Here, A has nonnegative stationary independent increments, and r is a nondecreasing continuous function. The solution X is a Hunt process. Paper considers the local time L of X at 0. L may be the occupation time of $\{0\}$ if the latter is not zero identically. The more interesting case is where the occupation time of $\{0\}$ is zero but 0 is regular for $\{0\}$; then L is constructed as the limit of a sequence of weighted occupation times of $\{0\}$ for a sequence of Hunt processes X^n approximating X. The λ-potential of L is computed in terms of the Lévy measure of A and the function r.

1. Introduction. Throughout this paper $\mathbb{R}_+ = [0, \infty)$, $\mathscr{R}_+$ is the Borel subsets of $\mathbb{R}_+$, and $\mathbb{N} = \{0, 1, 2, \cdots\}$. We are given a function r on $\mathbb{R}_+$ which vanishes at the origin, is positive elsewhere, and is nondecreasing and continuous on $(0, \infty)$ [we are allowing a jump at 0]. Also given is a probability space $(W, \mathscr{G}, P)$ over which there is defined a stochastic process $A = \{A_t;\, t \geqq 0\}$ having nondecreasing right continuous paths of the pure jump type, and with stationary independent increments. Then,

$$E[e^{-\lambda A_t}] = \exp[-t \textstyle\int_0^\infty (1 - e^{-\lambda y})\beta(dy)] \tag{1.1}$$

for all $t \geqq 0$, $\lambda \geqq 0$; here β, called the Lévy measure of A, is a nonnegative σ-finite measure concentrated on $(0, \infty)$ and satisfying

$$\textstyle\int_0^\infty (y \wedge 1)\beta(dy) < \infty\,. \tag{1.2}$$

We define $\mathscr{G}_t = \sigma(A_s : s \leqq t)$ and put

$$\Omega = \bar{\mathbb{R}}_+ \times W\,, \qquad \mathscr{H}^0 = \bar{\mathscr{R}}_+ \times \mathscr{G}\,, \qquad \mathscr{H}_t^0 = \bar{\mathscr{R}}_+ \times \mathscr{G}_t\,. \tag{1.3}$$

For each $x \in \bar{\mathbb{R}}_+$ we write ε_x for the Dirac measure concentrating its unit mass at x and define

$$P^x = \varepsilon_x \times P\,. \tag{1.4}$$

We let $\mathscr{H}$ be the completion of $\mathscr{H}^0$ with respect to the family of probabilities $\{P^x : x \in \bar{\mathbb{R}}_+\}$ and define $\mathscr{H}_t$ to be the relative completion of $\mathscr{H}_t^0$ in $\mathscr{H}$ with respect to the same family. It is now clear that $\{\mathscr{H}_t;\, t \geqq 0\}$ is a right-continuous nondecreasing family of "complete" σ-algebras.

For each $\omega = (x, w) \in \Omega$ we consider the equation

$$X_t(x, w) = x + A_t(w) - \textstyle\int_0^t r(X_s(x, w))\,ds\,, \qquad t \geqq 0\,. \tag{1.5}$$

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It was shown in [3] and [4] that there is a unique solution, denoted by $X_t(\omega)$ which is right continuous and has left hand limits everywhere. We will shortly sketch the construction of this solution; but first we complete the stochastic description.

If not already rich, W can be enlarged to have a family of shift operators $\varphi_t\colon W \to W$ satisfying

$$A_{t+s}(w) = A_t(w) + A_s(\varphi_t w) . \tag{1.6}$$

Then, we define the shifts $\theta_t\colon \Omega \to \Omega$ by

$$\theta_t(x, w) = (X_t(x, w), \varphi_t w) \tag{1.7}$$

for every $t \geqq 0$ and $\omega = (x, w) \in \Omega$. It was shown in [3] and [4] that

$$X = (\Omega, \mathscr{H}, \mathscr{H}_t, X_t, \theta_t, P^x) \tag{1.8}$$

is a Hunt process.

To go back to the construction of the solution to (1.5), of which we will make repeated use, first suppose the Lévy measure β of A is finite. Then, for almost all $w \in W$, $t \to A_t(w)$ has only finitely many jumps in any finite interval. For any exceptional w we put $X_t(x, w) = +\infty$ for all $x \in \mathbb{R}_+$. For a "good" w, if $t_1, t_2, \cdots$ are the successive jump times of $t \to A_t(w)$ with $a_1, a_2, \cdots$ the corresponding magnitudes, then

$$x_t = q(x_{t_n}, t - t_n) , \qquad t_n \leqq t < t_{n+1} , \tag{1.9}$$

and

$$x_{t_{n+1}} = q(x_{t_n}, t_{n+1} - t_n) + a_{n+1} , \qquad n \in \mathbb{N} \tag{1.10}$$

(where $t_0 = 0$, $x_0 = x$, and q is the solution of (1.5) for $A \equiv 0$, that is,

$$q(x, t) = \inf\{y > 0 \colon \textstyle\int_y^x (1/r(z))\, dz \leqq t\}) \tag{1.11}$$

define the unique solution of (1.5) and $X_t(x, w) = x_t$ for $t < \infty$ and we set $X_\infty(x, w) = \infty$. Finally, for $x = +\infty$, we put $X_t(\infty, w) = +\infty$ for all t.

If $\beta(\mathbb{R}_+) = +\infty$, $t \to A_t(w)$ has infinitely many jumps in any open interval and the above construction does not work. In this case, we employ an approximation procedure as follows. For any $n \in \mathbb{N}$, $\beta([1/n, \infty)) < \infty$ and the measure β_n defined by

$$\beta_n(B) = \beta(B \cap [1/n, \infty)) , \qquad B \in \mathscr{R}_+ \tag{1.12}$$

is the Lévy measure of the pure jump nondecreasing Lévy process

$$A_t^n = \textstyle\sum_{s \leqq t} (A_s - A_{s-}) I_{\{A_s - A_{s-} \geqq 1/n\}} , \qquad t \geqq 0 , \tag{1.13}$$

over the probability space $(W, \mathscr{G}, P)$. Since the Lévy measure β_n of A^n is finite, the construction of the preceding paragraph goes through to obtain a solution $X_t^n(x, w)$ to

$$x_t = x + A_t^n(w) - \textstyle\int_0^t r(x_s)\, ds , \qquad t \geqq 0 \tag{1.14}$$

for every $\omega = (x, w) \in \Omega$. Then, letting $\theta_t^n(x, w) = (X_t^n(x, w), \varphi_t w)$, with $\mathscr{H}$, $\mathscr{H}_t$, P^x as before, we obtain that

$$X^n = (\Omega, \mathscr{H}, \mathscr{H}_t, X_t^n, \theta_t^n, P^x) \tag{1.15}$$

is a Hunt process. This was shown in [3] along with some further results concerning the properties of X^n. The next step, which we put without proof for ease of referencing, obtains the solution X for arbitrary σ-finite Lévy measures β. Proof may be found in [4].

(1.16) PROPOSITION. *For any $\omega \in \Omega$, as $n \to \infty$, $X_t^n(\omega)$ increases to $X_t(\omega)$ which is the unique solution to* (1.5). *Further, the convergence is almost surely uniform over finite intervals.*

In this paper we are interested in the emptiness of the store, namely the set $\{t: X_t = 0\}$. With this purpose in mind, we had already standardized the input process A and the release function r. Somewhat more generally, the input will be of the form $A_t + a, t$ for some $a \geqq 0$. But the drift component can easily be absorbed into the function r by redefining r and changing the position of the point zero. One more condition which we will be putting on r is to insure against X never becoming zero (otherwise we have no problem left). The condition is that

$$m(x) = \int_{(0,x)} \frac{1}{r(y)}\, dy < \infty \tag{1.17}$$

for some $x > 0$ (which implies that $m(x) < \infty$ for all $x \in \mathbb{R}_+$). Noting that $m(x)$ is the time required to empty a store of initial content x in the absence of further inputs, what we impose is no restriction. This condition, however, is not sufficient for insuring that X ever reach 0 (see below, case 4).

Our approach to investigating the local behavior of X at 0 is via a study of the hitting time of 0, local times at 0, and the inverses of those local times. In Section 2 we concentrate on the "time to emptiness," namely the hitting time $S = \inf\{t > 0: X_t = 0\}$. We compute its transform $f^\lambda(x) = E^x[\exp(-\lambda S)]$ for all x and λ, and give some approximations. In Section 3 we consider the problem of constructing local times. By a local time at 0 we mean a continuous additive functional whose support is the singleton $\{0\}$. It exists if and only if 0 is regular for $\{0\}$, i.e., if and only if $S = 0$ P^0-almost surely, which is also equivalent to having $f^\lambda(0) = 1$ for some (and therefore all) λ. Finally, in Section 4, we consider the inverses of the local times constructed. These inverses are increasing Lévy processes, and we identify their exponents.

