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COMMENTS ON A PROBLEM OF CHERNOFF AND PETKAU

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A new method is used to study the optimal stopping set corrected for discreteness introduced by Chernoff and studied by Chernoff and Petkau. The discrete boundary is asymptotically the optimal boundary for a Wiener process translated downward by a constant amount. This amount is shown to be an "excess over the boundary" term, and this method yields it as a simple integral involving the characteristic function of the random walk.

This paper consists of an application of ideas about boundary crossing by random walks to problems considered by Chernoff (1965), and Chernoff and Petkau (1976). In Chernoff (1961), a Bayes test for the sign of a normal mean leads, in the diffusion limit, to an optimal stopping problem for the Wiener process, whose solution can be shown to be given by stopping the first time a Wiener process crosses a certain boundary. The boundary is given as the solution to a free boundary problem. In Chernoff (1965) the discrete version of the same problem is considered. This can be embedded in the original problem by allowing stopping only at the discrete times $n\delta$, $n = 0, 1, \dots$. Once again it is possible to show that the optimal policy is to stop when the Wiener process crosses a certain boundary at the permitted times. The question is: what is the relation between the boundary $\tilde{x}(t)$ of the unrestricted problem, and $\tilde{x}_\delta(t)$ of the restricted problem? Chernoff showed that $\tilde{x}_\delta(t) = x(t) + \hat{Z}\delta^{1/2} + o(\delta^{1/2})$, where, according to Chernoff (1965), $\hat{Z} = -0.582$. The sign of $\hat{Z}$ makes the continuation region smaller for the discrete problem, as it ought to because policies available in the discrete problem are a subset of those available in the continuous problem.

The key step in the proof of this is to introduce an auxiliary problem in which a Wiener process is started at a point (z, t), $t < 0$, no payoff is made if stopping occurs before time 0, and at $t = 0$ stopping is enforced and a payoff $Z^2 1_{(Z<0)}$ is received, stopping is permitted only at times $t = 0, -1, \dots,$ and each observation costs a dollar. For this problem the optimal stopping boundary can be shown to be increasing and contained in $[-1, 0]$, and therefore it has a limit as $t \to -\infty$ which turns out to be $\hat{Z}$.

In Chernoff and Petkau (1976) the solution to the auxiliary problem is considered for a family of dichotomous random variables depending on a parameter p, and continuity of $\hat{Z}$ as a function of p is established, as well as a method for calculating $\hat{Z}$ when p is rational.

For the normal random walk, Siegmund remarked that $\hat{Z} = ER$, where R is a random variable whose distribution is the same as that of the asymptotic excess over the boundary (see below for precise definition). Chernoff's analytic machin-

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ery, in particular the probabilistic interpretation of the solution of a Weiner–Hopf equation [see Spitzer (1957, 1960)] shows this to be the case. The first part of this paper presents a more direct connection between the role of $\hat{Z}$ as the expected asymptotic excess over the boundary, and $\hat{Z}$ as the solution to the auxiliary problem. The second part identifies the quantity $\hat{Z}$ in the auxiliary problem for arbitrary random walk with $E|S_1|^4 < \infty$, and establishes the continuity as a function of p in Chernoff and Petkau's family of dichotomous random variables x. $\hat{Z}$ can be computed as a one dimensional integral involving the characteristic function of S_1.

Here is some notation that will be used in the proof of Theorem 1; however be warned that due to the extremely heavy notational demands imposed by Theorem 2 these definitions will only hold for the proof of Theorem 1. Let $S_n = \sum_{i=1}^n X_i$, be a random walk with $ES_1 = 0$, $ES_1^2 = 1$, and $E|S_1|^4 < \infty$. Let $\tau_a = \inf\{n: S_n > a\}$, $R_a = S_{\tau_a} - a$, $\tau_+ = \tau_0$, and R be a random variable such that $\mathscr{L}(R) = \lim_{a \to \infty} \mathscr{L}(R_a)$, where $\mathscr{L}(X)$ denotes the distribution of the random variable X, and where $a \to \infty$ through numbers of the form nd if S_1 is arithmetic with span d. This limit is known to exist from renewal theory and it is also known that $\lim_{a \to \infty} ER_a = ER$. It is easy to show that for nonarithmetic S_1, $ER = ES_{\tau_+}^2/2ES_{\tau_+}$ while for arithmetic S_1 with span d, $ER = ES_{\tau_+}^2/2ES_{\tau_+} - d/2$. The limit as $t \to -\infty$ of the optimal boundary of the auxiliary problem will be denoted $\hat{Z}$. (An arithmetic random variable takes values in a discrete subgroup of $\mathbb{R}$. A lattice is a coset of a discrete subgroup of $\mathbb{R}$.)

