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ON THE MEAN CONVERGENCE OF THE BEST LINEAR INTERPOLATOR OF MULTIVARIATE STATIONARY STOCHASTIC PROCESSES[1]

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It is shown that a necessary and sufficient condition, for the existence of a mean-convergent series for the linear interpolator of a q-variate stationary stochastic process $\{X_n\}$ with density matrix W, is that the Fourier series of the isomorph of the linear interpolator should converge in the norm of $L^2(W)$, and this happens if the past and future of the process are at positive angle. This provides a positive answer to a question of H. Salehi (1979) concerning the square summability of the inverse of W and improves upon the work of Rozanov (1960) and Salehi (1979).

Introduction. Let $\{X_n\}$ be a q-variate, discrete parameter, weakly stationary stochastic process (S.S.P.) with the spectral density matrix W. Assume that all the values of $\{X_n\}$ are known, except for the values X_k, $k \in T$, where T is a finite subset of the set of all integers Z. An extremely important problem in the theory of q-variate S.S.P.'s is to interpolate the unknown values of X_k, $k \in T$ by using the known values X_k, $k \in T'$, where $T' = Z \backslash T$. In this case, the natural thing to do is to find a mean-convergent series representation for the interpolator in the time domain in terms of X_k, $k \in T'$. The possibility of such a series representation was first studied by Rozanov [7]. In [7] it is shown that if $W \in L^{\infty}_{q\times q}$ and $W^{-1} \in L^2_{q\times q}$, then such a series can be found. Later, by using the Von Neumann's alternating projections, Salehi [10] found an expression for the linear interpolator of a q-variate S.S.P. under the assumption that $W \in L^{\infty}_{q\times q}$ and $W^{-1} \in L^1_{q\times q}$. But, the expression obtained in [10] depends upon the optimal factor of the spectral density, the reciprocal of the optimal factor and the innovations of the process and consequently it is not suited for applications, because the formula for the interpolator is not explicit in X_k, $k \in T'$. In 1979 [11], for $q = 1$ and a stationary random field, Salehi has found a mean-convergent series for the linear interpolator in terms of the known values X_k, $k \in T'$ under the assumption that $W \in L^{\infty}$ and $W^{-1} \in L^2$. In [11], among other interesting open questions in this field, it is asked, whether the condition $W^{-1} \in L^2$ can be replaced by $W^{-1} \in L^1$. In this paper we show that the answer is positive. Actually, among other results, we show that, for $1 \leq q < \infty$, both $W \in L^{\infty}_{q\times q}$ and $W^{-1} \in L^2_{q\times q}$ can be weakened and replaced by the condition that the past and the future of the process be at positive angle i.e. $\rho(W) < 1$.

Our method will rest on using our previous results [5] concerning the "unrav-

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eling" of the Kolmogorov's isomorphism between the spectral and time domains and its relation with the positivity of the angle between the past and the future of the process.

1. Notation and preliminaries. Throughout the paper, for a $q \times q$ matrix $A = (a_{ij})$, $\operatorname{tr} A = \sum_{i=1}^{q} a_{ii}$, $A^* = (\bar{a}_{ji})$, A^{-1} for the inverse of A if it exists. Functions will be defined on $[-\pi, \pi]$ and we identify this interval with the unit circle in the complex-plane in the natural way. Values of a function f defined on $[-\pi, \pi]$ or on the unit circle will be denoted by $f(\theta)$. dm denotes the normalized Lebesgue measure on $[-\pi, \pi]$ i.e. $dm(\theta) = (2\pi)^{-1}d\theta$ and $\int$ stands for $\int_{-\pi}^{\pi}$. For $1 \leq p \leq \infty$, L^p denotes the usual Lebesgue space of functions on the interval $[-\pi, \pi]$. $L^p_{q\times q}$ denotes the space of all $q \times q$ matrix-valued functions whose entries are in L^p.

In the following, we introduce a few concepts which are needed in this study. For further study and information concerning the general theory of q-variate S.S.P.'s [4] and for interpolation theory of such processes [8, 9, 11] are recommended.