The following is a cross-file account of our results concerning the "emptiness set" $E = \{t: X_t = 0\}$. We write $f(x) = f^1(x) = E^x[e^{-S}]$. Depending on f and the Lévy measure β of the input process A, there are four cases:

(1) $\beta(\mathbb{R}_+) < \infty$;
(2) $\beta(\mathbb{R}_+) = +\infty$ but $\int (1 - f)\, d\beta < \infty$;
(3) $\beta(\mathbb{R}_+) = +\infty$, $\int (1 - f)\, d\beta = +\infty$, but $f(0) = 1$;
(4) $\beta(\mathbb{R}_+) = +\infty$, $\int (1 - f)\, d\beta = +\infty$, $f(0) < 1$.

In the first case the input is a compound Poisson process. If the store is empty, it stays empty for a positive time, namely, until the time of the first input. Then a nonempty period starts and lasts for some time, and so on. The empty and nonempty intervals alternate; lengths of the empty periods are independent and identically distributed exponential random variables; and the lengths of the nonempty periods are independent and identically distributed variables independent of the empty-interval lengths. In this case the local time at 0 is simply the occupation time process $L = \{L_t;\, t \geqq 0\}$ with

$$L_t = \int_0^t I_{\{0\}}(X_s)\, ds\,, \qquad t \geqq 0\,. \tag{1.18}$$

In the second case, the set E does not contain any open intervals; it has an empty interior, and its every point is a limit point of E (namely, E is a generalized Cantor set). However, the Lebesgue measure of E is positive and the local time at 0 is still the occupation time as defined in (1.18). The technique we use to get this result is as follows. We compute, directly, the λ-potential of

$$L_t^n = \int_0^t I_{\{0\}}(X_s^n)\, ds \tag{1.19}$$

(to which case 1 applies); then express the same λ-potential in terms of the potential operator of X^n by using a general formula from Blumenthal and Getoor [1]; show that the latter converges to a nonzero potential; and observe that L^n defined by (1.19) converges to the occupation time of 0 by X. A simple sufficient condition for this case to hold is that $\beta(\mathbb{R}_+) = +\infty$ and $\lim_{x\downarrow 0} r(x) > 0$. As such this case subsumes the results of Kendall [5] and Prabhu and Rubinovitch [9] who assumed that $r = 1$ on $(0, \infty)$.

By far the most interesting case is the third. The set E has the same structure as in case 2 except that its Lebesgue measure is now zero. Since $f(0) = 1$, 0 is regular for $\{0\}$ and a local time L at 0 exists by the general result in Blumenthal and Getoor [2, Chapter V, Theorem (3.13)], or Maisonneuve [8]. But L is no longer an occupation time. In [2] and [8], it is characterized as an additive functional whose increasing is restricted to the set $E = \{t : X_t = 0\}$. The familiar result of Lévy [7] displays the Brownian local time as a right-derivative at $x = 0$ of the occupation times. Without some such connection, one may question whether the term "local time" is fully appropriate. Our construction of L in this case is to partially answer this. We find suitable constants c_n increasing to infinity, and consider the sequence of local times $c_n L^n$ (the nth for the process X^n) with L^n defined as in (1.19). We show that this sequence has a subsequence which converges (almost surely) to a continuous additive functional of X, and by identifying the λ-potential of the limiting functional, we show that it must be the local time L at 0 (normalized). The construction looks close to the usual ones starting with a given potential (cf. Blumenthal and Getoor [2, page 162 ff.]). Our method of using local times of X^n to get a local time for X is reminiscent of the method used by Stone [10] for diffusions.

Finally, case 4 is devoid of any interest, because $f(0) < 1$ implies that the hitting time S is infinite almost surely (see Remark (2.17)). Then, the set E is empty except possibly for the point $t = 0$ which is in E if $X_0 = 0$. Hence, effectively, we only have three cases.

2. Time to emptiness. Considering the content process X specified by (1.8) we define

$$S = \inf\{t > 0: X_t = 0\}. \tag{2.1}$$

In this section we will compute

$$f^\lambda(x) = E^x[e^{-\lambda S}], \tag{2.2}$$

for all $x \in \mathbb{R}_+$ and $\lambda \geqq 0$. Our computation of f^λ will first be for processes X corresponding to input processes A with finite Lévy measures. Then, the general case will be obtained by a passage to the limit using the construction of X as the limit of processes X^n each of which had, corresponding to its input processes, a finite Lévy measure.

(2.3) PROPOSITION. *Suppose $b = \beta(\mathbb{R}_+) < \infty$. Then, $f^\lambda(0) = 1$, and for $x > 0$, $f^\lambda(x)$ satisfies*

$$\begin{aligned} f^\lambda(x) = {} & \exp[-(\lambda + b)m(x)] \\ & + \textstyle\int_0^{m(x)} ds \exp[-(\lambda + b)s] \int_0^\infty \beta(dy) f^\lambda(y + q(x, s)). \end{aligned} \tag{2.4}$$

There is one and only one bounded solution of (2.4); *it is*

$$f^\lambda(x) = \textstyle\int_0^\infty R^\lambda(x, dy) \exp[-(\lambda + b)m(y)], \qquad x > 0, \tag{2.5}$$

where $R^\lambda = \sum_{n=0}^\infty (Q^\lambda)^n$ is the potential operator corresponding to the sub-Markovian kernel Q^λ given by

$$Q^\lambda g(x) = \textstyle\int_0^{m(x)} ds \exp[-(\lambda + b)s] \int_0^\infty \beta(dy) g(y + q(x, s)) \tag{2.6}$$

for all $x \geqq 0$ and bounded measurable functions g on $\mathbb{R}_+$.

REMARK. Note that $Q^\lambda(x, B)$ is the Laplace transform of the measure $Q(x, B, \cdot)$ where

$$Q(x, B, C) = P^x\{X_T \in B, T \in C, T < S\}, \qquad B, C \in \mathbb{R}_+,$$

wherein T is the time of first jump for X. Then, if $T_1, T_2, \cdots$ are the successive jump times of X, $R^\lambda(x, B)$ becomes the Laplace transform of the measure $R(x, B, \cdot)$ where

$$R(x, B, C) = E^x[\textstyle\sum_n 1_B(X_{T_n}) 1_C(T_n) I_{\{T_n < S\}}].$$

In particular, $R^0(x, \mathbb{R}_+)$ is the expected number, starting at x, of jumps of X before X reaches 0.

PROOF. Let T be the time of first jump for A. Since $b = \beta(\mathbb{R}_+) < \infty$, T is almost surely positive, has the exponential distribution with mean $1/b$, and the magnitude of the jump at T has the distribution β/b.

If $X_0 = 0$, then $X_t = 0$ for all $t \in [0, T)$ and therefore $S = 0$. Hence $f^\lambda(0) = 1$

for all $\lambda \geqq 0$. Next assume $x > 0$; on the set $\{T > m(x)\}$ we have $S = m(x)$ and on the set $\{T \leqq m(x)\}$ we have $S = T + S \circ \theta_T$ P^x—almost surely in both cases. Using the strong Markov property at T, these arguments yield the equation (2.4).

For fixed $\lambda \geqq 0$, the kernel defined by (2.6) is sub-Markovian, and $Q^\lambda(x, \mathbb{R}_+) \leqq Q^0(x, \mathbb{R}_+) = 1 - \exp[-bm(x)] < 1$ for all $x > 0$. It follows that R^λ is positive and that $R^\lambda(x, \mathbb{R}_+) \leqq R^0(x, \mathbb{R}_+) < \infty$ for all $x > 0$. The second term on the right side of (2.4) is $Q^\lambda f^\lambda(x)$; hence, (2.4) is equivalent to

$$f^\lambda = g + Q^\lambda f^\lambda \tag{2.7}$$

with an obvious definition for g. It is now clear that $R^\lambda g$ is a solution to (2.7); that is, (2.5) is a solution to (2.4).