The auxiliary problem is as defined above, except that the random walk generated by S_1 is run rather than a Wiener process.

THEOREM 1. $\hat{Z} = -ES_{\tau_+}^2/2ES_{\tau_+}$.

REMARK. According to Siegmund (1985, Chapter X),

$$-\frac{ES_{\tau_+}^2}{2ES_{\tau_+}} = \tfrac{1}{6}ES_1^3 - \frac{1}{\pi}\int_0^\infty t^{-2}\mathrm{Re}\log\{2[1 - f(t)]/t^2\}\,dt,$$

where Re denotes real part, and $f(s) = E\exp(isS_1)$. This expression, for the normal distribution, is found in Chernoff (1965).

PROOF. It must be shown in general that the optimal policy is given by a stopping boundary $Z(t)$ where $Z(t) \to \hat{Z}$ as $t \to -\infty$. Assume this for the time being and we return to it at the end of the proof.

Suppose the optimal boundary is given by $Z(t)$, $t = 0, -1, \dots$ and that the random walk starts at time $-n_0$. Let $T = \inf\{n: S_n > Z(n + n_0)\} \wedge n_0$. Recall that the same boundary is optimal regardless of starting position. Then, Chernoff and Petkau show that the problem is equivalent to picking $Z(t)$ to minimize

$$E\big(S_T^2; T < n_0\big) + E\big(S_T^2; T = n_0, S_T \geq 0\big). \tag{1}$$

Now, assume that S_1 is nonlattice. A starting position can be chosen so that S_n crosses $Z(t)$ when $|Z - \hat{Z}| < \varepsilon$, with probability $> 1 - \varepsilon$. Moreover, this can be chosen so that $\hat{Z}$ is very much larger than the starting position. The small variation of Z over the region where most of the crossing occurs makes it clear that S_T has approximately the distribution $\hat{Z} + R$. This can be made rigorous using simplified versions of an argument presented in Hogan (1984). The idea in this case is clear: the nearly constant boundary Z can be replaced by the constant boundary $\hat{Z}$. It is necessary to show that the second term in (1) can be made arbitrarily small. By the definition of S_T, and the observed fact that $Z(t) \in [-k, 0]$ for some k and for all t, if the random walk starts at $-z$, $z \geq 0$, then $S_T \leq R_z + k$, where here R_z is the excess above 0 starting from $-z$. Then using problem 5, page 232 of Spitzer (1976), and Theorem 2.4 of Woodroofe (1982) it follows that $S_T^2 1_{\{S_T > 0\}}$ is uniformly integrable, as possible starting positions vary over negative numbers. Therefore, by (1), the problem is equivalent to minimizing

$$E(\hat{Z} + R)^2 + \eta,$$

where η can be made arbitrarily small by a suitable choice of starting position. The solution to this problem is to take $\hat{Z} = -ER$.

In the arithmetic case, let d be the span of the distribution of S_1. A starting position determines a lattice on which the random walk lives. Let $\Delta(\hat{Z})$ denote the distance from the random walk's starting position to the closest lattice value smaller than $\hat{Z}$. Excess over $\hat{Z} - \Delta(\hat{Z})$ has a known asymptotic distribution. The problem is essentially to minimize $E(\hat{Z} + Y)^2$, where again Y is the excess over the optimal boundary. Now, though, $Y + \Delta(Z)$ has the fixed, known limiting distribution $\mathscr{L}(R)$. Consequently,

$$\begin{aligned} E(\hat{Z} + Y)^2 &= E(Y + \hat{Z} + \Delta(\hat{Z}) - \Delta(\hat{Z}))^2 \\ &= E(Y + \Delta(\hat{Z}) - ER + ER + \hat{Z} - \Delta(\hat{Z}))^2 \\ &= E(Y + \Delta(\hat{Z}) - ER)^2 + (ER + \hat{Z} - \Delta(\hat{Z}))^2. \end{aligned}$$

