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BOUNDARY CROSSING PROBABILITIES FOR SAMPLE SUMS AND CONFIDENCE SEQUENCES[1]

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By making use of the martingale $\int_0^\infty \exp(yW(t) - (t/2)y^2)\, dF(y)$, Robbins and Siegmund have evaluated the probability that the Wiener process $W(t)$ would ever cross certain moving boundaries. In this paper, we study this class of boundaries and make use of certain moment generating function martingales to obtain boundary crossing probabilities for sums of i.i.d. random variables. Invariance theorems for these boundary crossing probabilities are proved, and some applications to confidence sequences and power-one tests are also given.

1. Introduction. Suppose $X_1, X_2, \cdots$ are i.i.d. random variables with mean 0 and variance 1. Throughout this paper, we shall let S_n denote the sample sum $X_1 + \cdots + X_n$ $(S_0 = 0)$. By the zero-one law, $P[S_n > b_n \text{ i.o.}]$ is either 0 or 1 for any sequence (b_n) of real numbers. Following Lévy, we say that (b_n) belongs to the *upper class* if the above probability is 0 and to the *lower class* if the above probability is 1. (Actually, the usual usage of the terms upper and lower class refers to the sequence $n^{-\frac{1}{2}}b_n$ rather than the sequence b_n itself, but here it is more convenient to consider the original sequence b_n instead.)

Suppose (b_n) belongs to the upper class. It would be interesting to find the boundary crossing probability $P[S_n \geqq b_n$ for some $n \geqq m]$. Such boundary crossing probabilities have statistical applications in power-one tests of one-sided hypotheses and in confidence sequences for the unknown parameters of parametric families of distributions. These statistical applications have been considered by Darling and Robbins [2], [3], [4], [5], Lai [7], [8], Robbins [11] and Robbins and Siegmund [12].

In this paper, we shall study boundary crossing probabilities for S_n by using suitably chosen martingales. Let us consider the following simple example which sheds light on how martingales can be used. Suppose $P[X_1 = 1] = p = 1 - P[X_1 = -1]$ with $p \leqq \frac{1}{2}$. Then for $b = 1, 2, \cdots$, it is well known that $P[S_n \geqq b$ for some $n \geqq 1] = (p/(1-p))^b$, and the usual proof is to use the Markov property of the random walk S_n. Alternatively, this result can also be proved by a martingale type of argument used by Wald [15]. Let $p_0 = p$, $p_1 = 1 - p$, $f_{in}(X_1, \cdots, X_n) = p_i^{(n+S_n)/2}(1-p_i)^{(n-S_n)/2}$ $(i = 0, 1)$, and consider the sequential

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test which stops with

$$\begin{aligned} N &= \inf\{n \geqq 1 : f_{1n}(X_1, \cdots, X_n)/f_{0n}(X_1, \cdots, X_n) \geqq ((1-p)/p)^b\} \\ &= \inf\{n \geqq 1 : S_n \geqq b\} = \inf\{n \geqq 1 : S_n = b\}. \end{aligned}$$

Wald's argument gives us

$$\begin{aligned} P[N < \infty] &= \sum_{n=1}^{\infty} \int_{[N=n]} f_{0n} = (p/(1-p))^b \sum_{n=1}^{\infty} \int_{[N=n]} f_{1n} \\ &= (p/(1-p))^b P_1[N < \infty] \\ &= (p/(1-p))^b \end{aligned}$$

where P_1 is the probability measure under which $X_1, X_2, \cdots$ are i.i.d. with $P_1[X_1 = 1] = p_1 = 1 - P_1[X_1 = -1]$.

We observe that the likelihood ratio $f_{1n}(X_1, \cdots, X_n)/f_{0n}(X_1, \cdots, X_n)$ used above is a martingale. Now let (b_n) be a strictly increasing sequence of positive integers and consider $P[S_n \geqq b_n$ for some $n \geqq 1]$. Define $N^* = \inf\{n \geqq 1 : S_n \geqq b_n\}$. While $\{S_{N \wedge n}, n \geqq 1\}$ is a Markov chain, the other stopped random walk $\{S_{N^* \wedge n}, n \geqq 1\}$, though Markovian, does not have stationary transition probabilities since the stopping boundary (b_n) is changing with time. Though we cannot use standard Markov chain techniques in the case of a moving boundary, martingale arguments enable us to study $P[N^* < \infty]$ for a certain class of sequences (b_n).

In Section 2, we shall consider the case where $X_1, X_2, \cdots$ are $N(0, 1)$ random variables. The sequence $(S_1, S_2, \cdots)$ has the same joint distribution as the sequence $(W(1), W(2), \cdots)$, where $W(t)$ is the standard Wiener process, and the martingale $\int_0^\infty \exp(yW(t) - (t/2)y^2)$ used in [14] gives the probability that the process $W(t)$ would ever cross a certain class of moving boundaries which we shall study in Section 2. In Section 3, we shall consider the case where $X_1, X_2, \cdots$ have a moment generating function φ which is finite on $(0, \alpha)$ for some $0 < \alpha \leqq \infty$, and we shall make use of the martingale $\int_0^\alpha (\varphi(y))^{-n} \exp(yS_n)\, dF(y)$ to obtain confidence sequences and power-one tests for the unknown parameters of the binomial, Poisson and gamma distributions. In Section 4, we shall prove certain invariance theorems relating to the boundary crossing probabilities for the Wiener process.

2. The normal case and the Robbins-Siegmund boundaries. Let $X_1, X_2, \cdots$ be i.i.d. $N(0, 1)$ random variables. Let F be any measure on $(0, \infty)$ and define

$$f(x, t) = \int_0^\infty \exp\left(xy - \frac{t}{2} y^2\right) dF(y). \tag{1}$$

Since $f(S_n, n)$, $n \geqq 1$, is a nonnegative martingale, it follows that (cf. [11], page 1400) for any $\varepsilon > 0$,

$$\begin{aligned} P[f(S_n, n) \geqq \varepsilon &\text{ for some } n \geqq m] \\ &\leqq P[f(S_m, m) \geqq \varepsilon] + \varepsilon^{-1} \int_{[f(S_m, m) < \varepsilon]} f(S_m, m)\, dP. \end{aligned} \tag{2}$$

Suppose the measure F satisfies $f(x, h) < \infty$ for some $h \geqq 0$ and all real x, and is therefore finite on any bounded interval. Assume that F is nontrivial, i.e.,

$F(0, \infty) > 0$. Given any $\varepsilon > 0$ and $t \geqq h$, the equation $f(x, t) = \varepsilon$ has a unique solution $x = A_F(t, \varepsilon)$. We shall call the function $t \mapsto A_F(t, \varepsilon)$, $t \geqq h$, a *Robbins–Siegmund boundary* (abbreviated R–S boundary). Robbins and Siegmund [14] have shown that if we replace S_n by the Wiener process $W(t)$, then equality holds for the corresponding version of (2), i.e.,

$$\begin{aligned} &P[f(W(t), t) \geqq \varepsilon \ \text{ for some } \ t \geqq h] \\ &\qquad = P[f(W(h), h) \geqq \varepsilon] + \varepsilon^{-1} \int_{[f(W(h),h)<\varepsilon]} f(W(h), h)\, dP\,. \end{aligned}$$

This relation can be written in terms of the R–S boundary $A_F(t, \varepsilon)$ as follows:

$$\begin{aligned} &P[W(t) \geqq A_F(t, \varepsilon) \ \text{ for some } \ t \geqq h] \\ &\qquad = 1\,, \qquad \text{if} \quad h = 0 \quad \text{and} \quad \varepsilon \leqq \textstyle\int_0^\infty dF \\ &\qquad = \varepsilon^{-1} \textstyle\int_0^\infty dF\,, \qquad \text{if} \quad h = 0 \quad \text{and} \quad \varepsilon > \textstyle\int_0^\infty dF \\ &\qquad = 1 - \Phi\left(\frac{A_F(h, \varepsilon)}{h^{\frac{1}{2}}}\right) + \varepsilon^{-1} \textstyle\int_0^\infty \Phi\left(\frac{A_F(h, \varepsilon)}{h^{\frac{1}{2}}} - yh^{\frac{1}{2}}\right) dF(y)\,, \\ &\qquad\qquad \text{if} \quad h > 0\,, \end{aligned} \tag{3}$$

where Φ denotes the distribution function of the standard normal distribution.