To show that (2.5) is the only bounded measurable solution, note first that any other solution to (2.7) can differ from $R^\lambda g$ by only a measurable function h satisfying

$$h = Q^\lambda h\,, \qquad 0 \leqq h \leqq 1\,. \tag{2.8}$$

But this implies that $h = (Q^\lambda)^n h \leqq (Q^\lambda)^n 1$ whereas

$$\lim_{n\to\infty} (Q^\lambda)^n 1 = 0 \tag{2.9}$$

for all $\lambda > 0$ since

$$\sup_x Q^\lambda(x, \mathbb{R}_+) = \frac{b}{\lambda + b} < 1$$

so that $(Q^\lambda)^n 1 \leqq (b/(\lambda + b))^n 1 \to 0$. Hence $h = 0$ is the only solution of (2.8). □

From some points of view, especially if the function r and the measure β are of simple form, the following provides a more tractable characterization for f^λ. In general however, the solution of the integro-differential equation appearing next can best be obtained by the technique of the preceding proposition. We are writing D for the differential operator.

(2.10) COROLLARY. *Suppose $b = \beta(\mathbb{R}_+) < \infty$. Then, for any fixed $\lambda \geqq 0$, the function f^λ has a nonpositive continuous derivative on $(0, \infty)$ and satisfies*

$$-r(x)Df^\lambda(x) = \lambda f^\lambda(x) + \textstyle\int_0^\infty \beta(dy)[f^\lambda(x) - f^\lambda(x + y)] \tag{2.11}$$

for $x > 0$ with the boundary condition $f^\lambda(0) = 1$.

PROOF. Clearly the function m and therefore $x \to q(x, s)$ are differentiable with continuous derivatives on $(0, \infty)$. Thus, it follows from the form of Q^λ given by (2.6) that, if g is differentiable with a continuous derivative then so is $Q^\lambda g$, and noting that $m'(x) = 1/r(x)$ and $q(x, m(x)) = 0$, we obtain

$$\begin{aligned} r(x)DQ^\lambda g(x) = {} & \exp[-(\lambda + b)m(x)] \textstyle\int_0^\infty \beta(dy)g(y) \\ & + \textstyle\int_0^{m(x)} ds \exp[-(\lambda + b)s] \int_0^\infty \beta(dy)g'(y + q(x, s))q_1(x, s)r(x)\,, \end{aligned}$$

where q_1 is the partial derivative of q with respect to the first argument (and q_2 below with respect to the second).

Note that $m(q(x, s)) = m(x) - s$, and therefore $r(x)q_1(x, s) = -q_2(x, s)$. Putting this into the above formula and integrating by parts, we obtain

$$r(x)DQ^\lambda g(x) = \int \beta(dy)g(x + y) - (\lambda + b)Q^\lambda g(x) .$$

Replacing g by $(Q^\lambda)^{n-1}g$, we see that for $n \geqq 1$

$$r(x)D(Q^\lambda)^n g(x) = \int \beta(dy)(Q^\lambda)^{n-1}g(x + y) - (\lambda + b)(Q^\lambda)^n g(x) . \tag{2.12}$$

In particular, (2.12) holds for $g(x) = \exp[-(\lambda + b)m(x)]$, and since $R^\lambda g = \sum (Q^\lambda)^n g$ is finite, then (2.12) shows that $\sum D(Q^\lambda)^n g$ converges. Therefore, on $(0, \infty)$, $Df^\lambda = DR^\lambda g$ exists and is equal to $\sum D(Q^\lambda)^n g$. Taking the indicated sums in (2.12) we obtain the desired result.

Next we consider the general case where β is not finite. Considering the process X^n defined by (1.15) we note that the preceding applies to compute, for any $\lambda \geqq 0$, the function

$$f_n^\lambda(x) = E^x[e^{-\lambda S_n}] , \qquad x \geqq 0 , \tag{2.13}$$

where

$$S_n = \inf\{t > 0 : X_t^n = 0\} . \tag{2.14}$$

Once the f_n^λ are known, the next proposition may be used to compute f^λ for the general case.

(2.15) PROPOSITION. *The sequence of stopping times S_n is increasing, and*

$$\lim_{n\to\infty} S_n = S \qquad \textit{on } \{X_0 > 0\} .$$

Thus, for any $\lambda \geqq 0$ and $x > 0$,

$$f^\lambda(x) = \lim_{n\to\infty} f_n^\lambda(x) . \tag{2.16}$$

PROOF. It is clear that (2.16) follows from the first statement. Since $X_t^n \leqq X_t^{n+1}$ for all $n \in \mathbb{N}$ and $t \geqq 0$, we must have $S_n \leqq S_{n+1}$ for all $n \in \mathbb{N}$. Hence $\lim_n S_n = S_0$ exists and is a stopping time. Since $X_t^n \leqq X_t$ for all n, $S_0 \leqq S$. We next show the reverse inequality on $\{X_0 > 0\}$.

On the set $\{S_n < \infty\}$ we have $X_{S_n}^n = 0$; therefore, since the X^n increase, $X_{S_n}^m = 0$ on $\{S_n < \infty\}$ for all $m \leqq n$. By the quasi-left-continuity of X^m this implies that $X_{S_0}^m = \lim_n X_{S_n}^m = 0$ almost surely on $\{S_0 < \infty\}$ for any $m \in \mathbb{N}$. This in turn implies that

$$X_{S_0} = \lim_m X_{S_0}^m = 0 \qquad \text{on } \{S_0 < \infty\} .$$

Thus, on $\{0 < S_0 < \infty\}$, $S_0 \geqq S$.

We have shown that $S = S_0$ on $\{S_0 > 0\}$; this completes the proof after noting that on $\{X_0 > 0\}$ we have $S_n \geqq m(X_0)$ for all n so that $S_0 \geqq m(X_0) > 0$.

(2.17) REMARK. For any $w \in W$ and $t \geqq 0$, $X_t(x, w) \leqq X_t(y, w)$ whenever $x \leqq y$; therefore, $S(x, w) \leqq S(y, w)$ for $x \leqq y$. It follows that $x \to f^\lambda(x)$ is nonincreasing (and this is equally true of each f_n^λ). This fact, coupled with the monotonicity of the convergence of f_n^λ to f^λ implies that the convergence to f^λ

is uniform over $[\varepsilon, \infty)$ for any $\varepsilon > 0$. If $f^\lambda(0) = 1$ we can further take $\varepsilon = 0$. Otherwise, if $f^\lambda(0) < 1$ we shall shortly see that $f^\lambda(0) = 0 \neq \lim_n f_n{}^\lambda(0) = 1$.

In principle, this completely specifies f^λ. But it is useful to find alternative solutions and we need to get at least some bounds. The following is aimed at this. One recurring idea is that, if the release function r were to be replaced by a function $\hat{r} \leqq r$ then the solution $\hat{X}_t$ corresponding to $\hat{r}$ would be greater than the original X_t. This is easy to see from the interpretation of $r(x)$ as the rate of output when the content is x.

(2.18) PROPOSITION. *For any* $x > 0$, $y \geqq 0$, $\lambda > 0$

$$f^\lambda(x) - f^\lambda(x + y) \leqq y\varphi^\lambda \circ r(x) \tag{2.19}$$

where $\varphi^\lambda(c)$ *is the unique solution of*

$$cz = \lambda + \int \beta(dy)(1 - e^{-yz}) \tag{2.20}$$

for any $c > 0$, $\lambda > 0$.

PROOF. For each $x \in \mathbb{R}_+$ define $T_x = \inf\{t > 0 : X_t \in [0, x]\}$. Since the discontinuities of X are always upward jumps, $X(T_x) = x$ almost surely on $\{X_0 > x, T_x < \infty\}$. On the set $\{X_0 > x\}$ we have $S > T_x$ and therefore $S = T_x + S \circ \theta_{T_x}$. Applying the strong Markov property at T_x and noting that $X(T_x) = x$ almost surely on $\{X_0 > x\}$, we have

$$E^{x+y}[e^{-\lambda S}] = E^{x+y}[e^{-\lambda T_x}]E^x[e^{-\lambda S}], \tag{2.21}$$

for all $x, y \in (0, \infty)$.

Let $X_t^c(x, w)$ be the solution of the equation (1.5) with r replaced by r' defined as $r'(0) = 0$, $r'(u) = c$ for $u > 0$. Then, $X^c = (\Omega, \mathscr{H}, \mathscr{H}_t, X_t^c, \theta_t^c, P^x)$ is again a Hunt process and we define

$$S^c = \inf\{t > 0 : X_t^c = 0\}.$$

Then, for any $x, y > 0$ and $w \in W$

$$T_x(x + y, w) \leqq S^{r(x)}(y, w) \tag{2.22}$$

by the fact that $r \geqq r(x) = c$ on $[x, \infty)$. This implies that

$$E^{x+y}[e^{-\lambda T_x}] \geqq E^y[e^{-\lambda S^{r(x)}}]. \tag{2.23}$$

On the other hand, for the process X^c the expression (2.21) becomes

$$E^{x+y}[e^{-\lambda S^c}] = E^y[e^{-\lambda S^c}]E^x[e^{-\lambda S^c}]$$

which implies that

$$E^x[e^{-\lambda S^c}] = e^{-x\varphi^\lambda(c)} \tag{2.24}$$

for all $x \geqq 0$ for some constant $\varphi^\lambda(c)$.