The problem of minimizing this is equivalent to that of minimizing $(ER + \hat{Z} - \Delta(\hat{Z}))^2$, as the first term is approximately the same regardless of the value of $\hat{Z}$. It is easy to see that this is done by taking $\hat{Z} = ER + d/2$ for the following reason. $\hat{Z} - \Delta(\hat{Z})$ denotes the largest lattice value smaller than $\hat{Z}$. $\hat{Z}$ should certainly be chosen to make this the closest lattice value to $-ER$, and, for a lattice with span d, $\hat{Z} = -ER + d/2$ is a recipe for doing that. The proof is finished by observing that, by the arithmetic renewal theorem,

$$ER - \frac{d}{2} = \frac{ES_{\tau_+}^2}{2ES_{\tau_+}}.$$

We return to the point raised at the beginning of the proof. First, Lemmas 3.2 of Chernoff (1965) and 3.2 of Chernoff and Petkau (1976) hold for a general mean 0, variance 1 random walk and it suffices to show that the optimal policy is given by a stopping boundary $Z(t)$ with $Z(t)$ increasing. It need only be shown that $Z(t)$ is bounded below to establish that $\hat{Z} = \lim_{t \to -\infty} Z(t)$ exists and is finite. Consider the policy that consists of stopping the first time the random walk exceeds 0. Formula 3.5 of Chernoff and Petkau (1976) shows that the reward using this policy starting from the initial position $-y$ is

$$y^2 - E(S_t^2;\ S_t > 0),$$

where t is the first time the process is above 0, or time 0, whichever comes first. Above it is shown that the quantities $E(S_t^2;\ S_t > 0)$ are uniformly bounded as starting positions vary over negative numbers. When $-y$ is such that y^2 is larger than this bound it pays to continue, and this provides a lower bound for Z. □

Chernoff and Petkau consider the same stopping problem for a family of random walks generated by dichotomous random variables

$$S_1^p = \begin{cases} (p/(1-p))^{1/2} = b(p) & \text{with probability } 1-p \\ -((1-p)/p)^{1/2} = -a(p) & \text{otherwise,} \end{cases}$$

$ES_1^p = 0$, $E(S_1^p)^2 = 1$. It is possible to give a pathwise construction of this process that shows that up to any stochastically bounded time t the process is continuous in probability as a function of p, i.e., for a fixed p_0, and for p close to p_0 the paths up to t can be made arbitrarily close to those of p_0 except for a set of small probability. This seems reasonably clear so the proof is omitted. Another fact about these processes that should be noted is that S_1^p is nonarithmetic iff p is irrational. Here is some notation that will be necessary:

$$\tau_+^p = \inf\{n\colon S_n^p > 0\} \quad \text{and by induction,}$$

$$\tau_+^{p(1)} = \tau_+^p,$$

$$\tau_+^{p(n)} = \inf\left\{j > \tau_+^{p(n-1)}\colon S_j^p > S^p_{\tau_+^{p(n-1)}}\right\} \quad \text{for } n > 1,$$

$$\bar{\tau}_+^p = \inf\{n\colon S_n^p \geq 0\},$$

$$\tau_\varepsilon^p = \inf\{n\colon S_n^p > \varepsilon\}.$$

If p is irrational, $\bar{\tau}_+^p = \tau_+^p$ with probability 1, for

$$P\{S_n^p = 0\} = 0 \quad \forall\, n.$$

The symbol p will often be omitted when this can be done unambiguously.

Here are some consequences of Theorem 1, observed in Chernoff and Petkau (1976), applied to this family of random walks.

(1) $\hat{Z} \geq -b/2$ (part of Theorem 3.3).

PROOF. Since $S_{\tau_+} \leq b$,

$$\hat{Z} = -\frac{ES_{\tau_+}^2}{2ES_{\tau_+}} \geq -\frac{b\,ES_{\tau_+}}{2ES_{\tau_+}} = -\frac{b}{2},$$

where $b = b(p)$ as above. Jensen's inequality produces the upper bound

$$\hat{Z} \leq -\frac{(ES_{\tau_+})^2}{2ES_{\tau_+}} = \frac{-ES_{\tau_+}}{2} \leq \frac{-ES_1^+}{2} = -\frac{b}{2}(1-p).$$

(2) If $p = 1/n$, $\hat{Z} = -b/2$ (Section 4, page 882). For then $-a > b$ and a/b is an integer. Under these conditions it is easy to see that $S_{\tau_+} = b$. For $p = (n-1)/n$, n an integer, $\hat{Z}$ can be calculated using some identities that follow from the Wiener–Hopf factorization.

THEOREM 2. *$E_pS^2_{\tau_+}/E_pS_{\tau_+}$ is a continuous function of p.*

PROOF. Note first that, by the path continuity property, continuity is obvious at a value of p for which $P\{\bar{\tau}^p_+ = \tau^p_+\} = 1$, i.e., p irrational. In this case the numerator and denominator are separately continuous. This accomplishes the main purpose of Chernoff and Petkau's result, which is to show that a calculation which can be done for p rational can be extended to irrational p by continuity. Therefore, it suffices to establish continuity at some fixed, rational p.