Hence by a martingale approach, Robbins and Siegmund [14] have obtained a class of boundary crossing probabilities for the Wiener process $W(t)$. Extensions of their method to study boundary crossing probabilities for other Markov processes $X(t)$ are given in [9] and [10], where we characterize space-time martingales of the form $u(X(t), t)$ and make use of them to evaluate boundary crossing probabilities.

The class of all R–S boundaries has the following nice properties:

(A) If g is a R–S boundary, then so is $g + a$ for any real number a.

(B) If $g(t) = A_F(t, \varepsilon)$, $t \geqq h$, then the function $v(t) = g(t + a)$, $t \geqq (h - a)^+$, is a R–S boundary for any real number a.

(C) If $g(t) = A_F(t, \varepsilon)$, $t \geqq h$, then the function $u(t) = c^{\frac{1}{2}} g(t/c)$, $t \geqq ch$, is a R–S boundary for any positive number c.

(D) Every R–S boundary is a concave function. (This property can be easily proved by using the Schwarz inequality.)

In [14], Robbins and Siegmund have proved the following limit theorem: Let $X_1, X_2, \cdots$ be i.i.d. random variables with mean 0 and variance 1. If $t^{-\frac{1}{2}} A_F(t, \varepsilon)$ is ultimately nondecreasing, then

$$\begin{aligned} &\lim_{m\to\infty} P[S_n \geqq m^{\frac{1}{2}} A_F(n/m, \varepsilon) \ \text{ for some } \ n \geqq hm] \\ &\qquad = P[W(t) \geqq A_F(t, \varepsilon) \ \text{ for some } \ t \geqq h]\,. \end{aligned} \tag{4}$$

Hence for large m, the probability that the sequence $(S_n, n \geqq hm)$ would ever cross the R–S boundary $u(t) = m^{\frac{1}{2}} A_F(t/m, \varepsilon)$ is approximately equal to

$$1 - \Phi(A_F(h, \varepsilon)/h^{\frac{1}{2}}) + \varepsilon^{-1} \int_0^\infty \Phi(A_F(h, \varepsilon)/h^{\frac{1}{2}} - yh^{\frac{1}{2}})\, dF(y) \qquad \text{if} \quad h > 0\,,$$

with no parametric assumptions about the X_i's. The statistical significance of this fact has been discussed in [11].

We shall devote the rest of this section to the study of the analytic properties, the asymptotic behavior and other characteristics of R–S boundaries. First let us consider the analytic properties. If $f(x, h) < \infty$ for all real x, then we can differentiate $f(x, t)$ with respect to x and t under the integral sign for $t \geqq h$, and using the implicit function theorem, it is easy to see that $A_F(\cdot, \varepsilon) \in C^\infty[h, \infty)$. In fact, the R–S boundary is not only of class C^∞, but also for $t > h$, there exists a neighborhood of t in which $A_F(\cdot, \varepsilon)$ can be expanded in a Taylor series. To show this, we note that the function $f(z, w)$ is an analytic function for all complex z and all complex w with $\mathscr{R}(w) > h$. Take any real number $t_0 > h$. Let $x_0 = A_F(t_0, \varepsilon)$. By the implicit function theorem for functions of several complex variables ([6], page 24), the equation $f(z, w) = \varepsilon$ (z, w complex) has a uniquely determined analytic solution $z = z(w)$ in a (complex) neighborhood of t_0 such that $z(t_0) = x_0$. The function $z(w)$ is therefore an analytic extension of $A_F(t, \varepsilon)$.

If g is a R–S boundary, then g is strictly increasing. Since g is concave, $(g(t) - g(h))/(t - h)$ is decreasing for $t > h$. The asymptotic behavior of $g(t) = A_F(t, \varepsilon)$ as $t \to \infty$ is closely related to y_F which is defined by

$$
\begin{aligned}
(5) \qquad y_F &= \sup\{y : y > 0, F(0, y) = 0\} \\
&= 0\,, \quad \text{if the above set is empty.}
\end{aligned}
$$

THEOREM 1. *Let F be a nontrivial measure on $(0, \infty)$ which satisfies $f(b, h) < \infty$ for some $h \geqq 0$ and some real number b, where f is defined by* (1). *Then for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} A_F(t, \varepsilon)/t = y_F/2$$

where y_F is defined by (5).

PROOF. We first prove that $\limsup_{t\to\infty} A_F(t, \varepsilon)/t \leqq y_F/2$. Suppose this were false. Then there exist $\alpha > \frac{1}{2}$, $\delta > y_F$ and a sequence $t_n \uparrow \infty$ such that $A_F(t_n, \delta) > \alpha\delta t_n$ for all n. First consider the case where $y_F > 0$. Then $F(0, y_F) = \lim_{y\uparrow y_F} F(0, y) = 0$. Since $\delta > y_F$, we have $0 < F(0, \delta) = F[y_F, \delta)$. Therefore

$$
\begin{aligned}
\varepsilon &\geqq \int_{(0,\delta)} \exp\left(yA_F(t_n, \varepsilon) - \frac{t_n}{2} y^2\right) dF(y) \\
&\geqq \exp\left\{y_F t_n\left(\alpha\delta - \frac{\delta}{2}\right)\right\} \cdot F[y_F, \delta) \to \infty \qquad \text{as } n \to \infty\,,
\end{aligned}
$$

leading to a contradiction. In the case where $y_F = 0$, we have $F(0, \delta) > 0$ and so we can choose $\delta' \in (0, \delta)$ such that $F[\delta', \delta) > 0$. Therefore

$$
\begin{aligned}
\varepsilon &\geqq \int_{[\delta',\delta)} \exp\left(yA_F(t_n, \varepsilon) - \frac{t_n}{2} y^2\right) dF(y) \\
&\geqq \left(\exp\left\{\delta' t_n\left(\alpha\delta - \frac{\delta}{2}\right)\right\}\right) F[\delta', \delta) \to \infty \qquad \text{as } n \to \infty\,,
\end{aligned}
$$

again leading to a contradiction. Hence we must have $\limsup_{t\to\infty} A_F(t, \varepsilon)/t \leqq y_F/2$.