Now putting (2.24) into (2.23) and that into (2.21) we get

$$f^\lambda(x + y) \geqq f^\lambda(x) \exp[-y\varphi^\lambda \circ r(x)]$$

from which (2.19) follows. That $\varphi^\lambda(c)$ satisfies (2.20) was shown by Kendall [5]. □

In fact the proof of Proposition (2.33) below can be given first for the special case $r(x) = c$ for $x > 0$ by using (2.24). This yields the estimate (2.30), and thus (2.20), without the appeal to Kendall's result.

It is clear that the solution $\varphi^\lambda(c)$ of (2.20) is finite. Therefore, (2.24) implies that $P^y\{S^c < \infty\} > 0$ for all y. Since $m(x) < \infty$ by hypothesis, $r(x) > 0$ for every $x > 0$, and (2.22) now implies that

(2.25) $$P^{x+y}\{T_x < \infty\} \geqq P^y\{S^{r(x)} < \infty\} > 0\,, \qquad x > 0\,.$$

At $x = 0$ these ideas yield the following information.

(2.26) COROLLARY. *If* $r(0+) = \lim_{x\downarrow 0} r(x) > 0$, *then* $f^\lambda(0) = 1$ *and*

(2.27) $$Df^\lambda(0) = \lim_{x\downarrow 0} \frac{1 - f^\lambda(x)}{x} \leqq \varphi^\lambda(r(0+)) < \infty\,.$$

If $r(0+) = 0$, *then*

(2.28) $$\lim_{x\downarrow 0} \frac{1 - f^\lambda(x)}{x} = +\infty\,.$$

PROOF. If $r(0+) = c > 0$ we have $S \leqq S^c$ and hence

(2.29) $$f^\lambda(x) \geqq \exp[-x\varphi^\lambda(c)]\,.$$

This shows that $f^\lambda(0) = 1$ and coupled with the monotonicity of f^λ shows that (2.27) holds.

In all cases, $S(x, w) \geqq m(x)$ for all x and w. Therefore, $f^\lambda(x) \leqq \exp[-\lambda m(x)]$, which implies that

(2.30) $$\liminf_{x\to 0} \frac{1 - f^\lambda(x)}{x} \geqq \frac{\lambda}{r(0+)}\,.$$

Hence, if $r(0+) = 0$, (2.28) holds. □

Unfortunately, (2.28) does not settle the question of whether $f^\lambda(0)$ is 1 or not when $r(0+) = 0$.

The following brings together a number of observations concerning this matter. In particular, this contains the statement that the case 4, where $f(0) < 1$, is totally without interest.

(2.31) PROPOSITION. *The following are equivalent*: (a) 0 *is regular for* $\{0\}$; (b) $f^\lambda(0) = 1$; (c) $f^\lambda(x) > 0$ *for some* $x > 0$; (d) $f^\lambda(x) > 0$ *for all* x; (e) $P^x\{S < \infty\} > 0$; (f) $P^x\{S < \infty\} > 0$ *for all* x.

PROOF. The only nontrivial implication is (e) $\Rightarrow$ (b). By Proposition 2.15, $S_n \nearrow S$ almost surely. Since β is infinite, $S_n < S$ almost surely; and on the set $\{S > S_n\}$ we have $S = S_n + S \circ \theta_{S_n}$. Thus,

(2.32) $$E^x[e^{-\lambda S} \mid \mathscr{H}_{S_n}] = \exp(-\lambda S_n) f^\lambda(X_{S_n})\,.$$

By the martingale convergence theorem, the left side converges to $\exp[-\lambda S]$, whereas, on the right side, the first factor converges to $\exp[-\lambda S]$. Hence, on the set $\{S < \infty\}$, we have $f^\lambda(X_{S_n}) \to 1$ almost surely. If (e) holds, then $\{S < \infty\}$ is not empty, and this shows that $f^\lambda(0) = 1$. □

(2.33) PROPOSITION. *For any fixed $\lambda \geqq 0$, f^λ is uniformly continuous and differentiable on $[0, \infty)$. The derivative Df^λ is continuous and satisfies*

$$(2.34) \qquad -r(x)Df^\lambda(x) = \lambda f^\lambda(x) + \int_0^\infty \beta(dy)[f^\lambda(x) - f^\lambda(x+y)]$$

for every $x > 0$. If $f^\lambda(0) < 1$, then $f^\lambda(x) = 0$ for all $x \geqq 0$. If $f^\lambda(0) = 1$, then $Df^\lambda(0)$ is finite or infinite according as $r(0+) > 0$ or $r(0+) = 0$.

PROOF. If $f^\lambda(0) < 1$ then $f^\lambda(x) = 0$ for all $x \geqq 0$ by the preceding proposition and all the claims are true. Suppose $f^\lambda(0) = 1$. Then, the convergence of f_n^λ to f^λ is uniform (by Remark 2.17), and each f_n^λ is continuous (by Corollary (2.10)). Hence f^λ is continuous. Thus, f^λ is uniformly continuous on the compact $[0, 1]$; and the estimate (2.19) shows that f^λ is uniformly continuous on $[1, \infty)$. Hence f^λ is uniformly continuous.

Concerning the differentiability, the situation at $x = 0$ is covered by Corollary (2.26). If β is finite, (2.34) follows from Proposition (2.10). Suppose β is not finite; Corollary (2.10) applies to f_n^λ and we can write

$$(2.35) \qquad -r(x)Df_n^\lambda(x) = \lambda f_n^\lambda(x) + \int \beta(dy)I_{[1/n,\infty)}(y)[f_n^\lambda(x) - f_n^\lambda(x+y)]$$

for all $x > 0$.

By Proposition (2.15), $f_n^\lambda \downarrow f^\lambda$, and by Proposition (2.18),

$$(2.36) \qquad 0 \leqq f_n^\lambda(x) - f_n^\lambda(x+y) \leqq \inf(1, cy)$$

for some constant $c < \infty$ for all $x \geqq \varepsilon > 0$ and $y \geqq 0$. In view of (1.2) the bounded convergence theorem applies to show that the last term of (2.35) converges to the last term in (2.34). This and (2.17) imply that the right-hand side of (2.35) converges to the right side of (2.34) uniformly for x in $[\varepsilon, \eta]$, $0 < \varepsilon < \eta$. Over $[\varepsilon, \eta]$ then, the left side of (2.35) converges to some function $-rg^\lambda$ uniformly. By (2.35) and the continuity of f_n^λ, the functions Df_n^λ are continuous; hence, using the bounded convergence theorem on the Riemann integrals $\int_{(\varepsilon,y)} Df_n^\lambda(x)\,dx$, it follows that f^λ is differentiable over $[\varepsilon, \eta]$. As the derivatives Df_n^λ converged to $-g^\lambda$, we must have $-g^\lambda = Df^\lambda$. □

The following is a supplementary result. Since f^λ is a regular excessive function, this shows its continuity on $(0, \infty)$ as a corollary.

(2.37) PROPOSITION. *Every regular excessive function is continuous on* $(0, \infty)$.

PROOF. Define $T = \inf\{t: X_t < x_0\}$ for $x_0 > 0$ fixed. By (2.25), $P^x\{T < \infty\} > 0$ for all $x > x_0$. Let g be a regular excessive function. Then, almost surely, $t \to g(X_t)$ is continuous everywhere $t \to X_t$ is. Pick $\omega = (x, w)$ such that $x > x_0$, $T(x, w) = t_0 < \infty$, and $t \to g(X_t(\omega))$ is continuous everywhere $t \to X_t(\omega)$ is. Since the only jumps of X are upward, $t \to X_t(\omega)$ is continuous and is equal to x_0 at t_0, and hence

$$\lim_{t \uparrow t_0} g \circ X_t(\omega) = g(x_0) = \lim_{t \downarrow t_0} g \circ X_t(\omega) . \tag{2.38}$$

Since $t \to X_t(\omega)$ has no downward jumps, and by the definition of $t_0 = \inf\{t: X_t(\omega) < x_0\}$, for some $t_1 > t_0$ and $x_1 < x_0$, the path $t \to X_t(\omega)$ achieves every value in the interval $[x, x_1]$ as t increases to t_1. This fact put together with (2.38) implies that g is continuous at x_0. Since $x_0 > 0$ is arbitrary this completes the proof.