The heuristic for why this works is that no amount of messing around near zero and continuing on the part of the random walk can affect the ratio considered here, although numerator and denominator separately can change. The simplest example of this phenomenon is

$$\frac{E_pS^2_{\bar{\tau}_+}}{E_pS_{\bar{\tau}_+}} = \frac{E_pS^2_{\tau_+}}{E_pS_{\tau_+}}.$$

This will be used below. It is an example of the heuristic because to get from $\bar{\tau}^p_+$ to τ^p_+ the random walk hits at 0 and then goes on to perform an identical copy of τ^p_+. Formally,

$$\begin{aligned} E_p(S_{\tau_+}) &= E_p(S_{\bar{\tau}_+};\ \bar{\tau}_+ = \tau_+) + E_p(S_{\tau_+};\ \bar{\tau}_+ < \tau_+) \\ &= E_p(S_{\bar{\tau}_+}) + P_p\{\bar{\tau}_+ < \tau_+\}E_p(S_{\tau_+}) \end{aligned}$$

or

$$E_p(S_{\tau_+}) = \frac{E_p(S_{\bar{\tau}_+})}{P_p\{\bar{\tau}_+ = \tau_+\}}.$$

Similarly,

$$E(S^2_{\tau_+}) = \frac{E_p(S^2_{\bar{\tau}})}{P_p\{\tau_+ = \bar{\tau}_+\}},$$

which establishes the claim.

The second example of this heuristic is

$$E_p(S_{\tau_\varepsilon}) = c(\varepsilon)E_pS_{\tau_+} + O(\varepsilon)$$

and

$$E_p(S^2_{\tau_\varepsilon}) = c(\varepsilon)E_pS^2_{\tau_+} + O(\varepsilon).$$

To see this write

$$\begin{aligned} E_p(S_{\tau_\varepsilon}) &= E(S_{\tau_+};\ \tau_+ = \tau_\varepsilon) + \int_0^\varepsilon \left[E(S_{\tau_{\varepsilon-x}}) + x\right]P\{S_{\tau_+} \in dx\} \\ &= E(S_{\tau_+}) + O(\varepsilon) + \int_0^\varepsilon E(S_{\tau_+};\ \tau_{\varepsilon-x} = \tau_+)P\{S_{\tau_+} \in dx\} \\ &\quad + \int_0^\varepsilon \int_0^{\varepsilon-x} E(S_{\tau_{\varepsilon-x-y}})P\{S_{\tau_+} \in dy\}P\{S_{\tau_+} \in dx\} \\ &= E(S_{\tau_+}) + O_1(\varepsilon) + \left(E(S_{\tau_+}) + O_2(\varepsilon)\right)P\{\tau_\varepsilon > \tau_+\} \\ &\quad + \int_0^\varepsilon \int_0^{\varepsilon-x} E(S_{\tau_{\varepsilon-x-z}})P\{S_{\tau_+} \in dz\}P\{S_{\tau_+} \in dx\}, \end{aligned}$$

where $|O_1(\varepsilon)|$, $|O_2(\varepsilon)| < \varepsilon$. Continuing in this manner establishes the claim with $c(\varepsilon) = 1 + \sum_{n=1}^{\infty} P\{\tau_\varepsilon > \tau_+^{(n)}\}$. $ES_{\tau_\varepsilon}^2$ follows analogously.

Now the result follows easily. Let $q \in [0,1)$. The path continuity property makes it clear that

$$S_{\tau_+^q}^q 1_{(\tau_+^p = \bar{\tau}_+^p)} \to S_{\tau_+^p}^p 1_{(\tau_+^p = \bar{\tau}_+^p)},$$

as $q \to p$. Convergence takes place as long as $S_{\bar{\tau}^p}^p > 0$. For the other paths, $S_{\tau_+^p}^q$ is close to zero, possibly above and possibly below. By these two observations

$$ES_{\tau_+^q}^q = ES_{\bar{\tau}_+^p}^p + ES_{\tau_+^q}^q \int_{-\delta}^{0} c(\varepsilon) P\{S_{\bar{\tau}_+^p}^q \in dx\} + o(1),$$

where $o(1) \to 0$ as $q \to p$, and $E(S_{\tau_+^q}^q)$ satisfies a similar equation. The fact about the ratios is now obvious. □

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