In the case $y_F = 0$, we have already proved that $\lim_{t\to\infty} A_F(t, \varepsilon)/t = 0$. Let $y_F > 0$ and assume that $\liminf_{t\to\infty} A_F(t, \varepsilon)/t < y_F/2$. Then there exist $\delta \in (0, 1)$ and a sequence $t_n \uparrow \infty$ such that $A_F(t_n, \varepsilon) < \delta y_F t_n/2$ for all n. Hence for $y \geqq y_F$, $yA_F(t_n, \varepsilon) < \delta t_n y^2/2$ and so by the dominated convergence theorem,

$$\varepsilon \leqq \int_{[y_F,\infty)} \exp((\delta - 1)t_n y^2/2)\, dF(y) \to 0 \qquad \text{as } n \to \infty ,$$

leading to a contradiction. Therefore the desired conclusion follows. □

Theorem 1 shows that if g is a R–S boundary, then $g(t) = O(t)$ as $t \to \infty$. It is easy to see from (3) that $\lim_{h\to\infty} P[W(t) \geqq g(t) \text{ for some } t \geqq h] = 0$. Therefore $P[W(t) < g(t) \text{ for all large } t] = 1$, and so all R–S boundaries belong to the upper class.

In [13], [14], we have seen linear R–S boundaries ct ($c > 0$), together with examples of R–S boundaries which are asymptotic to $(2t \log\log t)^{\frac{1}{2}}$, or $(ct \log t)^{\frac{1}{2}}$, or $t^\alpha (\frac{1}{2} < \alpha < 1)$. We now exhibit some other R–S boundaries below:

(a) If $dF(y) = \exp(-\exp(1/y))\, dy$, $0 < y < \infty$, then

$$A_F(t, \varepsilon) = t/(\log t - 2\log_2 t - \log 2 + o(1)) \qquad \text{as } t \to \infty ,$$

where we write $\log_2 t = \log\log t$, etc.

(b) If $dF(y) = dy/y(\log|y|)^{1+\delta}$, $0 < y < 1/e$, and $= 0$ elsewhere, where $\delta > 0$, then

$$A_F(t, \varepsilon) = \{2t[(1 + \delta)(\log_2 t - \log 2) + \tfrac{1}{2}\log_3 t + \tfrac{1}{2}\log(1 + \delta) + \log(\varepsilon/\pi^{\frac{1}{2}}) + o(1)]\}^{\frac{1}{2}} \qquad \text{as } t \to \infty .$$

(c) If F is any measure on $(0, \infty)$ such that $f(b, h) < \infty$ for some real b and $h \geqq 0$, and $F(\{2c\}) = \varepsilon e^{-2cd}$ ($c > 0$), $F(0, 2c) = 0$, then

$$A_F(t, \varepsilon) = ct + d + o(1) \qquad \text{as } t \to \infty .$$

(d) If $dF(y) = dy$, $2c < y < \infty$ and $= 0$ elsewhere, then

$$A_F(t, \varepsilon) = ct + \frac{1}{2c}(\log t + \log c\varepsilon + o(1)) \qquad \text{as } t \to \infty .$$

Let F be a nontrivial measure on $(0, \infty)$ such that $f(b, h) < \infty$ for some real b and $h \geqq 0$. Suppose $g(t)$ is a real-valued function defined for all large t and $\lim_{t\to\infty} f(g(t), t) = \varepsilon > 0$. Then $g(t) \sim A_F(t, \varepsilon)$ as $t \to \infty$. Now let $\xi(y)$ be a real-valued function on $[0, \infty)$ such that $\xi(y) = O(\exp(\beta y^2))$ as $y \to \infty$ for some $\beta > 0$ and $\lim_{y\downarrow y_F} \xi(y) = \xi(y_F)$. Then

$$\lim_{t\to\infty} \int_0^\infty \xi(y) \exp\left(yA_F(t, \varepsilon) - \frac{t}{2}y^2\right) dF(y) = \varepsilon\xi(y_F) .$$

Hence defining the measure F_ξ on $(0, \infty)$ by $dF_\xi(y) = |\xi(y)|\, dF(y)$, we have $A_{F_\xi}(t, \varepsilon|\xi(y_F)|) \sim A_F(t, \varepsilon)$ if $\xi(y_F) \neq 0$. In particular, letting ξ be of compact support and $\xi(y_F) = 1$, it follows that there exists a measure G of bounded support such that $A_G(t, \varepsilon) \sim A_F(t, \varepsilon)$ as $t \to \infty$.

THEOREM 2. *Let F be a nontrivial measure on $(0, \infty)$ which satisfies $f(b, h) < \infty$ for some real b and $h \geqq 0$. Suppose that $y_F = 0$, and that there exist $a > 0$ and a nonnegative Borel function ψ on $(0, a)$ such that for any Borel subset A of $(0, a)$, $F(A) = \int_A \psi(x)\,dx$. Let ε be any positive number.*

(i) *If $\inf_{x\in(0,a)} \psi(x) > 0$, then $A_F(t, \varepsilon) = O((t \log t)^{\frac{1}{2}})$ as $t \to \infty$.*

(ii) *If $\sup_{x\in(0,a)} \psi(x) < \infty$, then $(t \log t)^{\frac{1}{2}} = O(A_F(t, \varepsilon))$ as $t \to \infty$.*

(iii) *If $\inf_{x\in(0,a)} \psi(x) > 0$ and $\sup_{x\in(0,a)} \psi(x) < \infty$, then $A_F(t, \varepsilon) \sim (t \log t)^{\frac{1}{2}}$ as $t \to \infty$.*

(iv) *If $\lim_{x\to 0+} \psi(x) = L$ exists and is a finite positive number, then*

$$A_F(t, \varepsilon) = \{t(\log t + 2 \log (\varepsilon/L(2\pi)^{\frac{1}{2}}) + o(1))\}^{\frac{1}{2}}\,.$$

PROOF. For notational convenience, we shall write $A(t) = A_F(t, \varepsilon)$. Now $\lim_{t\to\infty} A(t)/t = y_F/2 = 0$, and so for all large t, $4A(t)/t < a$. It is not hard to show that

$$\text{(6)} \qquad \lim_{t\to\infty} e^{A^2(t)/2t} \int_0^{4A(t)/t} \exp\left\{-\frac{t}{2}\left(y - \frac{A(t)}{t}\right)^2\right\} \psi(y)\,dy = \varepsilon\,.$$

To prove (i), let $\psi(x) \geqq K > 0$ for all $x \in (0, a)$. Then it easily follows from (6) that

$$\varepsilon \geqq \limsup_{t\to\infty} (2\pi)^{\frac{1}{2}} K t^{-\frac{1}{2}} \exp(A^2(t)/2t)\,,$$

and so as $t \to \infty$, $A(t) = O((t \log t)^{\frac{1}{2}})$. Similarly to prove (ii), we let $0 \leqq \psi(x) \leqq K_1$ for all $x \in (0, a)$ and obtain from (6) that

$$\varepsilon \leqq \liminf_{t\to\infty} (2\pi)^{\frac{1}{2}} K_1 t^{-\frac{1}{2}} \exp(A^2(t)/2t)\,,$$

and so $(t \log t)^{\frac{1}{2}} = O(A(t))$ as $t \to \infty$.

To prove (iii), let $0 < K < \psi(x) < K_1$ for all $x \in (0, a)$. Take $C_1 > \log\{(\varepsilon/K)(2\pi)^{-\frac{1}{2}}\}$ and $C_2 < \log\{(\varepsilon/K_1)(2\pi)^{-\frac{1}{2}}\}$. Then it follows from our preceding argument that for all large t,

$$C_2 + (\log t)/2 \leqq A^2(t)/2t \leqq C_1 + (\log t)/2$$

and therefore $A(t) \sim (t \log t)^{\frac{1}{2}}$ as $t \to \infty$.