We end this section by pointing out a deficiency in our computations. This concerns the probabilities

$$F(x) = P^x\{S < \infty\} , \qquad x \geqq 0 . \tag{2.39}$$

We have seen in Proposition (2.31) that 0 is regular for $\{0\}$ if and only if $F(x) > 0$ for some x, and therefore for all x. If β is finite, or if $r(0+) > 0$, we have $F(x) > 0$. The remaining case of doubt is when

$$\beta(\mathbb{R}_+) = \infty , \qquad r(0+) = 0 . \tag{2.40}$$

Since $m(x) < \infty$, we must of course have $r(x)/x \to \infty$. It follows from Proposition (2.33) that, with

$$g(x) = -DF(x) , \qquad n(x) = \beta((x, \infty)) ,$$

we have

$$r(x)g(x) = \int_0^\infty n(y)g(x+y)\, dy . \tag{2.41}$$

If this equation has a continuous bounded solution $g \neq 0$, then $F(x) > 0$ and 0 is regular for $\{0\}$. Otherwise, if the only continuous solution to this is $g = 0$, we have $F \equiv 0$.

A related problem of less interest concerns the recurrence properties of the point 0, namely, whether $F(x) = 1$ or not. In [3] we had given a solution to this problem when β is finite. There, we also have a complete solution of the limiting behavior when β is infinite for the case where $r(x) = cx$ (which is of no interest to us here—then $m(x) = \infty$ and $F(x) = 0$ for all x).

3. Local time at zero. We are interested in the existence and characterization questions for a local time at $x = 0$. It is known that, cf. Blumenthal and Getoor [2, Chapter V, Theorem (3.13)], a local time at $x = 0$ exists if and only if 0 is regular for $\{0\}$, and all local times at 0 are constant multiples of each other. In this section we will obtain the local times in the first three cases mentioned in the introduction. In view of Proposition (2.31) these are the cases where $P^x\{S < \infty\} > 0$ for some x, and the remaining case is totally without interest since $P^x\{S = \infty\} = 1$ for all x in that case 4. The main results of this section are Theorems (3.1), (3.10), (3.17), and (3.62). As in the preceding section we first consider the case $\beta(\mathbb{R}_+) < \infty$ and then take limits to obtain the case $\beta(\mathbb{R}_+) = \infty$.

(3.1) THEOREM. *If* $\beta(\mathbb{R}_+) < \infty$, *then* $L = \{L_t;\, t \geqq 0\}$ *defined by*

$$L_t = \int_0^t I_{\{0\}}(X_s)\, ds\,, \qquad t \geqq 0\,, \tag{3.2}$$

is a local time and its λ-*potential*

$$u^\lambda(x) = E^x \int_0^\infty \exp[-\lambda t]\, dL_t\,, \qquad \lambda > 0\,, \tag{3.3}$$

is given by

$$u^\lambda(x) = \frac{f^\lambda(x)}{\lambda + \int \beta(dy)(1 - f^\lambda(y))}\,, \qquad x \in \mathbb{R}_+\,. \tag{3.4}$$

PROOF. Let $\{L_t\}$ be defined by (3.2) and u^λ by (3.3). That it is a local time is evident once we show that $u^\lambda(x)$ is not identically zero. So, the only thing we need to show is (3.4).

Since $L_S = 0$ and $X_S = 0$ almost surely on $\{S < \infty\}$

$$\begin{aligned} u^\lambda(x) &= E^x \int_S^\infty e^{-\lambda t}\, dL_t \\ &= E^x[e^{-\lambda S} E^{X(S)} \int_0^\infty e^{-\lambda t}\, dL_t] = f^\lambda(x) u^\lambda(0) \end{aligned} \tag{3.5}$$

for all $x \in \mathbb{R}_+$. Let T be the time of first jump for A. On the set $\{X_0 = 0\}$, $L_t = t$ for all $t < T$ and thus

$$\begin{aligned} \int_0^\infty e^{-\lambda t}\, dL_t &= \int_0^T e^{-\lambda t}\, dt + \int_T^\infty e^{-\lambda t}\, dL_t \\ &= \frac{1}{\lambda}(1 - e^{-\lambda T}) + e^{-\lambda T}(\int_0^\infty e^{-\lambda t}\, dL_t) \circ \theta_T\,. \end{aligned}$$

Using the strong Markov property at T and the normality of X, we get

$$\begin{aligned} u^\lambda(0) &= \frac{1}{\lambda}[1 - \int_0^\infty ds\, be^{-bs}e^{-\lambda s}] + \int_0^\infty ds\, e^{-bs} \int_0^\infty \beta(dy) e^{-\lambda s} u^\lambda(y) \\ &= \frac{1}{\lambda + b}[1 + \int \beta(dy) u^\lambda(y)] \end{aligned} \tag{3.6}$$

where we wrote $b = \beta(\mathbb{R}_+)$. Putting (3.5) into (3.6) we obtain an equation involving $u^\lambda(0)$ only. Solving for $u^\lambda(0)$ out of that we get

$$u^\lambda(0) = [\lambda + \int (1 - f^\lambda)\, d\beta]^{-1}\,. \tag{3.7}$$

The conclusion follows from (3.7) and (3.5). □

Next we consider the cases where $\beta(\mathbb{R}_+) = +\infty$. The preceding theorem applies to show that, for each $n \in \mathbb{N}$,

$$L_t^n = \int_0^t I_{\{0\}}(X_s^n)\, ds\,, \qquad t \geqq 0 \tag{3.8}$$

defines a local time at zero for the Markov process X^n and its λ-potential is given as

$$u_n^\lambda(x) = E^x \int_0^\infty e^{-\lambda t}\, dL_t^n = [\lambda + \int_{1/n}^\infty (1 - f_n^\lambda)\, d\beta]^{-1} f_n^\lambda(x) \tag{3.9}$$

(where f_n^λ is as defined by (2.13)).

Below we write f, f_n, etc. for f^1, f_n^1, etc. for simplicity. We note in passing that $\int (1 - f^\lambda)\, d\beta$ is either finite for all $\lambda > 0$ or else is infinite for all $\lambda > 0$.

The following theorem settles the second case. We recall that, by Corollary (2.26), a simple sufficient criterion for the condition of this theorem is that $r(0+) > 0$.

(3.10) THEOREM. *If* $\int (1 - f)\, d\beta < \infty$, *then*

$$L_t = \int_0^t I_{\{0\}}(X_s)\, ds\,, \qquad t \geqq 0 \tag{3.11}$$

defines a local time whose λ-potential is

$$u^\lambda(x) = \frac{f^\lambda(x)}{\lambda + \int (1 - f^\lambda)\, d\beta}\,, \qquad x \in \mathbb{R}_+\,, \tag{3.12}$$

for all $\lambda > 0$.

PROOF. If $\beta(\mathbb{R}_+) < \infty$, this is already proved. Suppose now that $\beta(\mathbb{R}_+) = +\infty$ and observe that the finiteness of $\int (1 - f)\, d\beta$ implies that $f(0) = 1$, and therefore, 0 is regular for $\{0\}$. So, the existence of a local time is guaranteed, and we want to show that it is a constant multiple of L defined by (3.11); namely, we need to show that L does not vanish identically.

Since $X^n \uparrow X$ as $n \to \infty$, L_t^n decreases as n increases. By the bounded convergence theorem, we must have

$$L_t = \lim_n L_t^n \tag{3.13}$$

for all $t \geqq 0$. Therefore, u_n^λ decreases and

$$u^\lambda(x) = \lim_n u_n^\lambda(x) \qquad \text{for all } x \in \mathbb{R}_+,\ \lambda > 0\,. \tag{3.14}$$

By Proposition (2.15), as $n \to \infty$, $f_n^\lambda \downarrow f^\lambda$ on $(0, \infty)$ and therefore $I_{[1/n,\infty)}(1 - f_n^\lambda) \uparrow (1 - f^\lambda)$. By the condition of our theorem, the bounded convergence theorem applies to show that the last term of (3.9) converges to the right hand side of (3.12). This and (3.14) shows that (3.12) holds. Since $u^\lambda(0) > 0$, L does not vanish identically. □

(3.15) REMARK. The same proof shows also that, if $\int (1 - f)\, d\beta = +\infty$, then

$$\int_0^\infty I_{\{0\}}(X_s)\, ds = 0 \tag{3.16}$$

almost surely; that is, the occupation time of $\{0\}$ is zero.