Let $\{X_n; n \in Z\} \subset H^q$, be a q-variate S.S.P., where H^q is the Cartesian product of a Hilbert space H with itself q times. For each process $\{X_n\} \subset H^q$ and integers $-\infty \leq k \leq \ell \leq \infty$, we define $M_k^{\ell}(X) = \overline{\operatorname{sp}}\{X_n; k \leq n \leq \ell\}$, where $\overline{\operatorname{sp}}\{\cdots\}$ stands for the closed linear span of elements of $\{\cdots\}$ in the metric of H^q. $M(X) = M_{-\infty}^{\infty}(X)$ is called the time space (domain) of the process $\{X_n\}$. The spectral space (domain) corresponding to the spectral density matrix W of the process is denoted by $L^2(W)$ and is defined by $L^2(W) = \{\Phi;\ \Phi$ a $q \times q$ matrix-valued function with $\|\Phi\|_W^2 = \int \operatorname{tr} \Phi(\theta)\, W(\theta)\Phi(\theta)^*\, dm(\theta) < \infty\}$. It is known [4] that $L^2(W)$ with the inner product $((\Phi, \Psi))_W = \int \operatorname{tr} \Phi W \Psi^*\, dm$ is a Hilbert space. The map V: $M(X) \to L^2(W)$ defined by $VX_n = e^{-in\theta}I$, $n \in Z$, where I is the $q \times q$ identity matrix, can be extended to an isomorphism between $M(X)$ and $L^2(W)$, [4]. This extension is also denoted by V and is called the Kolmogorov's isomorphism between the time and spectral domains. Under this isomorphism to each $Y \in M(X)$ there corresponds a unique function $\Phi \in L^2(W)$, which is called the *isomorph* of $Y \in M(X)$ in $L^2(W)$.

2. Interpolation of a q-variate S.S.P. From here on, we assume that our S.S.P. $\{X_n\}$ with the spectral density matrix W is *full-rank minimal* i.e. $W^{-1} \in L^1_{q\times q}$. Let $T \subset Z$ be a finite set and $k \in T$. In the following theorems we obtain conditions and formulas for the linear interpolator of X_k. Because of stationarity of the process, without of loss of generality we assume that $0 \in T$ and $k = 0$.

Let $M_{T'} = \overline{\operatorname{sp}}\{X_k;\ k \in T'\}$ in H^q. Then the best linear interpolator of X_0 denoted by $\hat{X}_0$ is defined by $\hat{X}_0 = (X_0 \,|\, M_{T'})$, where $(X_0 \,|\, M_{T'})$ denotes the orthogonal projection of X_0 onto the subspace $M_{T'}$ of H^q. Since $\hat{X}_0 \in M(X)$, because of the isomorphism between the spectral and time domains there exists a function $\Phi \in L^2(W)$ which is the isomorph of $\hat{X}_0$. In order to find a formula for $\hat{X}_0$ in the time domain, we must have the explicit form of $\Phi \in L^2(W)$. But it is shown in [8, page 101] that $\Phi(\theta) = I - (\sum_{k\in T} D_k e^{ik\theta}) W^{-1}(\theta)$ a.e. (θ), where D_k, $k \in T$, are constant $q \times q$ matrices and can be obtained by solving the following system of linear equations;

$$\begin{cases} \sum_{k\in T} D_k C_{k-t} = 0 \quad \text{for} \quad t \in T\setminus\{0\}, \\ \sum_{k\in T} D_k C_k = I, \end{cases} \tag{1}$$

where $C_k = \int e^{-ik\theta} W^{-1}(\theta)\, dm(\theta)$, $k \in Z$, i.e. the kth Fourier coefficient of the inverse of W. It follows from the system of equations (1) that Φ, the isomorph of $\hat{X}_0$, has a Fourier series representation as:

$$\Phi(\theta) \sim \sum_{k\in T'} B_k e^{-ik\theta}, \tag{2}$$

and its Fourier coefficients B_k, $k \in T'$, can be written explicitly in terms of C_k's and D_k's. Actually, we have

$$B_k = -\sum_{\ell\in T} D_\ell C_{\ell-k}, \quad k \in T'. \tag{3}$$

In view of the relation (2) and the isomorphism between the spectral and time domains, the temptation of writing $\hat{X}_0 = \sum_{k\in T'} B_k X_k$ cannot be resisted. However, in general, this is not correct, as the infinite series $\sum_{k\in T'} B_k X_k$ may not converge. The rest of this paper is devoted to finding conditions on W such that the series $\sum_{k\in T'} B_k X_k$ converges in the norm of H^q. The next important and simple lemma is an immediate consequence of the isomorphism between spectral and time domains [5].

For $\Phi \in L^2(W)$ we denote its kth Fourier coefficient by

$$\Phi_k = \int e^{-ik\theta}\Phi(\theta)\, dm(\theta),\ k \in Z.$$

We note that for a general W, Φ_k is not necessarily well-defined. But, in this paper we only deal with W's such that W and $W^{-1} \in L^1_{q\times q}$, in this case by a simple use of Cauchy-Schwartz inequality it can be shown that $\Phi \in L^1_{q\times q}$ i.e. Φ_k, $k \in Z$ is well-defined.