To prove (iv), since $\lim_{x\to 0+} \psi(x) = L \in (0, \infty)$, it follows from our proof of (i) and (ii) that

$$\lim_{t\to\infty} (2\pi)^{\frac{1}{2}} L t^{-\frac{1}{2}} \exp(A^2(t)/2t) = \varepsilon$$

and so the desired conclusion follows. □

3. Moment generating functions and boundary crossing probabilities for sums of i.i.d. random variables. Suppose $X_1, X_2, \cdots$ are i.i.d. such that $\varphi(y) = E\exp(yX_1) < \infty$ for all $y \in (0, \alpha)$, where $0 < \alpha \leqq \infty$. Let $\varepsilon > 0$ and let F be any measure on $(0, \infty)$ with support contained in $(0, \alpha)$. Let $b_n = \inf\{x : \int_0^\infty (\varphi(y))^{-n} \exp(xy)\,dF(y) \geqq \varepsilon\}$. Since $\int_0^\infty (\varphi(y))^{-n} \exp(yS_n)\,dF(y)$ is a nonnegative martingale, we have

$$
\begin{aligned}
&P[S_n \geqq b_n \ \text{ for some } \ n \geqq m] \\
(7)\qquad &\qquad = P[\textstyle\int_0^\infty (\varphi(y))^{-n} e^{yS_n}\, dF(y) \geqq \varepsilon \ \text{ for some } \ n \geqq m] \\
&\qquad \leqq P[S_m \geqq b_m] + \varepsilon^{-1} \textstyle\int_{[S_m < b_m]} \int_0^\infty (\varphi(y))^{-m} \exp(yS_m)\, dF(y)\, dP \\
&\qquad \leqq \varepsilon^{-1} \textstyle\int_0^\infty dF\,.
\end{aligned}
$$

If $\varphi\colon (-\infty, \infty) \to (0, \infty]$ is the moment generating function (abbreviated mgf) of some random variable X (i.e., $\varphi(y) = E\exp(yX)$), and D is any subset of the real line such that $\varphi(y) < \infty$ on D, then we shall say that φ is *subnormal on D* if $\varphi(y) \leqq \exp(y^2/2)$ for all $y \in D$, and φ is *supernormal on D* if $\varphi(y) \geqq \exp(y^2/2)$ for all $y \in D$.

Suppose $X_1, X_2, \cdots$ are i.i.d. with mgf φ such that φ is finite and subnormal on $(0, \alpha)$ where $0 < \alpha \leqq \infty$. Let F be any nontrivial measure on $(0, \infty)$ with support contained in $(0, \alpha)$ and $f(x, m) < \infty$ for all real x, where f is defined by (1). Then $f(S_n, n)$, $n \geqq m$, is a nonnegative supermartingale, and so we have

$$
\begin{aligned}
&P[S_n \geqq A_F(n, \varepsilon) \ \text{ for some } \ n \geqq m] \\
(8)\qquad &\qquad \leqq P[S_m \geqq A_F(m, \varepsilon)] \\
&\qquad\qquad + \varepsilon^{-1} \textstyle\int_0^\infty \int_{[S_m < A_F(m,\varepsilon)]} \exp\left(yS_m - \dfrac{m}{2} y^2\right) dP\, dF(y) \\
&\qquad \leqq \varepsilon^{-1} \textstyle\int_0^\infty dF\,.
\end{aligned}
$$

Darling and Robbins [5] have shown that given any $\varepsilon > 0$ and any sequence b_n with $n^{-\frac{1}{2}} b_n \uparrow \infty$ and $b_n/n \downarrow 0$ as $n \uparrow \infty$ and $\sum_{n=1}^\infty n^{-\frac{3}{2}} b_n \exp(-b_n^2/2n) < \infty$, we can construct a probability density function ψ on $(0, \infty)$ such that by choosing m sufficiently large, we have $\int_0^\infty \psi(y) \exp(yb_n - (n/2)y^2)\, dy \geqq \varepsilon$ for all $n \geqq m$. Letting F be the probability measure on $(0, \infty)$ with density function ψ, we have from (8) that if $X_1, X_2, \cdots$ are i.i.d. with mgf φ which is subnormal on $(0, \infty)$, then

$$P[S_n \geqq b_n \ \text{ for some } \ n \geqq m] \leqq P[S_n \geqq A_F(n, \varepsilon) \ \text{ for some } \ n \geqq m] \leqq \varepsilon^{-1}\,.$$

The following examples deal with statistical applications of the inequalities (7) and (8).

EXAMPLE 1. Suppose $X_1, X_2, \cdots$ are i.i.d. Bernoulli random variables with parameter p, i.e., $P_p[X_1 = 1] = p$, $P_p[X_1 = 0] = 1 - p = q$. The mgf $E_p \exp(yX_1)$ is supernormal on $(0, \alpha_p)$, where α_p is some positive number depending on p, and is subnormal on $[2, \infty)$. Now let $Z_i = (X_i - p)/(pq)^{\frac{1}{2}}$. Then $E_p e^{yZ_1}$ is subnormal on $(0, \infty)$ if $p = \frac{1}{2}$; and if $p \in (\frac{1}{2}, 1)$, there exists $\lambda_p > 0$ such that $E_p e^{yZ_1}$ is subnormal on $(0, \lambda_p)$, while if $p \in (0, \frac{1}{2})$, there exists $\lambda_p{}^* > 0$ such that $E_p e^{yZ_1}$ is supernormal on $(0, \lambda_p{}^*)$. Given any $p_0 \in (0, 1)$, we can choose $k_0 \geqq (p_0 q_0)^{\frac{1}{2}}$ and $\beta_0 > 0$ such that writing $\tilde{X}_i = (X_i - p_0)/k_0$, we have the subnormality of $E_{p_0} \exp(y\tilde{X}_1)$ on $(0, \beta_0)$. (For example, we can choose $k_0 = \beta_0 = 1$). Let $\varepsilon > 0$, $\tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n$, and let F be any probability measure on $(0, \beta_0)$. Then if $p \leqq p_0$,

$$P_p[\tilde{S}_n \geqq A_F(n, \varepsilon) \text{ for some } n \geqq 1] \leqq P_{p_0}[\tilde{S}_n \geqq A_F(n, \varepsilon) \text{ for some } n \geqq 1] \leqq 1/\varepsilon .$$

By choosing $y_F = 0$, this result can be used for open-ended power-one tests of $H_0: p \leqq p_0$ versus $H_1: p > p_0$. Alternatively, we can let G be a probability measure on $(0, \infty)$ such that $y_G = 0$ and $\int_0^\infty \exp(xy)\, dG(y) < \infty$ for all $x \in (-\infty, \infty)$, and define $b_n = \inf\{x: \int_0^\infty e^{xy}(p_0 e^y + q_0)^{-n}\, dG(y) \geqq \varepsilon\}$. Then writing $S_n = X_1 + \cdots + X_n$, it follows from (7) that for all $p \leqq p_0$, $P_p[S_n \geqq b_n$ for some $n \geqq 1] \leqq P_{p_0}[S_n \geqq b_n$ for some $n \geqq 1] \leqq 1/\varepsilon$. Also if $p > p_0$, then $P_p[\lim_{n\to\infty} \int_0^\infty e^{yS_n}(p_0 e^y + q_0)^{-n}\, dG(y) = \infty] = 1$, since $y_G = 0$.