In this case also there can be a local time at 0 and the next theorem shows its construction from the occupation times L^n defined by (3.8). Below, by the local time at 0 we mean the one whose λ-potential is equal to f for $\lambda = 1$.

(3.17) THEOREM. *Suppose*

$$\beta(\mathbb{R}_+) = +\infty\,, \qquad \int (1 - f)\, d\beta = +\infty\,, \qquad f(0) = 1\,. \tag{3.18}$$

For each $n \in \mathbb{N}$ *let*

$$c_n = 1 + \int_{1/n}^\infty (1 - f_n)\, d\beta\,. \tag{3.19}$$

Then, there exists a sequence $\mathbb{K}$ *such that for all* $t \geqq 0$.

$$L_t = \lim_{n\to\infty; n\in\mathbb{K}} c_n L_t^n \tag{3.20}$$

exists almost surely, and $L = \{L_t;\ t \geqq 0\}$ *is the local time for* X *at* $x = 0$. *Its* λ-*potential is*

$$u^\lambda(x) = f(x) - (\lambda - 1)U^\lambda f(x)\,, \qquad x \in \mathbb{R}_+\,; \tag{3.21}$$

here U^λ *is the* λ-*potential* (*or, resolvent*) *of the process* X.

Proof will be broken down to several lemmas. It will further be seen that $c_n L_t^n \to L_t$ in probability as $n \to \infty$.

(3.22) LEMMA. *Let* U_n^λ *be the* λ-*potential operator for the process* X^n *and set*

$$v_n^\lambda = f_n - (\lambda - 1)U_n^\lambda f_n\,, \qquad \lambda > 0\,, \tag{3.23}$$

Then, v_n^λ *is the* λ-*potential of* $c_n L^n$, *and as* $n \to \infty$, v_n^λ *approaches the right hand side of* (3.21). *Further, the convergence is monotone decreasing for* $\lambda \leqq 1$.

PROOF. It is shown in [1, page 52] that v_n^λ is the λ-potential of *the* local time, for the process X^n, satisfying $v_n^1(x) = f_n(x)$ for all $x \in \mathbb{R}_+$. Noting the definition (3.19) of c_n, $v_n^1 = c_n u_n^1$; and hence, from the uniqueness of local times up to multiplication by constants, v_n^λ must be the λ-potential of $c_n L^n$.

As $n \to \infty$, $f_n \downarrow f$ and $X_t^n \uparrow X_t$; hence, $f_n \circ X_t^n \downarrow f \circ X_t$ for all $t \geqq 0$ almost surely. Therefore, by the bounded convergence theorem,

$$U_n^\lambda f_n(x) = E^x \textstyle\int_0^\infty e^{-\lambda t} f_n(X_t^n)\,dt \downarrow E^x \int_0^\infty e^{-\lambda t} f(X_t)\,dt = U^\lambda f(x)\,. \tag{3.24}$$

Thus, $v_n^\lambda \to v^\lambda = f - (\lambda - 1)U^\lambda f$ as $n \to \infty$. □

It follows from [1] that

$$v^\lambda(x) = f(x) - (\lambda - 1)U^\lambda f(x) \tag{3.25}$$

is the λ-potential of the local time L at zero satisfying $v^1(x) = f(x)$. We will show that this L is the limit of $c_n L^n$ as $n \to \infty$ over some set $\mathbb{K} \subset \mathbb{N}$.

(3.26) LEMMA. *For any* $\lambda > 0$,

$$\lim_{m,n\to\infty} \sup_{t\geq 0} e^{-\lambda t} |v_n^\lambda(X_t^n) - v_m^\lambda(X_t^m)| = 0 \tag{3.27}$$

almost surely.

PROOF. It is sufficient to show (3.27) with $v_m^\lambda(X_t^m)$ replaced by $v^\lambda(X_t)$. It follows from [1, page 53] that we can write

$$v_n^\lambda(x) = b_n^\lambda f_n^\lambda(x)\,, \qquad v^\lambda(x) = b^\lambda f^\lambda(x)\,, \tag{3.28}$$

where b_n^λ, b^λ are positive constants where, in view of Lemma (3.22),

$$\lim_n |b_n^\lambda - b^\lambda| = 0\,. \tag{3.29}$$

Now, we can write (dropping the superscripts λ)

$$\begin{aligned}|v_n(X_t^n) - v(X_t)| &\leqq |b_n f_n(X_t^n) - bf_n(X_t^n)| + |bf_n(X_t^n) - bf(X_t^n)| \\ &\quad + |bf(X_t^n) - bf(X_t)| \\ &\leqq |b_n - b| + b\|f_n - f\| + b|f(X_t^n) - f(X_t)| \,.\end{aligned}$$

For any $\tau > 0$, $0 \leqq X_t - X_t^n \leqq A_t - A_t^n \leqq A_\tau - A_\tau^n = a_n$ for all $t \leqq \tau$. Hence,

$$(3.30)\qquad \begin{aligned}&\sup_{t\leqq\tau} |v_n(X_t^n) - v(X_t)| \\ &\qquad \leqq |b_n - b| + b\|f_n - f\| + b\sup_x [f(x) - f(x + a_n)] \,.\end{aligned}$$

As $n \to \infty$, $A_\tau^n \uparrow A_\tau$ almost surely and therefore the last term of (3.30) vanishes, by the uniform continuity of f^λ (cf. Proposition (2.33)), almost surely. This together with (3.29) and (2.17) shows that, almost surely,

$$(3.31)\qquad \lim_{n\to\infty} \sup_{t\leqq\tau} |v_n(X_t^n) - v(X_t)| = 0 \,.$$

Hence, noting that

$$(3.32)\qquad \sup_{t>\tau} e^{-\lambda t}|v_n(X_t^n) - v(X_t)| \leqq e^{-\lambda\tau} 2|b_n - b|$$

we obtain

$$(3.33)\qquad \lim_{n,m\to\infty} \sup_{t\geqq 0} e^{-\lambda t}|v_n^\lambda(X_t^n) - v^\lambda(X_t)| \leqq 2e^{-\lambda\tau}$$

almost surely. This completes the proof since τ can be taken arbitrarily large. □

The proof of the following lemma is fashioned after that of Theorem (3.8) in [2, page 162]. However, since our L^n are for different processes, our situation is somewhat more complicated.

(3.34) LEMMA. *Let, for each* $n \in \mathbb{N}$,

$$(3.35)\qquad B_t^n = \int_0^t e^{-\lambda s} c_n \, dL_s^n \,, \qquad t \geqq 0$$

for fixed $\lambda > 0$. *Then, for any* $\delta > 0$,

$$(3.36)\qquad \lim_{n,m\to\infty} \sup_x P^x\{\sup_{t\geqq 0} |B_t^n - B_t^m| > \delta\} = 0 \,.$$

PROOF. Note that $E^x[B_\infty^n] = v_n^\lambda(x)$ and clearly B_t^n is $\mathscr{H}_t$ measurable. So,

$$(3.37)\qquad E^x[B_\infty^n \mid \mathscr{H}_t] = B_t^n + e^{-\lambda t} v_n^\lambda(X_t^n) \,.$$

Hence, if we define

$$(3.38)\qquad M_t^n = B_t^n + e^{-\lambda t} v_n^\lambda(X_t^n) \,,$$

then $(\mathscr{H}_t, M_t^n, P^x)$ is a nonnegative martingale for each $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$, and

$$M_\infty^n = \lim_{t\to\infty} M_t^n = E^x[B_\infty^n \mid \mathscr{H}_\infty] = B_\infty^n$$

almost surely. From the extension of Kolmogorov's inequality to martingales,

$$(3.39)\qquad \begin{aligned}P^x\{\sup_t [M_t^n - M_t^m]^2 \geqq \delta^2\} &\leqq \delta^{-2} E^x\{[M_\infty^n - M_\infty^m]^2\} \\ &= \delta^{-2} E^x\{[B_\infty^n - B_\infty^m]^2\} \,.\end{aligned}$$