3. Lemma. *Let Y be an arbitrary element of $M(X)$ and $\Phi \in L^2(W)$, with Fourier coefficients Φ_k, $k \in Z$, be the isomorph of Y under the map V. Then $Y = \sum_{k=-\infty}^{\infty} \Phi_k X_{-k}$ in H^q if and only if the Fourier series of Φ converges in the norm of $L^2(W)$.*

By an application of Lemma 3 to $\hat{X}_0$, it follows that, for the series $\sum_{k\in T'} B_k X_k$ to be convergent in H^q, it is enough to find conditions on W such that the Fourier series (2) of Φ is convergent in $L^2(W)$. These conditions on W are, generally, found by demanding the Fourier series of the *individual* function Φ to be convergent, cf. [4, 7, 11]. Here, we use a different approach, that is we find a condition on W such that the Fourier series of *every* function in $L^2(W)$ is convergent in its norm.

Several important problems in the theory of q-variate S.S.P.'s are related to the problem of "unraveling" of the Kolmogorov's isomorphism V. Namely, finding conditions on W such that given any $\Phi \in L^2(W)$ as the isomorph of any $Y \in M(X)$, it is possible to find a mean-convergent series for Y in terms of X_n; $n \in Z$. In [5] we have discussed the importance of such "unraveling" to the problem of prediction of a S.S.P. Here, we show its importance and application to the problem of interpolation of q-variate S.S.P.'s through the use of the measure of the angle between the past and the future of the process.

As a measure of angle between $M_{-\infty}^{0}(X)$ and $M_{1}^{\infty}(X)$ we define, $\rho(W) = \sup |((P, F))|$, where P and F vary over the unit balls of $M_{-\infty}^{0}(X)$ and $M_{1}^{\infty}(X)$, respectively. It is clear that $0 \leq \rho(W) \leq 1$. The past and future of $\{X_n\}$ is said to be at *positive angle* if $\rho(W) < 1$. We note that if $\rho(W) < 1$, then $W^{-1} \in L_{q\times q}^{1}$, cf. [6].

For ease of reference, in the following we shall state without proof some of the results of [5] which are basic to our present work. For a density function $W(\theta)$, $\theta \in [-\pi, \pi]$ we denote its smallest and largest eigenvalues by $w_1(\theta)$ and $w_q(\theta)$, $\theta \in [-\pi, \pi]$, respectively.

4. Theorem. *Let W be a $q \times q$ matricial spectral density function.*

(a) *If $\rho(W) < 1$, then the Fourier series of every $\Phi \in L^2(W)$ converges in the norm of $L^2(W)$.*
(b) *If the Fourier series of every $\Phi \in L^2(W)$ converges in the norm of $L^2(W)$ and $w_1^{-1} \in L^\infty$, then $\rho(W) < 1$.*
(c) *If $(w_q/w_1) \in L^\infty$, then $\rho(W) < 1$ if and only if $w_q = e^{u+\tilde{v}}$, where u and v are bounded real-valued functions with $\| v \|_\infty < (\pi/2)$ and $\tilde{v}$ denotes the harmonic conjugate of v.*

Now, by applying Theorem 4 and Lemma 3 to the isomorph of $\hat{X}_0$ and other variables of interest, we find some useful conditions on W for the mean convergence of series of the form $\sum B_k X_k$.

In the important special case, in which T is a singleton, i.e., $T = \{0\}$, the system of equations (1) reduces to the very simple equation $D_0 C_0 = I$ or $D_0 = C_0^{-1} = (\int W^{-1}\, dm)^{-1}$. In this case, we have from (3) that, $B_k = -D_0 C_{-k} = -C_0^{-1} C_{-k}$, $k \neq 0$.

The S.S.P. $\{Y_n\}$ defined by $Y_n = C_0(X_n - \hat{X}_n)$ is called the *normalized two-sided innovation* process of $\{X_n\}$, [3]. A formula for finding Y_n in terms of the normalized one-sided innovation process of $\{X_n\}$ and the Taylor coefficients of the reciprocal of the optimal factor of the spectral density W is given by Lemma 2.7 (b) [3]. In the following theorem, we find a simple expression for Y_n in terms of X_n, $n \in Z$ and the Fourier coefficients of the reciprocal of W. We note that for each integer n, $e^{-in\theta} W^{-1}$ is the isomorph of Y_n, [3].