EXAMPLE 2. Suppose $X_1, X_2, \cdots$ are i.i.d. with the gamma density $f_\theta(x)$, i.e.,

$$f_\theta(x) = \theta^{-\beta} x^{\beta-1} \exp(-x/\theta)/\Gamma(\beta) , \qquad x > 0 ,$$

where $\theta\,(> 0)$ is unknown and $\beta\,(> 0)$ is known. Let $Z_i = \beta^{-\frac{1}{2}}(\beta - (X_i/\theta))$. Then for $y \geqq 0$, $E_\theta \exp(yZ_1) \leqq \exp(y^2/2)$. Hence if $\varepsilon > 0$ and F is a probability measure on $(0, \infty)$, then for all $\theta > 0$, letting $S_n = X_1 + \cdots + X_n$, $P_\theta[S_n \leqq \theta(\beta n - \beta^{\frac{1}{2}} A_F(n, \varepsilon))$ for some $n \geqq 1] \leqq 1/\varepsilon$, and we therefore obtain a one-sided confidence sequence for the scale parameter of the gamma distribution.

EXAMPLE 3. Suppose $X_1, X_2, \cdots$ are i.i.d. Poisson random variables with mean λ. Then $E_\lambda \exp(yX_1) = \exp(\lambda(e^y - 1))$. To construct a power-one test of $H_0: \lambda \leqq \lambda_0$ versus $H_1: \lambda > \lambda_0\,(\lambda_0 > 0)$, we let F be a probability measure on $(0, \infty)$ with $y_F = 0$. Set $S_n = X_1 + \cdots + X_n$ and $b_n = \inf\{x: \int_0^\infty \exp(xy - n\lambda_0(e^y - 1))\, dF(y) \geqq \varepsilon\}$. We obtain from (7) that if $\lambda \leqq \lambda_0$, then

$$P_\lambda[S_n \geqq b_n \text{ for some } n \geqq 1] \leqq P_{\lambda_0}[S_n \geqq b_n \text{ for some } n \geqq 1] \leqq 1/\varepsilon .$$

The above methods for finding power-one tests of one-sided hypotheses can be extended to other stochastically increasing families of distributions.

4. Some invariance theorems. The main results of this section are contained in Theorem 3 and Theorem 4. A counter-example is also given to show that if we do not assume the regularity conditions of the type we impose in our theorems, then the desired result may fail to hold. Theorem 5 gives us another way of interpreting the result of Theorem 4.

Suppose $X_1, X_2, \cdots$ are i.i.d. with $EX_1 = 0$, $EX_1^2 = 1$ and $\varphi(y) = E\exp(yX_1) < \infty$ for all $y \in (0, \alpha)$, where $0 < \alpha \leqq \infty$. Let F be any finite nontrivial measure on $(0, \infty)$ with support contained in $(0, \rho\alpha)$ for some $\rho > 0$. For any real number $m \geqq \rho^2$, define

$$f_m(x, t) = \int_0^\infty \exp(xy - mt \log \varphi(y/m^{\frac{1}{2}}))\, dF(y) , \qquad t > 0 . \tag{9}$$

Then for all $n \geqq 1$, $Ef_m(S_n/m^{\frac{1}{2}}, n/m) = \int_0^\infty dF < \infty$, and so $f_m(S_n/m^{\frac{1}{2}}, n/m)$ is finite almost surely. Let $\tau > 0$ and $[\tau m]$ denote the largest integer $\leqq \tau m$. Then

$f_m(S_n/m^{\frac{1}{2}}, n/m)$, $n \geqq [\tau m]$, is a martingale, and so for every $\varepsilon > 0$,

$$\begin{aligned} &P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon \text{ for some } n \geqq \tau m] \\ (10) \qquad &\leqq P[f_m(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) \geqq \varepsilon] \\ &\quad + \varepsilon^{-1} \int_{[f_m(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) < \varepsilon]} f_m(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m)\, dP\,. \end{aligned}$$

This inequality enables us to prove the invariance theorem below.

THEOREM 3. *Suppose $X_1, X_2, \cdots$ are* i.i.d. *with $EX_1 = 0$, $EX_1^2 = 1$ and $\varphi(y) = E\exp(yX_1) < \infty$ for all $y \in (0, \alpha)$, where $0 < \alpha \leqq \infty$. Let F be a finite nontrivial measure on $(0, \infty)$ with support contained in $(0, \rho\alpha)$ for some $\rho > 0$. Then for any $\varepsilon > 0$ and $\tau \geqq 0$,*

$$\begin{aligned} \lim_{m\to\infty} P[\textstyle\int_0^\infty \exp(yS_n/m^{\frac{1}{2}} - n\log\varphi(y/m^{\frac{1}{2}}))\, dF(y) \geqq \varepsilon \text{ for some } n \geqq \tau m] \\ = P\left[\int_0^\infty \exp\left(yW(t) - \frac{t}{2}y^2\right) dF(y) \geqq \varepsilon \text{ for some } t \geqq \tau\right]. \end{aligned}$$

LEMMA 1. *Under the assumptions of Theorem 3, φ is a convex, strictly increasing and twice continuously differentiable function on $[0, \alpha)$, and $\varphi(y) = 1 + y^2/2 + \delta(y)$, where $\lim_{y\to 0+} \delta(y)/y^2 = 0$. Hence given any $k > 0$ and any $c > 0$, $[mt]\log\varphi(y/m^{\frac{1}{2}})$ converges to $ty^2/2$ as $m \to \infty$ uniformly for $t \in [0, k]$ and $y \in [0, c]$.*

LEMMA 2. *Under the assumptions of Theorem 3, if F has bounded support, then for any $k > \tau \geqq 0$, as $m \to \infty$,*

$$\max_{k \geqq t \geqq \tau} f_m(S_{[mt]}/m^{\frac{1}{2}}, [mt]/m) \to_d \max_{k \geqq t \geqq \tau} f(W(t), t)\,,$$

where f_m is defined by (9) and f by (1), and $\to_d$ denotes convergence in distribution.

PROOF. We shall show that given any sequence of real numbers m_ν such that $\rho^2 \leqq m_\nu \uparrow \infty$, we have

$$\max_{k \geqq t \geqq \tau} f_{m_\nu}(S_{[m_\nu t]}/m_\nu^{\frac{1}{2}}, [m_\nu t]/m_\nu) \to_d \max_{k \geqq t \geqq \tau} f(W(t), t) \qquad \text{as } \nu \uparrow \infty\,.$$

As in Theorem 13.8 of [1], we can construct processes $\{X^{(\nu)}(t), t \geqq 0\}$, $\nu = 1, 2, \cdots$, having for each ν the same distribution as $\{S_{[m_\nu t]}/m_\nu^{\frac{1}{2}}, t \geqq 0\}$, defined on a common probability space Ω, and a standard Wiener process $W(t)$ on the same space, such that for any subsequence (ν_j) increasing rapidly enough, we have

$$\max_{0 \leqq t \leqq k} |X^{(\nu_j)}(t) - W(t)| \to 0 \qquad \text{almost surely as } j \to \infty\,.$$