We next estimate this last term. Now,

$$
\begin{aligned}
K_{n,m} &= [B_\infty^n - B_\infty^m]^2 \\
&= 2 \int_0^\infty [dB_t^n - dB_t^m] \int_t^\infty [dB_s^n - dB_s^m] \qquad (3.40)\\
&= 2 \int_0^\infty [dB_t^n - dB_t^m][B_\infty^n \circ \theta_t^n - B_\infty^m \circ \theta_t^m]e^{-\lambda t}
\end{aligned}
$$

so that

$$
\begin{aligned}
E^x(K_{n,m}) &= 2E^x \int_0^\infty [dB_t^n - dB_t^m]e^{-\lambda t}[v_n^\lambda(X_t^n) - v_m^\lambda(X_t^m)] \\
&\leqq 2E^x \int_0^\infty e^{-\lambda t}|v_n^\lambda(X_t^n) - v_m^\lambda(X_t^m)|\, dB_t^n \\
&\qquad + 2E^x \int_0^\infty e^{-\lambda t}|v_n^\lambda(X_t^n) - v_m^\lambda(X_t^m)|\, dB_t^m \qquad (3.41)\\
&\leqq 2E^x[(B_\infty^n + B_\infty^m)(\sup_t e^{-\lambda t}|v_n^\lambda(X_t^n) - v_m^\lambda(X_t^m)|)] \\
&\leqq 2\{E^x[(B_\infty^n + B_\infty^m)^2]E^x[(\sup_t e^{-\lambda t}|v_n^\lambda(X_t^n) - v_m^\lambda(X_t^m)|)^2]\}^{\frac{1}{2}}
\end{aligned}
$$

where the last inequality used the Schwartz inequality. Noting that $(a+b)^2 \leqq 2a^2 + 2b^2$,

$$E^x[(B_\infty^n + B_\infty^m)^2] \leqq 2E^x[(B_\infty^n)^2] + 2E^x[(B_\infty^m)^2] \qquad (3.42)$$

whereas

$$
\begin{aligned}
E^x[(B_\infty^n)^2] &= 2E^x \int_0^\infty dB_t^n \int_t^\infty dB_s^n \\
&= 2E^x \int_0^\infty e^{-\lambda t}c_n\, dL_t^n \cdot e^{-\lambda t}v_n^\lambda(X_t^n) \\
&\leqq 2E^x \int_0^\infty e^{-2\lambda t}c_n\, dL_t^n \cdot v_n^\lambda(0) \qquad (3.43)\\
&= 2v_n^{2\lambda}(x)v_n^\lambda(0) \leqq 2v_n^\lambda(0)v_n^{2\lambda}(0) \\
&= 2b_n^\lambda b_n^{2\lambda}
\end{aligned}
$$

independent of x. Hence, from (3.42) and (3.43) and (3.29)

$$
\begin{aligned}
\lim_{n,m\to\infty} E^x[(B_\infty^n + B_\infty^m)^2] &\leqq \lim_{n,m\to\infty} [4b_n^\lambda b_n^{2\lambda} + 4b_m^\lambda b_m^{2\lambda}] \qquad (3.44)\\
&= 8b^\lambda b^{2\lambda} < \infty\,.
\end{aligned}
$$

Now it follows from (3.41), (3.44), and Lemma (3.26) that

$$E^x[K_{n,m}] \to 0 \qquad n, m \to \infty \qquad (3.45)$$

uniformly in x. This implies, through the definition (3.40) of $K_{n,m}$ and the estimate (3.39), that as $m, n \to \infty$

$$P^x\{\sup_t [M_t^n - M_t^m]^2 \geqq \delta^2\} \to 0 \qquad (3.46)$$

uniformly in x. By the definition (3.38) of M^n and Lemma (3.26), (3.46) in turn implies (3.36), thus concluding the proof of (3.34). □

Let Z^λ be a positive random variable which is independent of X and with distribution function $1 - e^{\lambda t}$, $t \geqq 0$. Let $X^\lambda = (X, Z^\lambda)$ be the process X "killed at time Z^λ," that is, $X_t^\lambda = X_t$ for $t < Z^\lambda$ and $X_t^\lambda = \infty$ for $t \geqq Z^\lambda$. [Of course, Z^λ must in general be defined on a larger space than Ω but we will not dwell on these technical points.] By the definition (3.35) it is clear that, for each

$n \in \mathbb{N}$, B^n is a continuous additive functional of (X^n, Z^λ); in other words, in terms of the process X^n alone,

$$B^n_{t+s} = B_t^n + e^{-\lambda t} B_s^n \circ \theta_t^n .$$

(3.47) LEMMA. *There exists a subsequence $\mathbb{K} \subset \mathbb{N}$ such that the sequence $\{B_t^k;\, k \in \mathbb{K}\}$ converges uniformly for $t \in [0, \infty)$ almost surely. The limit is a continuous additive functional $\{B_t\}$ of (X, Z^λ).*

PROOF. By Lemma (3.34) we can find a sequence $\mathbb{K} \subset \mathbb{N}$ *independent* of x such that

$$P^x\{\sup_t |B_t^k - B_t^j| > 1/2^k\} \leqq 1/2^k$$

for all x when $j, k \in \mathbb{K}$, $j \geqq k$. This implies the first statement.

Let Λ be the set of those $\omega \in \Omega$ for which $B_t^k(\omega)$, $k \in \mathbb{K}$, converge for all $t < Z^\lambda(\omega)$. Define

$$(3.48)\qquad \begin{aligned} B_t(\omega) &= \lim_{k \in \mathbb{K}} B_t^k(\omega) && t < Z^\lambda(\omega)\,, \quad \omega \in \Lambda \\ &= \lim_{u \uparrow Z^\lambda(\omega)} B_u(\omega) && t \geqq Z^\lambda(\omega)\,, \quad \omega \in \Lambda \\ &= 0 && t \geqq 0\,, \quad \omega \notin \Lambda\,. \end{aligned}$$

It is clear that $t \to B_t$ is continuous. Next we show that it is an additive functional of (X, Z^λ), namely, that it satisfies

$$(3.49)\qquad B_{t+s} = B_t + e^{-\lambda t} B_s \circ \theta_t$$

for each $t, s \geqq 0$ almost surely.

First we note that, for any $w \in W$, $x \to X_t^n(x, w)$ is nondecreasing; hence, $x \to B_t^n(x, w)$ is a nonincreasing function. Since $X_t \geqq X_t^n$, this implies that

$$B_s^n(X_t(x, w), \varphi_t w) \leqq B_s^n(X_t^n(x, w), \varphi_t w)$$

for all $s, t \geqq 0$, $(x, w) \in \Omega$. That is, in view of (1.7) defining θ_t and the similar definition of θ_t^n,

$$(3.50)\qquad B_s^n \circ \theta_t \leqq B_s^n \circ \theta_t^n .$$

Hence, on $\{Z^\lambda > t + s\} \cap \Lambda$, since $\theta_t^{-1}\Lambda \subset \Lambda$,

$$\begin{aligned} B_t + e^{-\lambda t} B_s \circ \theta_t &= \lim_k [B_t^k + e^{-\lambda t} B_s^k \circ \theta_t] \\ &\leqq \liminf_k [B_t^k + e^{-\lambda t} B_s^k \circ \theta_t^n] \\ &= \liminf_k B^k_{t+s} = B_{t+s} . \end{aligned}$$

By a similar reasoning for $\{Z^\lambda \leqq t + s\}$, we see that on the set Λ

$$(3.51)\qquad B_{t+s} \geqq B_t + e^{-\lambda t} B_s \circ \theta_t .$$

On the other hand,

$$(3.52)\qquad \begin{aligned} E^x[B_t] &= \lim_{k \in \mathbb{K}} E^x[B_t^k] \\ &= \lim_k E^x[B_\infty^k - E^{X^k(t)}(e^{-\lambda t} B_\infty^k)] \\ &= \lim_k v_k^\lambda(x) - e^{-\lambda t} \lim_k E^x[v_k^\lambda(X_t^n)] . \end{aligned}$$

By Lemmas (3.26) and (3.22), this becomes

$$E^x[B_t] = v^\lambda(x) - e^{-\lambda t}P_t v^\lambda(x) \tag{3.53}$$

where $\{P_t\}$ is the transition semi-group of X. Using (3.53) we also have

$$\begin{aligned} E^x[e^{-\lambda t}B_s \circ \theta_t] &= E^x[e^{-\lambda t}E^{X(t)}(B_s)] \\ &= e^{-\lambda t}P_t(v^\lambda - e^{-\lambda s}P_s v^\lambda)(x) \\ &= e^{-\lambda t}P_t v^\lambda(x) - e^{-\lambda(t+s)}P_{t+s}v^\lambda(x) . \end{aligned} \tag{3.54}$$

From (3.53) and (3.54), it follows that

$$E^x[B_{t+s}] = E^x[B_t + e^{-\lambda t}B_s \circ \theta_t] .$$

This together with (3.51) imply (3.49) and thus complete the proof of Lemma (3.47). □

We now proceed to the completion of the proof of Theorem (3.17). The continuous additive functional B defined in the preceding lemma depends on λ which was fixed starting with Lemma (3.34). To show this dependence we now write B^λ; then, B^λ is a continuous additive functional of (X, Z^λ) and has potential $v^\lambda(x) = E^x[B_\infty{}^\lambda] = E^x[B^\lambda(Z^\lambda)]$.