5. Theorem. *Let $\{X_n\}$ be a full-rank minimal q-variate* S.S.P. *with the normalized two-sided innovation process $\{Y_n\}$ and the spectral density matrix W. Then,*

(a) $Y_n = \sum_{k=-\infty}^{\infty} C_k X_{n-k}$, *if and only if the Fourier series of W^{-1} converges in the norm of $L^2(W)$.*
(b) $Y_n = \sum_{k=-\infty}^{\infty} C_k X_{n-k}$, *if $\rho(W) < 1$.*
(c) $Y_n = \sum_{k=-\infty}^{\infty} C_k X_{n-k}$, *if $W \in L_{q\times q}^{\infty}$, $W^{-1} \in L_{q\times q}^{2}$.*

Theorem 5 (a) and (b) follow immediately from Lemma 3, and Theorem 4 (a), respectively. Theorem 5(c) is an immediate consequence of the Riesz-Fischer Theorem and the boundedness of W.

6. Remark. Since $Y_0 = C_0(X_0 - \hat{X}_0)$ or $\hat{X}_0 = X_0 - D_0 Y_0$, it follows from Theorem 5 that, $\hat{X}_0 = X_0 - D_0 \sum_{k=-\infty}^{\infty} C_k X_{-k} = X_0 - D_0 C_0 X_0 - \sum_{k\neq 0}^{\infty} D_0 C_k X_{-k} = \sum_{k\neq 0}^{\infty} (-D_0 C_{-k}) X_k = \sum_{k\neq 0}^{\infty} B_k X_k$. Thus, the best linear interpolator $\sum_{k\neq 0}^{\infty} B_k X_k$ is convergent in H^q, if and only if the series for Y_n is convergent in H^q. This shows that the conditions of Theorem 5 are sufficient for the convergence of the representation of $\hat{X}_0$.

The next theorem provides conditions for the convergence of $\sum_{n\in T'} B_k X_k$, when T is not necessarily $\{0\}$. Its proof follows from Lemma 3 and Theorem 4 applied to the function $\Phi \in L^2(W)$ with Fourier series and coefficients as in (2) and (3).

7. Theorem. *Let* $\{X_n\}$ *be a full-rank minimal q-variate* S.S.P. *with the spectral density matrix* W. *Then,*

(a) $\hat{X}_0 = \sum_{k\in T'} B_k X_k$, *if and only if the Fourier series of* Φ *converges in the norm of* $L^2(W)$.
(b) $\hat{X}_0 = \sum_{k\in T'} B_k X_k$ *if* $\rho(W) < 1$.
(c) $\hat{X}_0 = \sum_{k\in T'} B_k X_k$ *if* $W \in L^{\infty}_{q\times q}$ *and* $W^{-1} \in L^2_{q\times q}$.

8. Remarks. (a) There are many unbounded matricial spectral density functions such that $\rho(W) < 1$. This can be seen either from Theorem 4(c) or the Helson-Szegö Theorem [1, 2] on characterization of scalar density functions W with $\rho(W) < 1$. It is shown in [1] that $\rho(W) < 1$, if and only if $W = e^{u+\tilde{v}}$, where u and v are bounded real-valued functions with $\| v \|_\infty < (\pi/2)$. Also, it is known that if $\rho(W) < 1$, then both W and $W^{-1} \in L^{1+\varepsilon}$ for ε sufficiently small [12, page 81] (This shows that W^{-1} is not necessarily in L^2.) It follows from the Helson-Szegö's Theorem that the class of density functions defined by $W(\theta) = |1 + e^{i\theta}|^{\alpha} |1 - e^{i\theta}|^{-\beta}$, $\beta > 0$, $\alpha \geq 1/2$ and $\alpha + \beta < 1$, has the property that $\rho(W) < 1$. But, these densities are neither bounded nor their reciprocals belong to L^2. Thus, it follows from Theorem 4(c) that a matricial density function with $\rho(W) < 1$ is not necessarily in $L^{\infty}_{q\times q}$ nor $W^{-1} \in L^2_{q\times q}$. Thus, Theorem 7(b) provides a positive answer to Salehi's question [11, page 841] and improves upon the work of Rozanov [7] on this problem.

(b) We note that in Theorems 5 and 7, the Fourier coefficients of W^{-1} are playing a role similar to that of Fourier coefficients of the reciprocal of the optimal factor of W, cf. [3, 4]. Considering the difficulties of finding the optimal factor of W [4], it becomes clear that the task of implementing the algorithm for the linear interpolation is much easier than that of the linear prediction of S.S.P.'s.

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