Clearly for each ν, $\max_{\tau \leqq t \leqq k} f_{m_\nu}(S_{[m_\nu t]}/m_\nu^{\frac{1}{2}}, [m_\nu t]/m_\nu)$ has the same distribution as $\max_{\tau \leqq t \leqq k} f_{m_\nu}(X^{(\nu)}(t), [m_\nu t]/m_\nu)$. Now F has bounded support, say $F[c, \infty) = 0$. By Lemma 1, $[m_{\nu_j} t]\log\varphi(y/m_{\nu_j}^{\frac{1}{2}})$ converges to $ty^2/2$ as $j \to \infty$ uniformly for $t \in [\tau, k]$ and $y \in [0, c]$. Also for almost every $\omega \in \Omega$, $X^{(\nu_j)}(t, \omega) \to W(t, \omega)$ as $j \to \infty$ uniformly for $t \in [\tau, k]$. Since $\exp(yW(t, \omega) - (t/2)y^2)$ is continuous in y, t and F is a finite measure, it is obvious that

$$\begin{aligned} &\int_{(0,c)} \exp(yX^{(\nu_j)}(t, \omega) - [m_{\nu_j} t]/\log\varphi(y/m_{\nu_j}^{\frac{1}{2}}))\, dF(y) \\ &\qquad \to \int_{(0,c)} \exp\left(yW(t, \omega) - \frac{t}{2}y^2\right) dF(y) \end{aligned}$$

as $j \to \infty$ uniformly for $t \in [\tau, k]$. Therefore the desired conclusion follows. $\square$

PROOF OF THEOREM 3. First consider the case $\tau = 0$. If $\varepsilon \leqq \int_0^\infty dF$, then clearly $P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon$ for some $n \geqq 0] = 1 = P[f(W(t), t) \geqq \varepsilon$ for some $t \geqq 0]$.

Now assume that $\varepsilon > \int_0^\infty dF$. Take any $k > 0$, $c > 0$ and let F_c be the measure on $(0, \infty)$ defined by $F_c[c, \infty) = 0$, $F_c(0, x) = F(0, x)$ for $0 < x \leqq c$. Since $\varepsilon > \int_{(0,c)} dF$, $A_{F_c}(0, \varepsilon) > 0$ and so $P[\max_{0 \leqq t \leqq k} (W(t) - A_{F_c}(t, \varepsilon)) = 0] = 0$. It then follows that

$$
\begin{aligned}
&P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon \ \text{ for some } \ n \geqq 0] \\
&\quad \geqq P[\max_{0 \leqq t \leqq k} \textstyle\int_{(0,c)} \exp(yS_{[mt]}/m^{\frac{1}{2}} - [mt] \log \varphi(y/m^{\frac{1}{2}}))\, dF(y) \geqq \varepsilon] \\
&\quad \to P\left[\max_{0 \leqq t \leqq k} \textstyle\int_{(0,c)} \exp\left(yW(t) - \frac{t}{2} y^2\right) dF(y) \geqq \varepsilon\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{as} \quad m \to \infty \quad \text{(by Lemma 2)} \\
&\quad \geqq P[W(t) \geqq A_{F_c}(t, \varepsilon) \ \text{ for some } \ t > 0] \\
&\qquad - P[W(t) \geqq A_{F_c}(t, \varepsilon) \ \text{ for some } \ t \geqq k] \\
&\quad \to \varepsilon^{-1} \textstyle\int_0^\infty dF\,, \qquad \text{letting} \quad k \to \infty \quad \text{and then} \quad c \to \infty\,.
\end{aligned}
$$

Therefore $\liminf_{m \to \infty} P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon$ for some $n \geqq 0] \geqq \varepsilon^{-1} \int_0^\infty dF$. On the other hand, since $f_m(S_n/m^{\frac{1}{2}}, n/m)$, $n \geqq 0$, is a nonnegative martingale, $P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon$ for some $n \geqq 0] \leqq \varepsilon^{-1} \int_0^\infty dF$. Hence we have proved the case where $\tau = 0$.

Hereafter we shall assume that $\tau > 0$. For any $c > 0$, define

$$
\begin{aligned}
\tilde{f}_m{}^c(x, t) &= \textstyle\int_{[c,\infty)} \exp(xy - mt \log \varphi(y/m^{\frac{1}{2}}))\, dF(y)\,, \\
f_m{}^c(x, t) &= \textstyle\int_{(0,c)} \exp(xy - mt \log \varphi(y/m^{\frac{1}{2}}))\, dF(y)\,, \qquad (11) \\
f^c(x, t) &= \textstyle\int_{(0,c)} \exp\left(xy - \frac{t}{2} y^2\right) dF(y)\,.
\end{aligned}
$$

Take any $\delta \in (0, \varepsilon)$. Then we have

$$
\begin{aligned}
&P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon \ \text{ for some } \ n \geqq \tau m] \\
&\quad \leqq P[f_m{}^c(S_n/m^{\frac{1}{2}}, n/m) \geqq \delta \ \text{ for some } \ n \geqq \tau m] \qquad (12) \\
&\qquad + P[\tilde{f}_m{}^c(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon - \delta \ \text{ for some } \ n \geqq \tau m]\,.
\end{aligned}
$$

Since $\tilde{f}_m{}^c(S_n/m^{\frac{1}{2}}, n/m)$, $n \geqq 1$, is a nonnegative martingale, we have

$$
\begin{aligned}
(13) \quad &P[\tilde{f}_m{}^c(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon - \delta \ \text{ for some } \ n \geqq \tau m] \\
&\quad \leqq (\varepsilon - \delta)^{-1} E\tilde{f}_m{}^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) = (\varepsilon - \delta)^{-1} \textstyle\int_{[c,\infty)} dF\,.
\end{aligned}
$$

By (10), we have

$$
\begin{aligned}
&P[f_m{}^c(S_n/m^{\frac{1}{2}}, n/m) \geqq \delta \ \text{ for some } \ n \geqq \tau m] \\
&\quad \leqq P[f_m{}^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) \geqq \delta] \qquad (14) \\
&\qquad + \delta^{-1} \textstyle\int_{[f_m{}^c(\ ) < \delta]} f_m{}^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m)\, dP\,.
\end{aligned}
$$

By Lemma 2, $f_m^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) \to_d f^c(W(\tau), \tau)$ as $m \to \infty$, and so we have

$$f_m^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) I_{[f_m^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) < \delta]} \to_d f^c(W(\tau), \tau) I_{[f^c(W(\tau), \tau) < \delta]} .$$

From this, it follows that

$$\begin{aligned}\lim_{m\to\infty} E(f_m^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) I_{[f_m^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) < \delta]}) &= E(f^c(W(\tau), \tau) I_{[f^c(W(\tau), \tau) < \delta]}) \\ &= \int_{(0,c)} \Phi(A_{F_c}(\tau, \delta)/\tau^{\frac{1}{2}} - y\tau^{\frac{1}{2}})\, dF(y) .\end{aligned}$$

Moreover we have

$$\begin{aligned}\lim_{m\to\infty} P[f_m^c(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) \geqq \delta] &= P[f_m^c(W(\tau), \tau) \geqq \delta] \\ &= 1 - \Phi(A_{F_c}(\tau, \delta)/\tau^{\frac{1}{2}}) .\end{aligned}$$

Hence from (12), (13), (14), we obtain

$$\begin{aligned}&\limsup_{m\to\infty} P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon \text{ for some } n \geqq \tau m] \\ &\quad \leqq 1 - \Phi(A_{F_c}(\tau, \delta)/\tau^{\frac{1}{2}}) + \delta^{-1} \int_{(0,c)} \Phi(A_{F_c}(\tau, \delta)/\tau^{\frac{1}{2}} - y\tau^{\frac{1}{2}})\, dF(y) \\ &\qquad + (\varepsilon - \delta)^{-1} \int_{[c,\infty)} dF .\end{aligned} \tag{15}$$