Let $\mu > 0$, Z^μ be an exponentially distributed variable with parameter μ which is independent of X and Z^λ. Then,

$$\begin{aligned} E^x[B^\lambda(Z^\lambda \wedge Z^\mu)] &= E^x[B^\lambda(Z^\lambda)] - E^x[B^\lambda(Z^\lambda) - B^\lambda(Z^\mu); Z^\mu < Z^\lambda] \\ &= v^\lambda(x) - \mu U^{\lambda+\mu}v^\lambda(x) . \end{aligned} \tag{3.55}$$

Using the resolvent equation and the definition (3.25) of v^λ,

$$v^\lambda + \mu U^{\lambda+\mu}v^\lambda = v^{\lambda+\mu} . \tag{3.56}$$

Hence, from (3.55) and (3.56),

$$E^x[B^\lambda(Z^\lambda \wedge Z^\mu)] = E^x[B^\mu(Z^\lambda \wedge Z^\mu)] = v^{\lambda+\mu} .$$

It follows from the symmetry of this expression in λ and μ that $B_t{}^\lambda = B_t{}^\mu$ almost surely on the set $\{Z^\lambda \wedge Z^\mu > t\}$. By the independence of X, Z^λ, Z^μ this implies that there exists a continuous additive functional $C = \{C_t; t \geqq 0\}$ of the process X such that, for each $\lambda > 0$,

$$\begin{aligned} B_t{}^\lambda &= C_t \qquad \text{if} \quad t < Z^\lambda , \\ &= C_{Z^\lambda} \qquad \text{if} \quad t \geqq Z^\lambda . \end{aligned}$$

Then, for all x and $\lambda > 0$,

$$E^x \int_0^\infty e^{-\lambda t}\, dC_t = E^x[B_\infty{}^\lambda] = v^\lambda(x) . \tag{3.57}$$

Since v^λ is also the λ-potential of the local time L at 0 satisfying $v^1(x) = f(x)$, from the uniqueness of potentials, it follows that $C_t = L_t$ for all t almost surely. We have thus shown that

$$\lim_{k \in \mathrm{K}} c_k L_t{}^k = L_t$$

for all $t \geqq 0$ almost surely. Since the limit L does not depend on the sequence $\mathbb{K}$ chosen, we conclude that (3.20) holds. Of course, (3.21) is just (3.25) and (3.57). This completes the proof of Theorem (3.17). $\square$

In addition, it follows from the proof that, the λ-potential u^λ of L can also be obtained as

$$u^\lambda(x) = \lim_n c_n u_n^\lambda(x) \tag{3.58}$$

with c_n as defined by (3.19) and u_n^λ by (3.9).

4. Inverses of local times. In this section we limit ourselves to the cases where 0 is regular for $\{0\}$. In all these three cases we had obtained local times L for the process X at 0. We define

$$\tau_t = \inf\{s : L_s > t\}, \qquad t \geqq 0. \tag{4.1}$$

It follows that, for any bounded Borel measurable function $h : [0, \infty] \to [0, \infty]$ with $h(\infty) = 0$, we have

$$\int_0^\infty h(t)\, dL_t = \int_0^\infty h(\tau_t)\, dt \tag{4.2}$$

almost surely. In particular, then

$$u^\lambda(x) = E^x \int_0^\infty e^{-\lambda t}\, dL_t = E^x \int_0^\infty e^{-\lambda \tau_t}\, dt. \tag{4.3}$$

Each τ_t is a stopping time and

$$\tau_{t+s} = \tau_t + \tau_s \circ \theta_{\tau_t} \tag{4.4}$$

for all $t, s \geqq 0$ almost surely. Since $X_{\tau_t} = 0$ almost surely on $\{\tau_t < \infty\}$, the strong Markov property at τ_t implies that, considered as a process over $(\Omega, \mathscr{H}, P^x)$ for any fixed x, (τ_t) has stationary and independent increments. This, together with the fact that $\tau_0 = S$, implies that

$$E^x[\exp(-\lambda\tau_t)] = f^\lambda(x)\exp(-tg(\lambda)), \qquad t, \lambda \geqq 0 \tag{4.5}$$

for some exponent g, that is,

$$g(\lambda) = a\lambda + \int_{(0,\infty]} (1 - e^{-\lambda s})\nu(ds) \tag{4.6}$$

for some constant $a \geqq 0$, called the drift rate, and some measure ν on $(0, \infty]$, called the Lévy measure of (τ_t) (such that $s \wedge 1$ is integrable with respect to ν). On the other hand, (4.3) and (4.5) imply that

$$g(\lambda) = 1/u^\lambda(0), \tag{4.7}$$

where u^λ is the λ-potential of L.

We now use (4.6) and (4.7) together to identify the drift term a and the Lévy measure ν in each of the three cases. First we define

$$F(x, B) = P^x\{S \in B\}, \qquad x \geqq 0,\ B \in \bar{\mathscr{R}}_+. \tag{4.8}$$

The following identifies a and ν in the first two cases:

(4.9) PROPOSITION. *Suppose* $\int (1-f)\, d\beta < \infty$. *Then,*

$$(4.10) \qquad a = 1\,, \qquad \nu(B) = \int_{(0,\infty)} \beta(dx) F(x, B)\,, \qquad B \in \bar{\mathscr{R}}_+\,.$$

PROOF. According to (3.4), (3.12), and (4.7),

$$\begin{aligned} g(\lambda) &= \lambda + \int (1 - f^\lambda)\, d\beta \\ &= \lambda + \int_{(0,\infty)} \beta(dx) \int_{(0,\infty]} F(x, ds)(1 - e^{-\lambda s})\,. \end{aligned}$$

Now (4.10) is immediate from Fubini's theorem. □

(4.11) REMARK. If β is finite, so is ν; and conversely.

Next is the identification of a and ν in the third case.

(4.12) PROPOSITION. *Suppose* $\int (1-f)\, d\beta = \infty$, $f(0) = 1$. *Then, with* c_n *defined as in* (3.19) *and* $F_n(x, B) = P^x\{S_n \in B\}$, $x \in \mathbb{R}_+$, $B \in \bar{\mathscr{R}}_+$, *we have*

$$(4.13) \qquad a = 0\,, \qquad \nu = \lim_{n\to\infty} \frac{1}{c_n} \int_{[1/n,\infty)} \beta(dx) F(x, \cdot)\,,$$

where the convergence is in the weak topology.

PROOF. Let $\nu_n(B)$ be defined as the quantity which is claimed to be converging to $\nu(B)$. Then,

$$(4.14) \qquad [c_n u_n^\lambda(0)]^{-1} = \lambda/c_n + \int_{(0,\infty]} \nu_n(ds)(1 - e^{-\lambda s})\,.$$

By Theorem (3.17) and Lemma (3.22), $c_n u_n^\lambda \to u^\lambda$ as $n \to \infty$. Hence, (4.6), (4.7), and (4.14) imply that $a = 0$ and that

$$(4.15) \qquad \int_{(0,\infty]} \nu(ds)(1 - e^{-\lambda s}) = \lim_n \int_{(0,\infty]} \nu_n(ds)(1 - e^{-\lambda s})\,.$$

If we define $m(s) = \nu((s, \infty])$, $m_n(s) = \nu_n((s, \infty])$, then (4.15) implies that the Laplace transform of the right continuous nonincreasing function m_n converges to the Laplace transform of m which is also right continuous and nonincreasing. It follows that $m_n(s) \to m(s)$ as $n \to \infty$, and this implies the claimed convergence of ν_n to ν in the weak sense. □

Finally, let $E = \{t : X_t = 0\}$. Since X is continuous at 0, the set E is almost surely closed. Its complement in $\mathbb{R}_+$, therefore, is a countable union of open intervals. Let F be the set of all left extremities of these contiguous intervals. Then, $E \backslash F$ is the smallest right closed set whose closure is E. From Maisonneuve's results in [8], it is known that

$$(4.16) \qquad E\backslash F = \{t : \tau_s = t \text{ for some } s\}\,.$$

Well-known results concerning the sample path properties of the additive process (τ_s) can now be used to study $E\backslash F$ and its closure E (which differs from $E\backslash F$ by the countable set F). A short resumé of some such results was given in the introduction.

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