Since $\lim_{c\to\infty} A_{F_c}(\tau, \delta) = A_F(\tau, \delta)$, we have upon letting $c \to \infty$ in (15) that

$$\begin{aligned}&\limsup_{m\to\infty} P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon \text{ for some } n \geqq \tau m] \\ &\quad \leqq 1 - \Phi(A_F(\tau, \delta)/\tau^{\frac{1}{2}}) + \delta^{-1} \int_0^\infty \Phi(A_F(\tau, \delta)/\tau^{\frac{1}{2}} - y\tau^{\frac{1}{2}})\, dF(y) .\end{aligned} \tag{16}$$

It is easy to see that $A_F(\tau, \delta) \uparrow A_F(\tau, \varepsilon)$ as $\delta \uparrow \varepsilon$, and so by letting $\delta \uparrow \varepsilon$ in (16), we obtain

$$\begin{aligned}&\limsup_{m\to\infty} P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon \text{ for some } n \geqq \tau m] \\ &\quad \leqq 1 - \Phi(A_F(\tau, \varepsilon)/\tau^{\frac{1}{2}}) + \varepsilon^{-1} \int_0^\infty \Phi(A_F(\tau, \varepsilon)/\tau^{\frac{1}{2}} - y\tau^{\frac{1}{2}})\, dF(y) .\end{aligned}$$

We can prove that

$$\begin{aligned}&\liminf_{m\to\infty} P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon \text{ for some } n \geqq \tau m] \\ &\quad \geqq 1 - \Phi(A_F(\tau, \varepsilon)/\tau^{\frac{1}{2}}) + \varepsilon^{-1} \int_0^\infty \Phi(A_F(\tau, \varepsilon)/\tau^{\frac{1}{2}} - y\tau^{\frac{1}{2}})\, dF(y)\end{aligned}$$

in a similar way as that of (i) of the following theorem. □

While Theorem 3 considers the case where F is a finite measure, the following theorem studies the situation when F may not be finite.

THEOREM 4. *Suppose* $X_1, X_2, \cdots$ *are* i.i.d. *with* $EX_1 = 0$, $EX_1^2 = 1$ *and* $\varphi(y) = E\exp(yX_1) < \infty$ *for all* $y \geqq 0$. *Let* F *be any nontrivial measure on* $(0, \infty)$ *such that* $f(b, h) < \infty$ *for some real* b *and* $h \geqq 0$, *where* f *is defined by* (1). *Let* $\varepsilon > 0$ *and* $\tau > h$.

(i)
$$\begin{aligned}&\liminf_{m\to\infty} P\Big[\int_0^\infty \exp(yS_n/m^{\frac{1}{2}} - n\log\varphi(y/m^{\frac{1}{2}}))\, dF(y) \geqq \varepsilon \\ &\qquad \textit{for some } n \geqq \tau m\Big] \\ &\quad \geqq P\left[\int_0^\infty \exp\left(yW(t) - \frac{t}{2}y^2\right) dF(y) \geqq \varepsilon \textit{ for some } t \geqq \tau\right].\end{aligned}$$

(ii) *If* φ *is supernormal on* $(0, \infty)$, *then*

$$\lim_{m\to\infty} P[\textstyle\int_0^\infty \exp(yS_n/m^{\frac12} - n\log\varphi(y/m^{\frac12}))\,dF(y) \geqq \varepsilon \text{ for some } n \geqq \tau m]$$
$$= P\left[\int_0^\infty \exp\left(yW(t) - \frac{t}{2}y^2\right)dF(y) \geqq \varepsilon \text{ for some } t \geqq \tau\right].$$

PROOF. (i) Take any $c > 0$, $\delta \in (0, 1)$ and $k > \tau + 1$. Then for all large m,

$$P[f_m(S_n/m^{\frac12}, n/m) \geqq \varepsilon \text{ for some } n \geqq \tau m]$$
$$\geqq P[\max_{\tau+\delta\leqq t\leqq k} f_m^c(S_{[mt]}/m^{\frac12}, [mt]/m) \geqq \varepsilon]$$
$$\to P\left[\max_{\tau+\delta\leqq t\leqq k} \int_{(0,c)} \exp\left(yW(t) - \frac{t}{2}y^2\right)dF(y) \geqq \varepsilon\right] \quad \text{as } m \to \infty.$$

The last relation above follows from Lemma 2. By letting $k \to \infty$ and $\delta \downarrow 0$, we obtain that

$$P\left[\max_{\tau+\delta\leqq t\leqq k} \int_{(0,c)} \exp\left(yW(t) - \frac{t}{2}y^2\right)dF(y) \geqq \varepsilon\right]$$
$$\to P\left[\int_{(0,c)} \exp\left(yW(t) - \frac{t}{2}y^2\right)dF(y) \geqq \varepsilon \text{ for some } t \geqq \tau\right]$$
$$= 1 - \Phi(A_{F_c}(\tau, \delta)/\tau^{\frac12}) + \varepsilon^{-1}\int_{(0,c)} \Phi(A_{F_c}(\tau, \varepsilon)/\tau^{\frac12} - y\tau^{\frac12})\,dF(y)$$
$$\to 1 - \Phi(A_F(\tau, \varepsilon)/\tau^{\frac12}) + \varepsilon^{-1}\int_0^\infty \Phi(A_F(\tau, \varepsilon)/\tau^{\frac12} - y\tau^{\frac12})\,dF(y) \quad \text{as } c \to \infty.$$

The desired conclusion then follows.

(ii) We assert that if φ is supernormal on $(0, \infty)$, then $f_m(S_{[\tau m]}/m^{\frac12}, [\tau m]/m) \to_d f(W(\tau), \tau)$ as $m \to \infty$. To prove this, given any sequence of positive numbers $m_\nu \uparrow \infty$, we construct processes $\{X^{(\nu)}(t), t \geqq 0\}$, $\nu = 1, 2, \cdots$, having for each ν the same distribution as $\{S_{[m_\nu t]}/m_\nu^{\frac12}, t \geqq 0\}$, defined on a common probability space Ω, such that for any subsequence (ν_j) increasing rapidly enough, $\max_{0\leqq t\leqq\tau} |X^{(\nu_j)}(t) - W(t)| \to_{\text{a.s.}} 0$ as $j \to \infty$. Hence for almost every $\omega \in \Omega$, $\lim_{j\to\infty} X^{(\nu_j)}(\tau, \omega) = W(\tau, \omega)$, $\sup_j |X^{(\nu_j)}(\tau, \omega)| < \infty$, and for all $y > 0$,

$$\lim_{j\to\infty} \exp(yX^{(\nu_j)}(\tau, \omega) - [\tau m_{\nu_j}]\log\varphi(y/m_{\nu_j}^{\frac12})) = \exp\left(yW(\tau, \omega) - \frac{\tau}{2}y^2\right).$$

Choosing $h_1 \in (h, \tau)$ and j_0 such that $\tau - 1/m_{\nu_j} > h_1$ for all $j \geqq j_0$, we have from the supernormality of φ on $(0, \infty)$ that $[\tau m_{\nu_j}]\log\varphi(y/m_{\nu_j}^{\frac12}) \geqq h_1 y^2/2$ if $y \geqq 0$ and $j \geqq j_0$. Also

$$\int_0^\infty \exp(y \sup_j |X^{(\nu_j)}(\tau, \omega)| - \tfrac12 h_1 y^2)\,dF(y) < \infty.$$

Therefore by the dominated convergence theorem,

$$\lim_{j\to\infty} f_{m_{\nu_j}}(X^{(\nu_j)}(\tau, \omega), [\tau m_{\nu_j}]/m_{\nu_j}) = f(W(\tau, \omega), \tau) \quad \text{a.s.}$$

It is then clear that $f_m(S_{[\tau m]}/m^{\frac12}, [\tau m]/m) \to_d f(W(\tau), \tau)$ as $m \to \infty$.

By (10), we have

$$P[f_m(S_n/m^{\frac12}, n/m) \geqq \varepsilon \text{ for some } n \geqq \tau m]$$
$$\leqq P[f_m(S_{[\tau m]}/m^{\frac12}, [\tau m]/m) \geqq \varepsilon] + \varepsilon^{-1}\int_{[f_m(\ )<\varepsilon]} f_m(S_{[\tau m]}/m^{\frac12}, [\tau m]/m)\,dP.$$

Since $f_m(S_{[\tau m]}/m^{\frac{1}{2}}, [\tau m]/m) \to_d f(W(\tau), \tau)$, it then follows that

$$\limsup_{m\to\infty} P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon \text{ for some } n \geqq \tau m] \leqq P[f(W(t), t) \geqq \varepsilon \text{ for some } t \geqq \tau]. \qquad \square$$

Let $X_1, X_2, \cdots, S_n, \varphi, F, \varepsilon, \tau$ be as in Theorem 4. If φ is not supernormal on $(0, \infty)$, then the conclusion in (ii) of Theorem 4 may not be true. Consider the following example. Let $X_1, X_2, \cdots$ be i.i.d. with $P[X_1 = 1] = P[X_1 = -1] = \frac{1}{2}$. Then $\varphi(y) = E\exp(yX_1) = \cosh y$ and φ is subnormal on the whole real line. Let F be a measure on $(0, \infty)$ defined by $dF(y) = \exp(y^\beta)\,dy$, $y > 0$, where $1 < \beta < 2$. Now $f(x, t) < \infty$ for all $t > 0$ and all real x, and we can choose $\tau > 0$ such that $P[W(t) \geqq A_F(t, \varepsilon)$ for some $t \geqq \tau] < 1$. On the other hand, $f_m(x, t) = \infty$ for all $m > 0$, $t \geqq 0$ and all real x, so that $P[f_m(S_n/m^{\frac{1}{2}}, n/m) \geqq \varepsilon$ for some $n \geqq \tau m] = 1$ for all $m > 0$. Instead of assuming φ to be supernormal on $(0, \infty)$, however, it is clear from our proof in (ii) of Theorem 4 that we can replace the supernormality of φ by the following more general condition:

$$\begin{aligned}&\text{There exists } m_0 > 0 \text{ and a function } g: (0, \infty) \to (0, \infty) \text{ such}\\ &\text{that } m \log \varphi(y/m^{\frac{1}{2}}) \geqq g(y) \text{ for all } y > 0 \text{ and } m \geqq m_0, \text{ and}\\ &\textstyle\int_0^\infty \exp(xy - tg(y))\,dF(y) < \infty \text{ for all real } x \text{ and } t > h.\end{aligned} \tag{17}$$

Suppose X is a random variable with $EX = 0$, $EX^2 = 1$ and $\varphi(y) = E\exp(yX) < \infty$ for $y \in (0, \alpha)$ where $0 < \alpha \leqq \infty$. Let F be any nontrivial measure on $(0, \infty)$ such that either F is a finite measure with bounded support, or if the support of F is unbounded, then $\alpha = \infty$, (17) holds and there exist $h \geqq 0$ and b real for which $f(b, h) < \infty$. Then for any $\varepsilon > 0$, $t > h$ and all large m, we can define $A_F^{(m)}(t, \varepsilon)$ as the unique solution $x = A_F^{(m)}(t, \varepsilon)$ of $f_m(x, t) = \varepsilon$. The function $A_F^{(m)}(t, \varepsilon)$ is concave, strictly increasing and continuous on (h, ∞) for all large m, say $m \geqq m_1$. It can be proved that if $t > h$, then $A_F^{(m)}(t, \varepsilon)$ converges to $A_F(t, \varepsilon)$ as $m \to \infty$ uniformly on every compact subset of (h, ∞). When φ is supernormal on $(0, \infty)$, $A_F^{(m)}(t, \varepsilon) \geqq A_F(t, \varepsilon)$, $t > h$, $m \geqq m_1$. The following theorem, which is an extension of Theorem 2 in [14], then provides us with another way of looking at Theorem 4.

THEOREM 5. *Let $X_1, X_2, \cdots$ be* i.i.d. *random variables such that $EX_1 = 0$, $EX_1^2 = 1$. Let $h \geqq 0$ and let $\{g_m\}_{m>0}$ be a family of continuous functions on (h, ∞) such that $g_m(t)$ converges to $g(t)$ as $m \to \infty$, the convergence being uniform on every compact subset of (h, ∞).*

(i) *For any $k > \tau > h$,*

$$\lim_{m\to\infty} P[S_n \geqq m^{\frac{1}{2}} g_m(n/m) \text{ for some } km \geqq n \geqq \tau m] = P[W(t) \geqq g(t) \text{ for some } k \geqq t \geqq \tau].$$

(ii) *Assume*:

$$\begin{aligned}&\text{There exist } \tau_1 > h \text{ and a continuous function } \psi: [\tau_1, \infty) \to\\ &[0, \infty) \text{ such that } g_m(t) \geqq \psi(t) \text{ for all } t \geqq \tau_1,\ m \geqq m_0, \text{ and}\\ &\psi(t)/t^{\frac{1}{2}} \text{ is nondecreasing and } \textstyle\int_{\tau_1}^\infty t^{-\frac{3}{2}}\psi(t)\exp(-\psi^2(t)/2t)\,dt < \infty.\end{aligned} \tag{18}$$

Then for any $\tau > h$,

$$\lim_{m\to\infty} P[S_n \geqq m^{\frac{1}{2}} g_m(n/m) \text{ for some } n \geqq \tau m] = P[W(t) \geqq g(t) \text{ for some } t \geqq \tau] .$$

(iii) *Let* $h = 0$. *Suppose*

(19) *There exist* $\tau_2 > 0$ *and a continuous function* $\tilde{\psi}: (0, \tau_2) \to [0, \infty)$ *such that* $g_m(t) \geqq \tilde{\psi}(t)$ *for all* $t \in (0, \tau_2)$, $m \geqq m_0$, *and* $\tilde{\psi}(t)/t^{\frac{1}{2}}$ *is nonincreasing and* $\int_{0+}^{\tau_2} t^{-\frac{3}{2}}\tilde{\psi}(t) \exp(-\tilde{\psi}^2(t)/2t)\, dt < \infty$.

Then for any $\tau > 0$,

$$\lim_{m\to\infty} P[S_n \geqq m^{\frac{1}{2}} g_m(n/m) \text{ for some } 1 \leqq n \leqq \tau m] = P[W(t) \geqq g(t) \text{ for some } 0 < t \leqq \tau] .$$

(iv) *Let* $h = 0$. *Suppose* (18) *and* (19) *are both satisfied. Then*

$$\lim_{m\to\infty} P[S_n \geqq m^{\frac{1}{2}} g_m(n/m) \text{ for some } n \geqq 1] = P[W(t) \geqq g(t) \text{ for some } t > 0] .$$

The same relations remain valid if S_n, $W(t)$ *are replaced by* $|S_n|$ *and* $|W(t)|$.

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